50 years of the Hirsch conjecture

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Algorithmic and Combinatorial Geometry, Budapest

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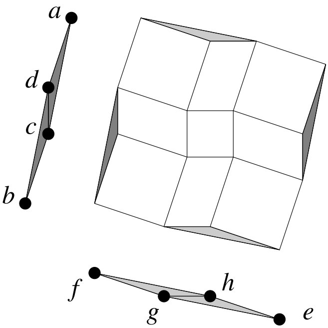
52 years of the Hirsch conjecture

(with focus on "partial counterexamples")



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Conjecture: Warren M. Hirsch (1957)

For every polytope P with f facets and dimension d,

$$\delta(P) \leq f - d.$$

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Some known cases

Hirsch conjecture holds for

- $d \leq 3$: [Klee 1966].
- $f - d \leq 6$: [Klee-Walkup, 1967] [Bremner-Schewe, 2008]
- 0-1 polytopes [Naddef 1989]
- Polynomial bound for network flow polytopes [Goldfarb 1992, Orlin 1997]
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Theorem [Kalai-Kleitman 1992]

For every d-polytope with f facets:

$$\delta(P) \leq f^{\log_2 d+2}.$$

and a subexponential simplex algorithm:

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There are random pivot rules for the simplex method which, for any linear program, yield an algorithm with expected complexity at most

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Given a linear program with d variables and f restrictions, we consider a random perturbation of the matrix, within a parameter ϵ.

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Why is $f - d$ a "reasonable" bound?

- It holds with equality in simplices ($f = d + 1$, $\delta = 1$) and cubes ($f = 2d$, $\delta = d$).
- If P and Q satisfy it, then so does $P \times Q$: $\delta(P \times Q) = \delta(P) + \delta(Q)$. In particular:

> For every $f \leq 2d$, there are polytopes in which the bound is tight (products of simplices).
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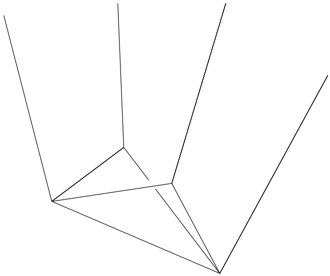
Unbounded polys. and regular triangulations

An unbounded d-polyhedron is polar to a regular triangulation of dimension $d - 1$.

Regular triangulations of dimension $d - 1$ with f vertices and diameter $f - d$ are easy to construct by "stacking" simplices one after another.



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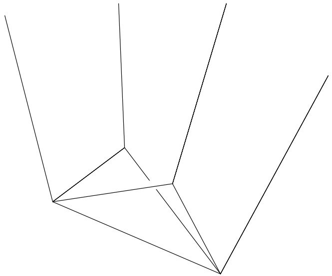

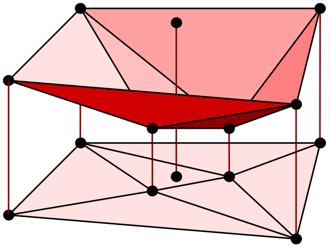

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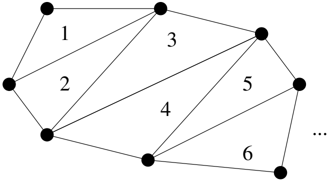

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It is possible to go from u to v so that at each step we abandon a facet containing u and we enter a facet containing v.

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Theorem [Klee-Walkup 1967]

Hirsch $\Leftrightarrow$ d-step $\Leftrightarrow$ non-revisiting.

Proof: Let $H(f, d) = \max\{\delta(P) : P \text{ is a } d\text{-polytope with } f \text{ facets}\}$. The basic idea is:

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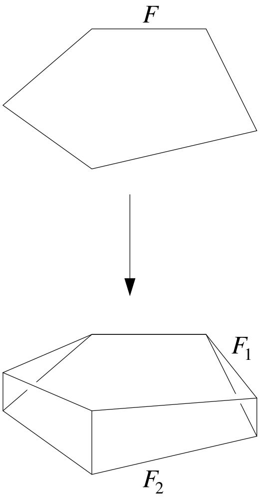

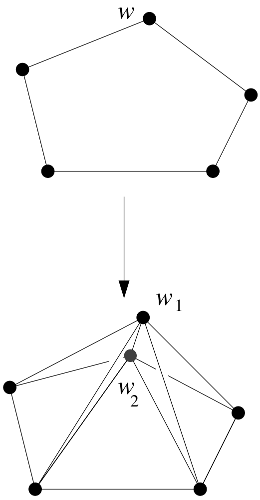

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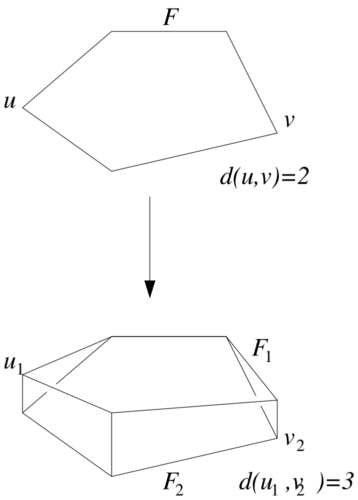

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The feasible region of a linear program can be an unbounded polyhedron, instead of a polytope.

Unbounded version of the Hirsch conjecture:

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Remark: this was the original conjecture by Hirsch.

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For any simplicial sphere of dimension $d-1$ with f vertices, the adjacency graph among $d-1$-simplices has diameter at most $f-d$.

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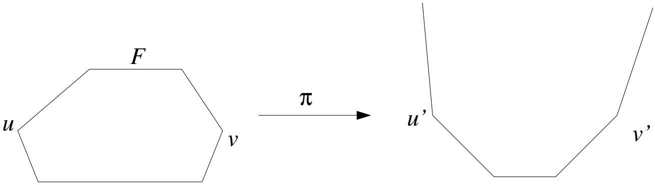

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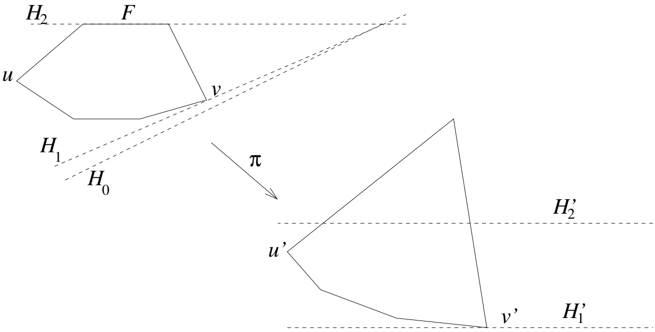

The Klee-Walkup Hirsch-tight (9,4)-polytope

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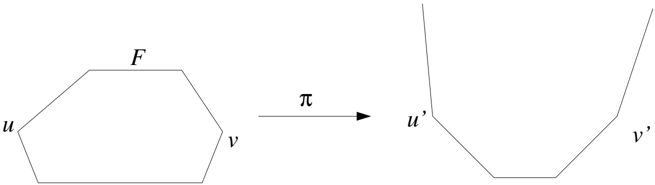
The "unbounded trick" is reversible

From an unbounded 4-polyhedron with 8 facets and diameter five we can get a bounded polytope with 9 facets and sme diameter:



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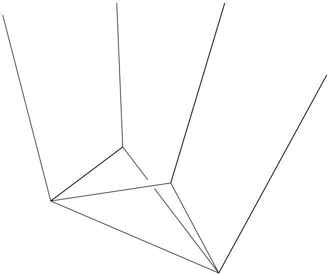
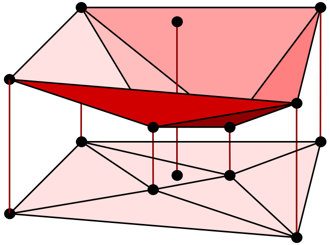
And remember that

"The polar of an unbounded 4-polyhedron with nine facets is a regular triangulation of eight points in $\mathbb{R}^3$".





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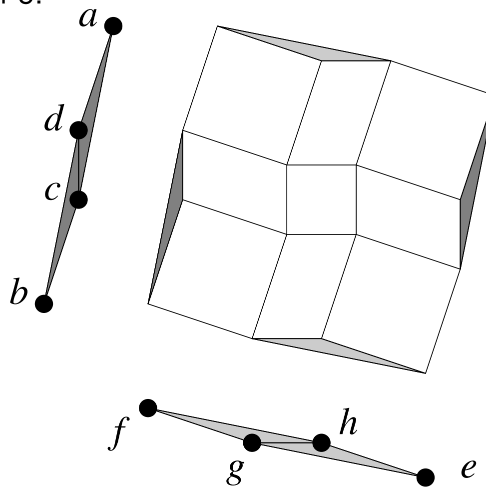
This is a (Cayley Trick view of a) 3D triangulation with 8 vertices and diameter 5:



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These are coordinates for it, derived from this description:

$$a := (-3, 3, 1, 2), \qquad e := (3, 3, -1, 2),$$
$$b := (3, -3, 1, 2), \qquad f := (-3, -3, -1, 2),$$
$$c := (2, -1, 1, 3), \qquad g := (-1, -2, -1, 3),$$
$$d := (-2, 1, 1, 3), \qquad h := (1, 2, -1, 3),$$
$$w := (0, 0, 0, -2).$$

The Mani-Walkup "always revisiting" simplicial 3-sphere

Mani and Walkup constructed a simplicial 3-ball with 20 vertices and with two tetrahedra *abcd* and *mnop* with the property that any path from *abcd* to *mnop* must revisit a vertex previously abandonded.

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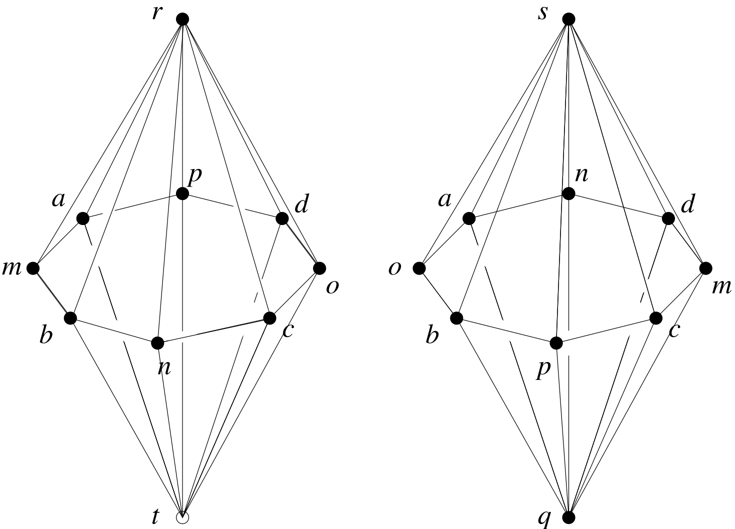

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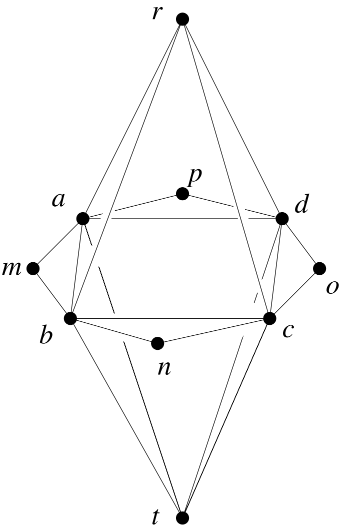

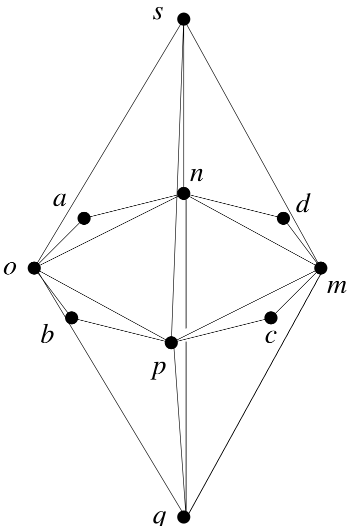

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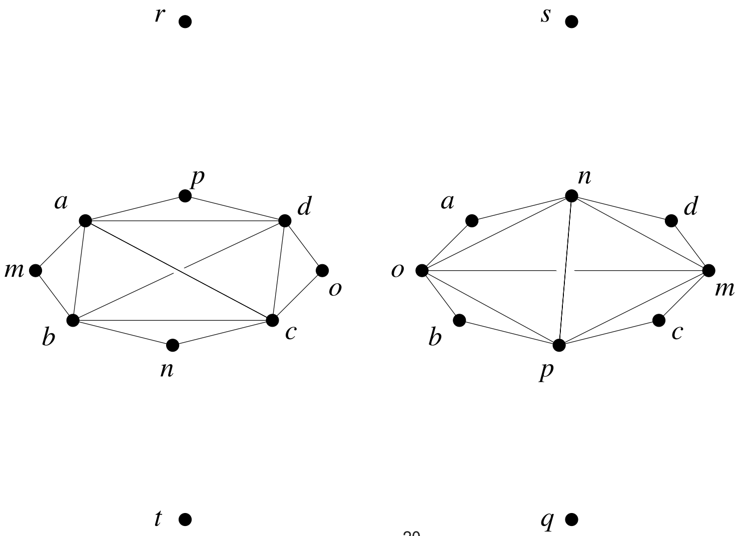

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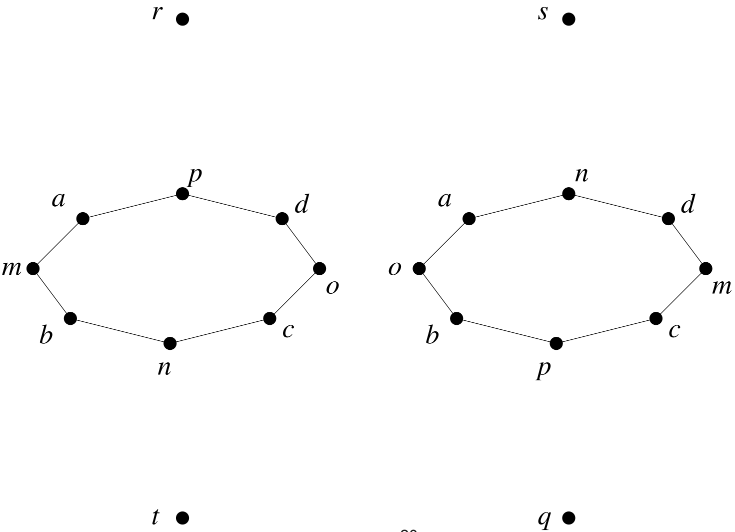

Hirsch-sharp polytopes

Hirsch tight

Politopes of dimension d, with f facets and diameter $f - d$.

- For $f \leq 2d$ they are easy to construct (e.g., products of simplices).
- For $d \leq 3$ (and $f > 2d$): they do not exist.
  $H(f, d) \sim \frac{d-1}{d}(f - d)$.
- $H(9, 4) = 5$ [Klee-Walkup 1967], but "only by chance": Out of the 1142 combinatorial types of polytopes with $d = 4$ and $f = 9$ only one has diameter 5 [Altshuler-Bokowski-Steinberg, 1980].
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2	$=$	$<$	$<$	$<$	$<$	$<$	$<$	$<$	$\cdots$
3	$=$	$<$	$<$	$<$	$<$	$<$	$<$	$<$	$\cdots$
4	$=$								
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Hirsch-sharpness for $f \leq 3d - 3$ [Klee-Holt]

When we wedge in a Hirsch-sharp polytope . . .

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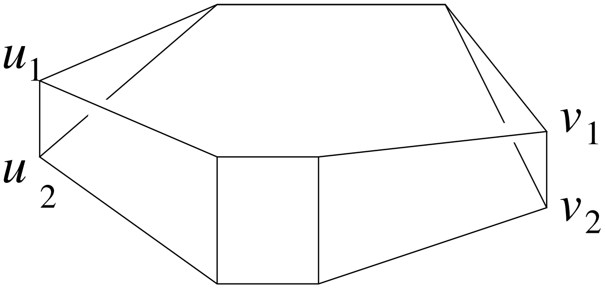

Hirsch-sharpness for $f \leq 3d - 3$ [Klee-Holt]

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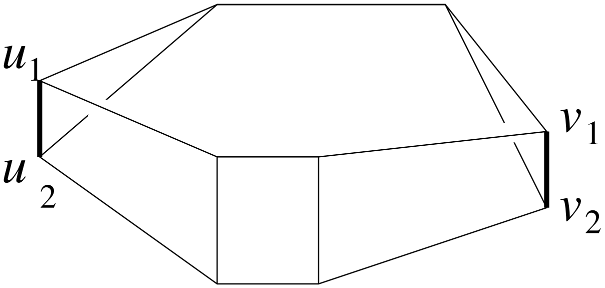

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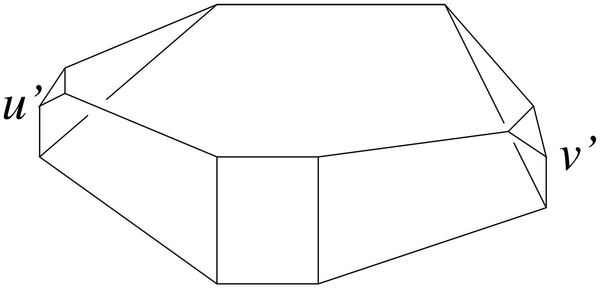

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When we glue two (simplicially) Hirsch-sharp polytopes along a facet . . . the new polytope is "Hirsch-sharp-minus-1". . . unless before glueing (at least) half of the neighbors of the glued faces were not part of Hirsch paths.

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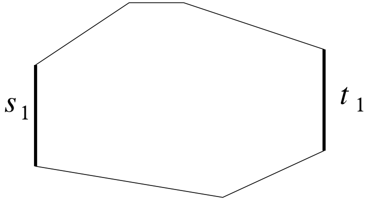

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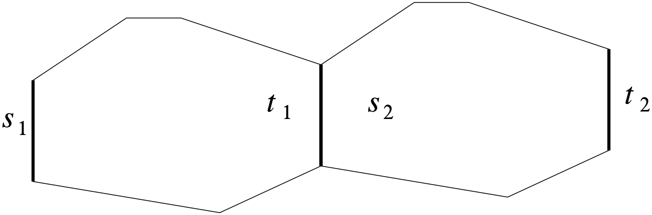

$$d(s_1, t_2) = d(s_1, t_1) + d(s_2, t_2) - 1$$

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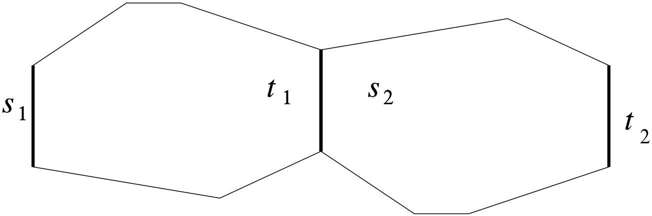

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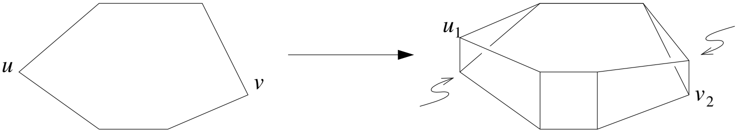

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Theorem [Holt-Fritzsche '05]

After wedging 4 times in the KW (9,4)-polytope, we can glue and preserve Hirsch-sharpness

Hirsch-sharpness for $d = 7$ [Holt]

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Same idea, but instead of based on forbiden neighbors, based on gluing along more than one simplex: Wedging three times on the KW (9,4)-polytope creates two "cliques of four simplices on eight vertices". We can glue on those eight vertices.

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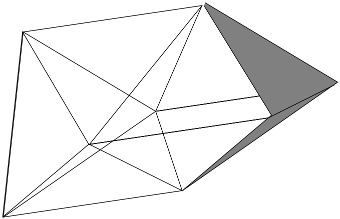

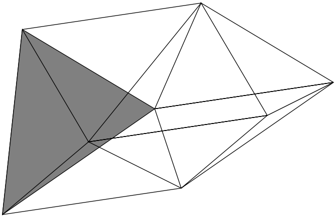

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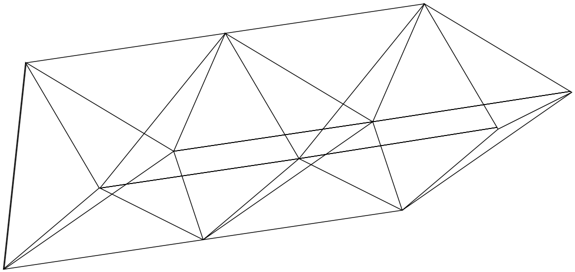

Network flow polytopes

Network

Directed graph, with demands (negative numbers) or supplies (positive numbers) associated to its vertices.

Transportation problem in a network

Minimize a certain linear functional ("cost") having one variable for each edge x_e and the restrictions:

- For each edge e $\quad 0 \leq x_e$.
- For each vertex v, the sum

$$\sum_{e \text{ exits } v} x_e - \sum_{e \text{ enters } v} x_e$$

equals the supply (positive) or demand (negative) at v.

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The flow polytope (set of feasible flows) in a network with V vertices and E edges has dimension $d \leq E - V$ and number of facets $f \leq E$.

Its diamater is polynomial:

Theorem [Cunningham '79, Goldfarb-Hao '92, Orlin '97]

Every network flow polytope has diameter bounded by $O(EV \log V)$, that is, $O(f^2 \log f)$.

Remark: these are very particular polytopes (e.g., their 2-faces have at most six sides), but extremely important in optimization.

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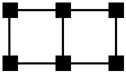

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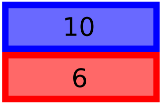

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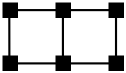

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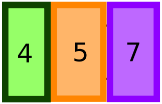

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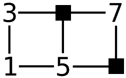

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Example

$m = n$; $a = b = (1, \ldots, 1) \Rightarrow$ Birkhoff polytope.

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3-way transportation polytopes

We now consider tables
with three dimensions.

3-way transportation polytopes

Definition

Given $a \in \mathbb{R}^l$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, the 1-marginal 3-way transportation polytope associated to them is defined in lmn non-negative variables $x_{i,j,k} \in \mathbb{R}_{\geq 0}$ with the $l+m+n$ equations

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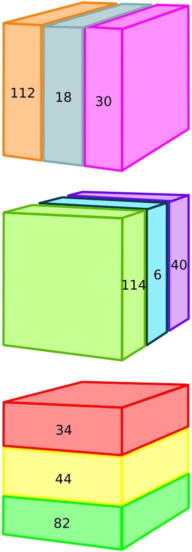

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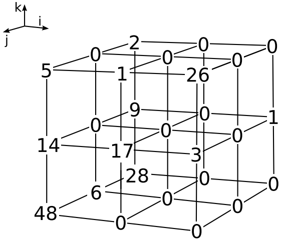

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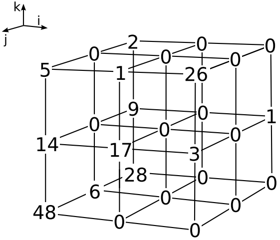

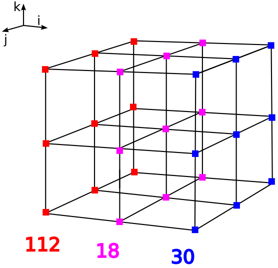

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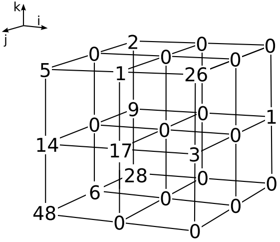

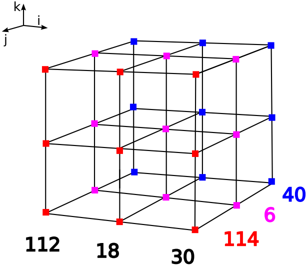

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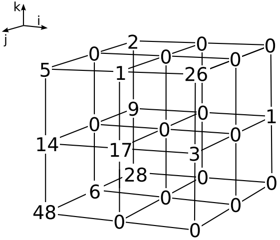

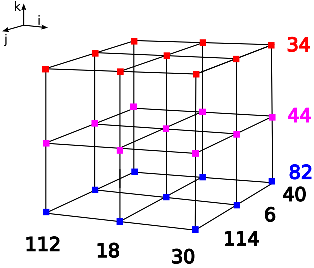

3-way transportation polytopes

Definition

Given $a \in \mathbb{R}^l$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, the 1-marginal 3-way transportation polytope associated to them is defined in lmn non-negative variables $x_{i,j,k} \in \mathbb{R}_{\geq 0}$ with the $l + m + n$ equations

$$\sum_{j,k} x_{i,j,k} = a_i \quad \forall i,$$

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2-marginal version

Same definition but with $lm + ln + mn$ equations.

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2-marginal version

Given three matrices $A \in \mathbb{R}^{lm}$, $B \in \mathbb{R}^{ln}$ and $C \in \mathbb{R}^{mn}$,

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Universality of 3-way transportation polytopes

Theorem [De Loera-Onn 2004]

Given any polytope P, defined via equations with rational coefficients,

- There is a 2-marginal 3-way transportation polytope isomorphic to P.
- There is a 1-marginal 3-way transportation polytope with a face isomorphic to P.
- Moreover, both can be computed in polynomial time starting from the description of P.

Theorema: De Loera-Kim-Onn-Santos 2007]

Every 1-marginal 3-way transportation polytope with f facets has diameter bounded by $4f^2$.

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The end

THANK YOU!