Convocatorias 2014
Proyectos de I+D "EXCELENCIA" y Proyectos de I+D+I "RETOS INVESTIGACIÓN"
Dirección General de Investigación Científica y Técnica
Subdirección General de Proyectos de Investigación

AVISO IMPORTANTE

En virtud del artículo 11 de la convocatoria **NO SE ACEPTARÁN NI SERÁN SUBSANABLES MEMORIAS CIENTÍFICO-TÉCNICAS** que no se presenten en este formato.

Lea detenidamente las instrucciones que figuran al final de este documento para rellenar correctamente la memoria científico-técnica.

Parte A: RESUMEN DE LA PROPUESTA/SUMMARY OF THE PROPOSAL

INVESTIGADOR PRINCIPAL 1 (Nombre y apellidos):

Francisco Santos Leal

INVESTIGADOR PRINCIPAL 2 (Nombre y apellidos):

TÍTULO DEL PROYECTO: Combinatoria y complejidad de estructuras geométricas discretas

ACRÓNIMO:

RESUMEN Máximo 3500 caracteres (incluyendo espacios en blanco):

Se propone el estudio de diversos problemas relacionados con la combinatoria y la complejidad de ciertos objetos geométricos. La elección de temas se ha hecho con vistas a las aplicaciones, algoritmos, y conexiones con otras áreas. Nuestros objetivos se estructuran en cuatro líneas de investigación, no totalmente disjuntas.

Línea 1: Complejos politopales y simpliciales. El principal objetivo en este bloque es entender ciertas cuestiones sobre poliedros y politopos. Estudiamos en especial sus grafos y la Conjetura de Hirsch Polinómica, el número y estructura de sus "missing faces" y el número de politopos y esferas simpliciales para cada dimensión y número de vértices.

Línea 2: Politopos reticulares y su relación con geometría algebraica. Los politopos reticulares (es decir, politopos con coordenadas enteras) aparecen frecuentemente en el contexto de la geometría tórica y tropical. Nos planteamos preguntas relacionadas con su enumeración y con la existencia de triangulaciones unimodulares.

Línea 3: Combinatoria de configuraciones. Aquí nos centramos en varias preguntas sobre configuraciones de puntos, ya sea en el plano o en dimensión superior: número de cruce, números de k-conjuntos y rectas o planos bisectores, j-caras, g-vectores. Nos interesamos por generalizaciones del Teorema de Erdos-Szekeres y por el número de cruce topológico del grafo completo.

Línea 4: Grafos geométricos. Aquí consideramos diversas estructuras en el plano, especialmente pseudotriangulaciones, multitriangulaciones y su generalización común a través de arreglos de pseudorrectas con soporte dado a priori. Nos interesamos también por propiedades de la envolvente convexa cuando la convexidad se define a través de un número finito de direcciones. Estudiamos también aplicaciones de coloración de grafos a la asignación de espectros de frecuencias en telecomunicación.

PALABRAS CLAVE: Geometría Discreta, Combinatoria, Politopo, Grafo, Complejidad,

TITLE OF THE PROJECT: Combinatorics and complexity of discrete geometric structures

ACRONYM:

SUMMARY Maximum 3500 characters (including spaces):

Several problems regarding the combinatorics and complexity of certain geometric objects will be studied. The topics are chosen with an eye on applications, algorithms, and relations to other areas. We structure our research goals in four (not completely disjoint) research lines:

Research Line 1: Polytopal and simplicial complexes. The aim in this block is to understand several combinatorial questions involving polytopes and polyhedra. In particular, we study their graphs and the polynomial Hirsch Conjecture, the number and structure of their missing faces, and the number of different simplicial polytopes or simplicial spheres for a given dimension and number of vertices.

Research Line 2: Lattice polytopes. Lattice polytopes (i.e., polytopes with integer vertices) play a role in several areas of mathematics. Most significantly, they arise in the context of toric and tropical geometry. We address questions related to their classification and to triangulations of them.

Research Line 3: Combinatorics of point configurations and topological graphs. Here we address several questions concerning point configurations, be it in the plane or in higher dimension: crossing number, number of k-sets and halving lines and planes, j-facets, g-vectors. We are also interested in generalizations of the Erdos-Szekeres Theorem and in further exploiting recent applications of k-edges and related concepts to the study of the topogical crossing number of the complete graph.

Research Line 4: Abstract and geometric graphs. Here we consider several geometric structures in the plane, especially pseudotriangulations, multitriangulations and their common generalization via arrangements of pseudo-lines with given support. We are also interested in questions regarding the convex hull of a point set when convexity is defined by a set of directions. Finally, we explore applications of graph coloring to spectrum assignment in telecommunications.

KEY WORDS: discrete geometry, combinatorics, complexity, polytopes, graphs

Parte C: DOCUMENTO CIENTÍFICO

C.1. PROPUESTA CIENTÍFICA

Remark: All subsections in C.1 refer to the bibliography included in section C.1.6, page "8 de 20 / Parte C"

C.1.1 Introduction

The goal of this project is the advance of our knowledge in several problems regarding the combinatorics and complexity of certain geometric objects. It is intended as a pure, basic, mathematical research, but the topics are chosen with an eye on applications, algorithms, and relations to other areas. Most notably, one of our lines of research tries to advance our knowledge regarding the diameters of graph polytopes. The principal investigator of this team has recently refuted the long-standing „Hirsch conjecture" [San12], a problem in polyhedral combinatorics with implications to the complexity of the simplex method in linear programming. But his solution leaves open the fundamental underlying question of how large can the diameter of a polytope be in terms of its definining variables and equations. The answer to this question, perhaps the most important open question in discrete and computational geometry right now, could have a great impact for linear programming algorithms.

The project is a continuation of the currently active project MTM2011-22792. The research team is the same, consisting of three full senior members (Francisco Santos, Pedro Ramos and David Orden) and one Ph D student financed by the FPI program from the spanish ministry of science. The first three are quite active in research production and the fourth has just finished her Master studies. We also have several external frequent collaborators. Although none of them is formally included as a member of this project, we would like to mention Günter Ziegler (Freie U. Berlin) [CSZ11+, SaZi13], Christian Haase (freie U. Berlin) [HPPS14+, BHHL+13+, HNPS08], Oswin Aichholzer (Technische U. Graz) [AHORRSS13, AHHHPSSV13, AAFRS13, AAFRS13b, AAFRS14+], and Vincent Pilaud (CNRS and Ecole Polytechnique) [PfPiSa12, PiSa12], related respectively to the four research lines in which we structure the project.

Research Line 1: Polytopal and simplicial complexes. The aim in this block is to understand several combinatorial questions involving polytopes and polyhedra. In particular, we study their graphs and the *polynomial Hirsch Conjecture*, the number and structure of their missing faces, and the number of different simplicial polytopes or simplicial spheres for a given dimension and number of vertices.

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C.1.2 Background and state of the art

The following paragraphs contain a description, for each of the research lines of the project, of the background and the state of the art of scientific knowledge. The full references of the works cited here are listed in Section 4.

Research Line 1: Polytopal and simplicial complexes.

The graph of a polytope is one of its most basic combinatorial features. Yet, our current understanding of it is quite defective. For 3-dimensional polytopes there is a good characterization of polytopal graphs (they are the 3-connected planar graphs [Zie94]) but starting in dimension 4 such a simple characterization fails and no general method to check polytopality of a graph is known. Members of the research team have recently stated polytopality of special classes of graphs [PfPiSa12]; one of our goals is to continue this research.

But the most important open questions on polytope graphs relate to its *diameter*. The long-standing *Hirsch conjecture* [Dan63, Zie94, KimSan10] (equivalent to the d-step conjecture [Alt85, KaKl92, Klee66, KlWa67]), stated that a graph with n facets and dimension d (that is, defined by n inequalities in d variables) cannot have graph diameter larger than n-d. Although the conjecture has been disproved recently by the PI of this project [San10, San12], the counter-examples produced so far still have their diameter growing as a linear function of d and n. This leaves two fundamental open questions: construct polytopes with superlinear diameter and prove polynomial upper bounds for polytope diameters. Both questions are probably too difficult to be posed explicitly as „goals“ in this project, but they certainly deserve attention. Two possible approaches to the second one are via 3-way *transportation polytopes* [dLKOS09], via abstractions of graph polytopes [San13] or extending the recent proof of the Hirsch bound by Adiprasito and Benedetti in the case of flag polytopes [AdBe13]

Another question we are interested in is the g-theorem on face numbers of simplicial polytopes. The g-theorem is a statement that describes precisely the possible f-vectors that simplicial polytopes can have (the f-vector of a polytope is the vector counting the numbers of faces of all dimensions. The „problem“ with this theorem is that the proofs known so far are quite algebraic: the original one by Stanley [Stan80] using cohomology of toric varieties, and one by McMullen [McMull2006] using the „algebra of polytopes“. We would very much like to find a combinatorial proof. A recent results in this direction is Murai and Nevo's recent proof of the Generalized Lower Bound Conjecture [MuNe13] for which Nevo was awarded the Erdos Prize in 2014.

Last, a third interesting question is about the number of combinatorially different simplicial polytopes and spheres of a given dimension and number of vertices. Although our recent preprint [NeSaWi14+] (with the afore mentioned Nevo and with S. Wilson) closes substantially the gap between the previous best lower and upper bounds, there is still room for work and improvement, specially regarding the number of geodesic spheres in odd dimension.

Research Line 2: Lattice polytopes.

Lattice polytopes are the basic link between combinatorics and algebraic geometry: Every lattice polytope gives rise to a *projective toric variety* and there is a very precise dictionary between the properties of the former and of the latter [KKMS73, Oda88, MS05, San05]. They also have connections to Representation Theory, Statistics, and Optimization [Rez08].

In particular, there are several interesting open questions about normality and resolution of singularities in toric varieties that can be translated into questions about covering properties of the polytope P and the behaviour of the semigroup of lattice points in the cone over P [BrGu02, BGHMW99, FiZi99]. One fundamental open question, to which two international workshops have been devoted in the last years (Oberwolfach 2007, AIM 2009), is the so-called Oda's question: *Is every smooth polytope „projectively normal"?* In particular*, does it admit a unimodular triangulation?* [HNPS08, San04].

Concerning unimodular triangulations, a fundamental result of Knudsen, Mumford, and Waterman [KKMS73] asserts that for every lattice polytope P there is a dilation constant k such that kP admits a unimodular triangulation. The open question(s) are to find good bounds for k in terms of the parameters of P (volume, number of vertices, dimension). It is even open the question of whether there is a constant k=k(d) that is valid for all the polytopes of a given dimension d. This was solved in the affirmative for d=3 (giving k(3)≤4) by Kantor and Sarkaria [KaSa03] but is open starting in dimension four. Recently, Santos and Ziegler [SaZi13] have extended the Kantor-Sarkaria results to show that *almost every* k is valid for d=3 (more precisely, k=1,2 are not valid, k=3,5 are undecided, every other k is valid). Also, Santos et al [HPPS14+] have given the first effective version of the KMW theorem, giving an explicit bound for the needed k in terms of the dimension and the volume of the polytope. But this bound is doubly exponential; it would be good to find a singly exponential one.

We are also very much interested in classification and structural questions about lattice polytopes with few lattice points. Even in the simplest case of empty simplices (lattice simplices with no lattice point other than its vertices) a full classification of them is known only in dimension 3 [Whi63]. In dimension four only a finiteness result is known saying that all but finitely many lattice simplices have width at most 2 [BBBK11].

In dimension three, the next cases after empty simplices (that is, lattice polytopes with five or six lattice points) has recently been completely solved by the PI and his Ph D Student Mónica Blanco [BlSa14+a, BlSa14+b]. Still, it would be interested to continue the classification (at least partially) up to eight points, since this would include all „distinct pair sum" polytopes, of interest for their connections with the representation of polynomials as sums of squares [ChLaRe02, Rez08].

<u>Research Line 3: Combinatorics of point configurations</u>

The computation of crossing numbers of graphs (or their asymptotic bounds) is an outstanding task in combinatorics, which has been explored in a long list of works [Vrto11] since the seminal paper of Zarankiewicz [Zar54]. Looking for the rectilinear crossing number of K_n, $cr(K_n)$, is among the most popular in this kind of problems. In the last ten years, major advances have been obtained in the problem, following results by Ábrego, Lovász et al. [AF05,LVWW04] that connected the study of cr(Kn) to the study of lower bounds on the number of at most k-edges that a set of n points can have (an at most k-edge is a line defined by two points of the set and leaving to one side at most k points of it), another central problem in discrete and computational geometry [BMP05]. See also [Ram10].

In a sequence of papers, some members of the applicant team and Ábrego et al. [AGOR07, AFLS08] obtained major improvements on the known lower bounds. Nevertheless, there is still a gap with respect to the best upper bound knows [ACFLS10]. More importantly, the only known structural property of optimal sets is that they have a triangular convex hull [AGOR07]. A deeper insight on the structure of optimal sets could lead to a proof of the two interesting conjectures, concerning symmetry and decomposability of those optimal point sets [ACFLS10].

Recently, these techniques have been succesfully extended to the more general setting of topological graphs by a team including one of the members of the applicant group [AAFRS13]. Computing the (general) crossing number of Kn is an old open problem whose history is nicely sumarized in [BW10]. 50-years-old Harary-Hill conjecture states that the crossing number is exactly the number of crossings obtained in so-called cylindrical drawings. In [AAFRS13], the conjecture was shown to be true for 2-page drawings of the complete graph. Shortly after, the same ideas were generalized to confirm the conjecture for monotone drawings [AAFRS13b,BKF13]. In a further step, the concept of shellable drawing of a graph has been recently stablished [AAFRS14+] as the key to further exploit k-edges of topological drawings. Shellable drawings include 2-page, monotone and cylindrical, and so Harary-Hill conjecture has been stablished for those families of graphs. Our objective in this problem is trying to extend the proof to a more general class of graphs, hopefully covering all possible drawings.

A particular instance of j-edges are those called halving edges which have, for n even, j=(n-2)/2. Finding the maximum number of halving edges is another important open problem in discrete geometry, first studied by Erdös, Lovász, Simmons, and Straus [Lov71, ELSS73], for which some partial results have been obtained in the last years [BR02, AFLS08b]. To the best of our knowledge, there are no results about the structural properties of sets attaining the maximum number of halving edges, similar to those developed in [AGOR07, AGOR09].

It is worth to mention that j-facets of point configurations are related to the g-vector of polytopes [ShaWel03], already mentioned in research line 1. Hence, work on this research line might also help the goal of working towards a combinatorial proof of the g-theorem. In particular, in [ORS10] members of this team have provided the first proof for the exact minimum number of halving linear planes (in the space) that does not use the Generalized Lower Bound Theorem. The main tool is using the concept of balanced lines for bicolored point sets, where a bicolored point set is basically the same as an affine Gale diagram, a classical tool in polytope combinatorics. It might be possible to continue that work and try to actually prove the Generalized Lower Bound Theorem, or the g-theorem, in this context.

Looking for the number of crossings when embedding the complete graph on a given point configuration is equivalent to looking for the number of empty convex quadrilaterals. This relates the previous type of problems to the classical Erdős-Szekeres problem: For any integer k, $k \geq 3$, determine the smallest positive integer N(k) such that any planar point set in general position that has at least N(k) points contains k points that are the vertices of a convex k-gon.

Since 1935, when Erdős and Szekeres proved the finiteness of N(k) using Ramsey theory, this and other related questions have attracted a huge interest from researchers in Combinatorics. In particular, it has been the main topic of the just finished collaborative research project ComPoSe („Combinatorics of Point Sets and Arrangements of Objects“) within the EuroGIGA call of the EuroCORES program by the ESF. Oswin Aichholzer, frequent collaborator of the present applying team, was the cordinating PI. The spanish team (with PI Ferran Hurtado) included the three main members of this research project, David Orden, Pedro Ramos and Francisco Santos.

Recently, Erdős-Szekeres problems have also arisen in the long-standing research area of games [Beck08]. In particular, the monochromatic Maker-Breaker game has been studied for empty convex k-gons [PaSa13], using a quite exhaustive case analysis. Other possibilities remain open, like bichromatic points or Avoider-Avoider versions.

Research Line 4: Abstract and geometric graphs

Apart of the connections that Research Line 3 has to geometric graph theory, in this research line we include questions on more structured graphs, tipically giving rise to nice decompositions in the plane, and questions on the number and properties of the graphs of such type that can be embedded with a given set of vertices in the plane. The model situation is that of triangulations of a convex n-gon. The number of them is the ubiquituous Catalan number and the different triangulations are connected to one another by flips, forming a graph that turns out to be polytopal (the graph of the *associahedron*, a polytope of interest in algebraic topology, representation theory, algebraic geometry, theory of algorithms, etc) [DRS10].

Generalizing that situation, members of the research team have studied triangulations of arbitrary point sets [San00, San07], pseudo-triangulations in the plane [RSS03, OS05, OSSS07, RSS08], multitriangulations [PiSan09] and quadrangulations [FHKO10]. Even more so, recently Vincent Pilaud has found, jointly with Michel Pocchiola, a common framework for most of these objects, based on studying pseudoline arrangements on sorting networks [PiPo12]. Several questions related to the graph of flips of such objects are still claiming attention, among which we may point out its study when the vertex and face degrees are bounded. A first step towards this study has been recently achieved by members of the team [AHORRSS13].

Among the structures above, triangulations, pseudo-triangulations, and quadrangulations can be viewed as a tessellation of the convex hull of the point set. This crucial notion can be generalized by redefining convexity to use a subset O of directions instead of all possible ones [FiWo04]. Before aiming to generalize the above structures, several new problems arise just considering this O-hull. For an example, how is it affected by transformations on the set O of directions.

Finally, we also consider some questions on applications of graphs. Graph coloring is one of the central problems in the area [JeTo95], which is well known for having applications to many different fields [PaMaXu99]. One of them is the spectrum assignment in telecommunications, where graph coloring constitutes one of the most widely used approaches [TZFS13]. However, in this survey Tragos et al. point out that the basic constraint of two connected vertices not being assigned the same color (corresponding to the same spectrum band) does not reflect the adjacent-channel interference, which has been shown to cause severe performance degradation in Cognitive Radio Networks. One member of this team (David Orden) has been asked to collaborate on this problem by the Telematic Services Engineering Research Group of the Universidad de Alcalá.

C.1.3 Previous results of the team on these lines of research can be summarized as:

- Related to research line 1, Francisco Santos has recently found counter-examples to the Hirsch Conjecture [San12, San10], one of the oldest and more fundamental questions in the area of geometric combinatorics. After the first counter-example was found, two follow-up papers explored the extent to which the ideas that led to it (based on the use of a class of so-called *prismatoids*) could be generalized [SST12,MSW12+]. He has also shortened the gap between lower and upper bounds for the numbers of polytopes and simplicial spheres [NeSaWi14+], and computed asymptotically exact upper and lower bounds for the diameters of pure simplicial complexes of given dimension and number of vertices [San13]
- Related to research line 2, [SaZi13] extends the Kantor-Sarkaria Theorem on unimodular triangulations of dilated 3-polytopes and [HPPS14+] contains the first effective

version on the KMW theorem on arbitrary dimension. Also, Blanco and Santos have initiated the classification of three-dimensional lattice polytopes with few lattice points [BlSa14+a, BlSa14+b].

- Related to research line 3, Oswin Aichholzer, David Orden and Pedro Ramos (together with J. García, 2007 and 2009) found structural properties of sets minimizing the rectilinear crossing number, which led to improved lower bounds on this invariant. Also, together with G. Salazar (2010) they have found a proof for the number of halving linear planes of a 3d vector configuration, independent of the Generalized Lower Bound Theorem. Pedro Ramos participated in the team that recently proved Harary-Hill conjecture for 2-page drawings [AAFARS13] and further extended it to so-called shellable drawings [AAFARS14]
- Related to research line 4, Pilaud and Santos (2009) have found very interesting structural properties of multi-triangulations. Also, David Orden and F. Santos (with several authors, 2003, 2004, 2005, 2007) have proved several properties of pseudo-triangulations. Finally, David Orden and P. Ramos (2013) have stated some results for the flip graph of triangulations whose maximum vertex degree is bounded by a constant. Santos et al. [SSW14+] have studied a generalization of the non-crossing relation for diagonals of the n-gon related to the Hilbert series of arbitrary Grassmannians.

C.1.4 General goals

Combinatorics, and geometric combinatorics in particular, has undergone an increase in relevance and connections to other parts of mathematics (algebraic geometry, representation theory, topology) and applied sciences (computer science, molecular biology, statistical mechanics, optimization, computer aided design and manufacturing, logistics,…). The IP of this project has solid records of research in pure geometric combinatorial problems, but with an eye to applications in other areas. His construction of disconnected triangulation spaces (2000-2006) gave interesting consequences in the context of algebraic geometry (toric Hilbert schemes) and combinatorial topology (generalized Baues conjecture). His recent counter-example to the Hirsch conjecture (2010, 2012) has had a big impact not only in the discrete geometry community but also in the optimization and operations research communities. The other two senior researchers in the team, Pedro Ramos and David Orden, have since long worked in similar topics and are also well-positioned in the international computational geometry community.

This project is designed to continue the fruitful lines of research undergone so far. As the title indicates, the common thread to all of them is their relation to "geometric complexity", be it in the enumerative sense (how many solutions a geometric problem has) or in the quantitative (how large or complex can be a geometric object, with respect to its algebraic description). They are problems, hence, implicitly related to computational and algorithmic aspects. In some of them the relation is explicit, and the goal is to find good algorithms to solve these problems. In others the goal is more structural or theoretical, but these theoretical results may also help improving certain algorithms or better understanding their complexity.

C.1.5 Specific goals

<u>Research Line 1: Polytopal and Simplicial Complexes</u>

Francisco Santos will be in charge of this line, which will include work by Student 1 (U of Cantabria) and of the rest of the team members. G. M. Ziegler (FU Berlin) is considered external collaborators for this line.

<u>Goal 1.1:</u> Answer questions related to the diameter of poytope graphs and the Polynomial Hirsch Conjecture:

- Try to extend the Adiprasito-Benedetti proof to further classes of polytopes, or abstract objects. Does the proof work at the level of „connected layer multifamilies" [San13]?

- Is there a quantitative version of the Adiprasito-Benedetti bound for non-flag polytopes?

Goal 1.2: Answer questions related to the structure and number of simplicial polytopes or spheres:

- Further close the gap between lower and upper bounds on the number of simplicial polytopes or spheres, or find interesting classes of them that match the lower bound.
- Find combinatorial proofs of UBT-type theorems, and explore the connections (via Gal duality) between the g-vector of a polytope and the numbers of j-facets of ist Gale transform (see also Research Line 3).

Research Line 2: Lattice Polytopes

Francisco Santos will be in charge of these research line, which includes the work of the Ph D student Mónica Blanco. Christian Haase and G. M. Ziegler (FU Berlin) are considered external collaborators for this line. In particular, the student M. Blanco is about to start a five months stay at FU Berlin.

Goal 2.1: Keep studying questions about unimodular triangulations of lattice polytopes. For example:

- Is there a singly-exponential upper bound (in terms of dimension and volume) for the KMW dilation factor of every polytope? The bound in [HPPS14+] is doubly exponential.
- Is there an analogue of “Dehn’s invariant” for lattice 3-polytopes, implying that two lattice 3-polytopes with the same Ehrhart function may not be equidecomposable into finitely many pieces? What about higher dimension?

Goal 2.2: Continue the classification of lattice three-polytopes and lattice empty d-simplices.

- Classify dps three-polytopes of dimension three (which are known to have at most eight lattice points) or at least show sufficient structural properties to solve some of the questions about them raised in [ChLaRe02, Rez08]. Specifically: does every maximal dps 3-polytope have **exactly** 8 lattice points?
- Classify or bound the number of empty four-simplices of width greater than two, which are known to be finitely many by [BBBK11]

Research Line 3: Combinatorics of point configurations

Pedro Ramos (in charge), David Orden and the Student 2 will work on this line. Oswin Aichholzer (T. U. Graz, Austria) is considered an external collaborator in it.

Goal 3.1: Find a winning strategy for the Erdős-Szekeres game in which A and B play in alternating turns, placing monochromatic points in the plane aiming to avoid the creation of an empty convex k-gon.

- The natural starting point will be the work [PaSa13]. Geometric and combinatorial properties will be exploited in order to find a simpler proof for the case k=5 and, hopefully, study the cases for k>5.

Goal 3.2: Improve the known bounds for certain variations of the crossing number of the complete graph, most specifically the topological crossing number and the sphere crossing number.

Research Line 4: Abstract and geometric graphs
David Orden (in charge), Pedro Ramos and the Student 2 will work on this task. Oswin Aichholzer (T. U. Graz) and Vincent Pilaud (CNRS-Ecole Polytechnique) are considered external collaborators in it.

Goal 4.1: Determine the connectivity of the graph of flips for geometric pseudo-triangulations with face-degree bounded by four, known as 4-PPTs. Consider the same problem for combinatorial pseudo-triangulations instead of geometric ones.

- The techniques used by members of the team in [AHORRSS13], together with the expertise in pseudo-triangulations [RSS03, OS05, OSSS07, RSS08] will be a useful background for research on this topic.

Goal 4.2: Study, from the combinatorial and algorithmic points of view, the problem of, given a set of directions O, finding the rotation angle which minimizes/maximizes the area of the O-hull: (1) When rotating the whole set O. (2) When changing the angles between some of the directions in O.

- The results and software obtained in [Exp09] will be used to get insight enough for trying to extend the results in [AGRSU12].

Goal 4.3: Define and solve an enriched graph coloring problem for a better modelling of spectrum assignment in Cognitive Radio Networks.

- The issues pointed out in [TZFS13] will be analyzed and current techniques for graph coloring [JeTo95] will be explored, in order to test their viability for the enriched problem. In addition, an automated-negotiation aproach [MKJA14] will be tested and compared for both the basic and the enriched problem.

C.1.6 References

[AF05] B. M. Ábrego and S. Fernández-Merchant. A lower bound for the rectilinear crossing number. Graphs and combinatorics 21:3 (2005), 293-300.

[AAFRS13] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, G. Salazar. The 2-page crossing number of Kn. Discrete & Computational Geometry 49 (4), 747-777, 2013.

[AAFRS13b] B.M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, and G. Salazar. More on the crossing number of Kn: Monotone drawings. Electronic Notes in Discrete Mathematics, 44:411-414, 2013. Special issue dedicated to LAGOS2013.

[AAFRS14+] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, G. Salazar. *Shellable drawings and the cylindrical crossing number of K_n.* Discrete & Computational Geometry, in press.

[ACFLS10] B. M. Ábrego, M. Cetina, S. Fernández-Merchant, J. Leaños, and G. Salazar. 3-symmetric and 3-decomposable drawings of Kn. Discrete Applied Mathematics 158:12 (2010), 1240-1258.

[AFLS08] B. M. Ábrego, S. Fernández-Merchant, J. Leaños, and G. Salazar. A central approach to bound the number of crossings in a generalized configuration, Electronic Notes in Discrete Mathematics 30 (2008), 273–278.

[AFLS08b] B. M. Ábrego, S. Fernández-Merchant, J. Leaños, and G. Salazar. The maximum number of halving lines and the rectilinear crossing number of Kn for n≤27. Electronic Notes in Discrete Mathematics 30 (2008), 261-266.

[AB13] K. Adiprasito and B. Benedetti, The Hirsch conjecture holds for normal flag complexes, Math. of Operations Research.

[AGOR07] O. Aichholzer, J. García, D. Orden, and P. A. Ramos. New lower bounds for the number of (at most k)-edges and the rectilinear crossing number of Kn. Discrete and Computational Geometry 38:1 (2007), 1-14.

[AGOR09] O. Aichholzer, J. García, D. Orden, and P. A. Ramos. New results on lower bounds for the number of (at most k)-facets, European Journal of Combinatorics 30:7 (2009), 1568-1574.

[AGRSU12] C. Alegría-Galicia, T. Garduño, A. Rosas-Navarrete, C. Seara, and J. Urrutia. Rectilinear convex hull with minimum area. Lecture Notes in Computer Science 7579 (2012), 226-235.

[AHHHPSSV13] Oswin Aichholzer, Thomas Hackl, Michael Hoffmann, Clemens Huemer, Attila Pór, Francisco Santos, Bettina Speckmann, Birgit Vogtenhuber, Maximizing Maximal Angles for Plane Straight-Line Graphs. Computational Geometry: Theory and Applications 46:1 (January 2013), 17-28. DOI: 10.1016/j.comgeo.2012.03.002

[AHORRSS13] O. Aichholzer, T. Hackl, D. Orden, P. Ramos, G. Rote, A. Schulz, and B. Speckmann. Flip Graphs of Bounded Degree Triangulations. Graphs and Combinatorics 29:6 (2013), 1577-1593.

[Alt85] A. Altshuler. The Mani-Walkup spherical counterexamples to the $W\sb \nu$-path conjecture are not polytopal. Math. Oper. Res. 10 (1985), no. 1, 158-159.

[BKF13] M. Balko, J. Kyncl, R. Fulek. *Monotone crossing number of complete graphs*, in: Proceedings of the XV Spanish Meeting on Computational Geometry, 2013.

[BBBK11] Margherita Barile, Dominique Bernardi, Alexander Borisov and Jean- Michel Kantor, On empty lattice simplices in dimension 4, Proc. Amer. Math. Soc. 139 (2011), 4247-4253

[BW10] L. Beineke and R. Wilson. *The early history of the brick factory problem*. Mathematical Intelligencer, 32:41--48, 2010.

[BlSa14a+] M. Blanco and F. Santos, Classification of lattice 3-polytopes with five points. Preprint, 19 pages, September 2014. arXiv:1409.6701

[BlSa14b+] M. Blanco and F. Santos, Classification of lattice 3-polytopes with six points, in preparation.

[Beck08] J. Beck. Combinatorial Games: Tic-Tac-Toe Theory. Cambridge University Press, 2008.

[BR02] A. Beygelzimer and S. Radziszowski. On halving line arrangements. Discrete Mathematics 257 (2002), 267-283.

[BHHL+13+] Tristram Bogart, Christian Haase, Milena Hering, Benjamin Lorenz, Benjamin Nill, Andreas Paffenholz, Günter Rote, Francisco Santos, Hal Schenck, Few smooth d-polytopes with N lattice points, Preprint, October 2010, 19 pages. Updated July 2013., arXiv:1010.3887. Accepted in Israel J. Math.

[BrGu02] W. Bruns and J. Gubeladze (2002). Unimodular covers of multiples of polytopes. Doc. Math. 7 (2002), 463-480.

[BGHMW99] W. Bruns, J. Gubeladze, M. Henk, A, Martin, and R. Weismantel. A counterexample to an integer analogue of Carathéodory's Theorem. J. reine angew. Math. 510 (1999), 179-185.

[BMP05] P. Brass, W. Moser, and J. Pach. Research Problems in Discrete Geometry. Springer-Verlag, New York, NY, 2005.

[CSZ11+] Cesar Ceballos, Francisco Santos, Günter M. Ziegler, Many non-equivalent realizations of the associahedron, Preprint, September 2011. arXiv:1109.5544 Accepted in Combinatorica.

[ChLaRe02] M. D. Choi, T. Y. Lam, B. Reznick. Lattice polytopes with distinct pair-sums. Discrete Comput. Geom. 27 (2002), 65-72.

[Dan63] G. B. Dantzig. Linear programming and extensions. Princeton University Press, 1963. Reimpreso en la serie Princeton Landmarks in Mathematics, Prin- ceton University Press, 1998.

[dLKOS09] J. A. De Loera, E. D. Kim, S. Onn, and F. Santos. Graphs of transportation polytopes, J. Combin. Theory Ser. A, 116:8 (2009), 1306-1325.

[DRS10] J. A. De Loera, J. Rambau, and F. Santos. Triangulations: Structures for Algorithms and Applications. 539 pp., Algorithms and Computation in Mathematics, Vol. 25, Springer-Verlag, ISBN: 978-3-642-12970-4 (2010).

[DS2000] J. Dongarra and F. Sullivan. Guest Editors' Introduction: The Top 10 Algorithms. Comput. Sci. Eng. 2, 22 (2000).

[ELSS73] P. Erdös, L. Lovász, A. Simmons, and E. G. Straus, Dissection graphs of planar point sets. In A survey of Combinatorial Theory, J. N. Srivastava et al., eds., North-Holland, 139-149 (1973).

[Exp09] J.D. Expósito Polo. Estudio y visualización de envolventes convexas y triangulaciones con direcciones restringidas en Geometría Computacional. Trabajo Fin de Carrera dirigido por David Orden Martín. Universidad de Alcalá, 2009.

[FHKO10] S. Felsner, C. Huemer, S. Kappes, and D. Orden. Binary labelings for plane quadrangulations and their relatives. Discrete Mathematics and Theoretical Computer Science 12:3 (2010), 115-138.

[FiWo04] E. Fink and D. Wood. Restricted-orientation convexity. Springer, 2004.

[FiZi99] R. T. Firla and G. M. Ziegler. Hilbert bases, unimodular triangulations, and binary covers of rational polyhedral cones. Discrete Comput. Geom. 21 (1999), 205–216.

[HNPS08] C. Haase, B. Nill, A. Paffenholz, and F. Santos. Lattice points in Minkowski sums, Electron J. Combin., 15 (2008), no. 1, Note 11, 5 pp.

[HPPS14+] Christian Haase, Andreas Paffenholz, Lindsay C. Piechnik, Francisco Santos. Existence of unimodular triangulations - positive results. Preprint, 82 pages, May 2014.

[JeTo95] T.R. Jensen and B. Toft. Graph coloring problems. Wiley-Interscience series in discrete mathematics and optimization (1995).

[KaKl92] G. Kalai and D. Kleitman. A quasi-polynomial bound on the diameter of convex polyhedra. Bulletin Amer. Math. Soc. 26, 315–316 (1992).

[KaSa03] J.-M. Kantor and K. S. Sarkaria. On primitive subdivisions of an elementary tetrahedron. Pacific J. Math. 211:1, 123-155 (2003).

[KKMS73] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat. Toroidal embeddings I. Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York (1973).

[KimSan10] E. D. Kim and Francisco Santos. An update on the Hirsch conjecture. Jahresbericht der Deutschen Mathematiker-Vereinigung, 112(2) (2010), 73–98.

[Klee66] V. Klee. Paths on polyhedra II. Pacific J. Math. 17 (1966), 249–262.

[KlWa67] V. Klee and D. Walkup. The d-step conjecture for polyhedra of dimension d < 6. Acta Math. 117 (1967), 53–78.

[Lee96] C. W. Lee. P.L.-spheres, convex polytopes, and stress. Discrete Comput. Geom. 15:4 (1996), 389-421.

[Lov71] L. Lovász. On the number of halving lines. Ann. Univ. Sci. Budapest Eötvös Sect. Math. 14 (1971), 107-108.

[LVWW04] L. Lovász, K. Vesztergombi, U. Wagner, and E. Welzl. Convex Quadrilaterals and k-Sets. In János Pach, editor, Towards a Theory of Geometric Graphs, number 342 in Contemporary Mathematics, pages 139–148. Amer. Math. Soc. (2004).

[MSW12+] Benjamin Matschke, Francisco Santos, Christophe Weibel, The width of 5-dimensional prismatoids. Preprint, February 2012, 28 pages. arXiv:1202.4701

[McMull06] P. McMullen. Polyhedra and polytopes: algebra and combinatorics. In Algebraic and geometric combinatorics, volume 423 of Contemp. Math., pages 235–267. Amer. Math. Soc., Providence, RI, 2006.

[MKJA14] I. Marsa-Maestre, M. Klein, C.M. Jonker, and R. Aydoğan. From problems to protocols: Towards a negotiation handbook. Decision Support Systems 60 (2014), 39–54.

[MuNe13] S. Murai and E. Nevo, On the generalized lower bound conjecture for polytopes and spheres, Acta Mathematica · Volume 210, Issue 1 , pp 185-202, 2103.

[NeSaWi14+] Eran Nevo, Francisco Santos, Stedman Wilson. Many triangulated odd-spheres, Preprint, 28 pages, August 2014. arXiv:1408.3501

[Oda88] T. Oda. Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete 15. Springer-Verlag, Berlin (1988).

[ORS10] D. Orden, P. Ramos, and G. Salazar. The number of generalized balanced lines, Discrete and Computational Geometry 44:4 (2010), 805:811.

[OS05] D. Orden and F. Santos. The polytope of non-crossing graphs on a planar point set. Discrete and Computational Geometry 33:2 (2005), 275-305.

[OSSS07] D. Orden, F. Santos, B. Servatius, and H. Servatius. Combinatorial Pseudo-Triangulations. Discrete Mathematics 307 (2007), 554-566.

[PaMaXu99] P.M. Pardalos, T. Mavridou, and J. Xue. The Graph Coloring Problem: A Bibliographic Survey in Handbook of Combinatorial Optimization (1999), 1077-1141.

[PaSa13] P. Kolipaka and A. Govindarajan. Two player game variant of the Erdős-Szekeres problem. Discrete Mathematicas and Theoretical Computer Science 15:3 (2013), 73-100.

[PaTo13] J. Pach, G. Tóth. Monochromatic empty triangles in two-colored point sets. Discrete Applied Mathematics 161 (2013) 1259–1261.

[PfPiSa12] J. Pfeifle, V. Pilaud, and F. Santos. Polytopality of Cartesian products of graphs. Israel J. Math 192(1) (2012), 121-144. DOI: 10.1007/s11856-012-0049-5

[PiSan09] V. Pilaud and F. Santos. Multi-triangulations as complexes of star polygons. Discrete and Computational Geometry 41:2 (2009), 284-317.

[PiSa12] Vincent Pilaud, Francisco Santos, The brick polytope of a sorting network. European J. Combin. 33:4 (2012), 632-662. DOI: 10.1016/j.ejc.2011.12.003

[PiPo12] V. Pilaud and M. Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks, Discrete Comput. Geom., 48(1):142-191, 2012 DOI:10.1007/s00454-012-9408-6

[Ram10] P. Ramos. Puntos, aristas y cruces. La Gaceta de la RSME 14:2, 2011, pp. 263-282.

[Rez08] B. Reznick. ``Let me tell you my favorite lattice-point problem'' in Integer Points in Polyhedra-Geometry, Number Theory, Representation Theory, Algebra, Optimization, Statistics, M. Beck, C. Haase, B. Reznick, M. Vergne, V. Welker, and R. Yoshida (eds.). Contemp. Math. { 452}, Amer. Math. Soc. 2008.

[RSS03] G. Rote, F. Santos, and I. Streinu. Expansive motions and the Polytope of Pointed Pseudo-Triangulations. In Discrete and Computational Geometry -- The Goodman-Pollack Festschrift (B. Aronov, S. Basu, J. Pach, M. Sharir, eds), Algorithms and Combinatorics 25, Springer Verlag, Berlin, June 2003, pp. 699-736.

[RSS08] G. Rote, F. Santos, and I. Streinu. Pseudo-triangulations - a survey. In „Surveys on Discrete and Computational Geometry: Twenty Years Later“, Edited by J. E. Goodman, J. Pach, and R. Pollack, Contemporary Mathematics 453, AMS, 2008. ISBN-13: 978-0-8218-4239-3

[San00] F. Santos, A point configuration whose space of triangulations is disconnected, J. Amer. Math. Soc. 13:3 (2000), 611-637.

[San04] F. Santos. On normal polytopes without regular unimodular triangulations. 2-page abstract in „Mini-workshop: Ehrhart Quasipolynomials: Algebra, Combinatorics, and Geometry. Abstracts from the mini-workshop held August 15-21, 2004. Organized by Jesús A. De Loera and Christian Haase“. Oberwolfach Reports. Vol. 1, no. 3. Oberwolfach Rep. 1 (2004), no. 3, 2071-2101.

[San05] F. Santos. Non-connected toric Hilbert schemes. Mathematische Annalen, 332:3 (2005), 645-665.

[San07] F. Santos, Geometric bistellar flips. The setting, the context and a construction. In Proceedings of the International Congress of Mathematicians, Madrid, August 22-30, 2006 (Marta Sanz-Solé, Javier Soria, Juan Luis Varona, Joan Verdera, eds.), European Mathematical Society, 2006, Vol III, pp. 931-962. ISBN 978-3-03719-022-7.

[San10] F. Santos. Sobre un contraejemplo a la Conjetura de Hirsch. La Gaceta de la RSME, Vol. 13 (2010), Num. 3, 525-538.

[San12] F. Santos. A counter-example to the Hirsch conjecture. Annals of Math. (2), 176 (July 2012), 383-412. DOI: 10.4007/annals.2012.176.1.7

[San13] Francisco Santos, Recent progress on the combinatorial diameter of polytopes and simplicial complexes, TOP 21:3 (October 2013), 426-460. DOI: 10.1007/s11750-013-0295-7

[SaZi13] F. Santos, G. M. Ziegler, Unimodular triangulations of dilated 3-polytopes, Trans. Moscow Math. Soc., 74 (2013), 293-311. DOI: 10.1090/S0077-1554-2014-00220-X

[SST12] Francisco Santos, Tamon Stephen, Hugh Thomas, Embedding a pair of graphs in a surface, and the width of 4-dimensional prismatoids. Discrete Comput. Geom. 47:3 (2012), 569-576. DOI: 10.1007/s00454-011-9361-9

[SSW14+] Francisco Santos, Christian Stump, Volkmar Welker. Noncrossing sets and a Grassmann associahedron. Preprint, March 2014, 34 pages. arXiv:1403.8133

[ShaWel03] M. Sharir and E. Welzl. Balanced lines, halving triangles, and the generalized lower bound theorem. In B. Aronov, S. Basu, J. Pach, M. Sharir (eds.) Discrete and Computational Geometry—The Goodman–Pollack Festschrift, pp. 789–798. Springer, Heidelberg (2003).

[Stan80] R. P. Stanley. The number of faces of a simplicial convex polytope. Advances in Math. 35 (1980), 236-238.

[TZFS13] E.Z. Tragos, S. Zeadally, A.G. Fragkiadakis, and V.A. Siris. Spectrum Assignment in Cognitive Radio Networks: A Comprehensive Survey. IEEE Communications surveys and tutorials, 15:3 (2013), 1108-1135.

[Vrto11] I. Vrt'o. Crossing numbers of graphs: A bibliography. ftp://ftp.ifi.savba.sk/pub/imrich/crobib.pdf Last access: September 18, 2014.

[Whi63] G. K. White. Lattice tetrahedra. Canadian J. Math. 16 (1964), 389--396.

[Zar54] K. Zarankiewicz. On a problem of P. Turán concerning graphs. Fund. Math. 41 (1954), 137-145.

[Zie94] G. M. Ziegler. Lectures on Polytopes. Springer-Verlag, New York, 1994.

C.2. IMPACTO ESPERADO DE LOS RESULTADOS

This is a basic research project. The expected benefit is the advancement of knowledge and the difusion of it will be done via international confencerences and workshops, their published proceedings where appropriate, seminars in universities and research centers and, most significantly, refereed journals. The three senior members of the team have proven experience and records of presence in all of these forms of research difusion.

For example, they have been awarded the maximum number of “sexenios de investigación” that they can have (3, 3 and 2, with Ph. D. Theses from 1995, 1995 and 2003 respectively) and they together have authored 20 papers in journals indexed in the Journal Citation Reports of Thomson ISI in the 4 years 2010 to 2013. See more details in their respective CVs.

Special mention deserves the impact had by recent research of the PI, who has delivered some twenty one-hour invited talks in different conferences in optimization, computational geometry, combinatorics, or computer algebra in the last four years.

The topics of research, although they all belong to the area of geometric combinatorics, have been chosen with an eye to applications in other sciences, or other areas of mathematics. The questions in Research Line 1 are of great interest to the optimization community, as proven by the fact that after disproving the Hirsch Conjecture F. Santos has been invited to numeruous workshops and conferences in that area. Research Line 2 has close connections to algebraic geometry (resolution of singularities, factoriazation of polynomials, toric and tropical geometry) but also to optimization (integer and and semidefinite programming) or statistics. Research Line 3 has applications to network analysis and design. Research Line 4 has applications to algorithms in computational geometry and to telecommunication network optimization.

C.3. CAPACIDAD FORMATIVA DEL EQUIPO SOLICITANTE

The team asks for two FPI student scholarships, one to be based at the University of Cantabria (advisor Francisco Santos) and one at Alcalá (advisors Pedro Ramos and/or David Orden).

The principal investigator Francisco Santos has directed five Ph. D. In the last fifteen years:

- Daciana Bochis (1999), Estereoedros de Dirichlet en dimensión tres.
- Miguel Azaola (2001) Subdivisiones poliedrales en corrango tres,
- David Orden (2003) Dos problemas de Combinatoria Geometrica: Triangulaciones eficientes del hipercubo; Grafos planos y rigidez. Accésit al **Premio de Investigación** del Consejo Social de la U Cantabria
- Pilar Sabariego (2008), Algunos problemas sobre teselaciones.
- Vincent Pilaud (2010), Multitriangulations, pseudotriangulations and some problems of realization de polytopes. (Cotutelada entre la U. de Cantabria y la U. de Paris VII, codirigida por Michel Pocchiola). **Premio extraordinario de doctorado** de la Universidad de Cantabria

Los doctores Bochis y Azaola ocupan actualmente puestos de responsabilidad en sendas **empresas tecnológicas** españolas de pirimer nivel (INDRA y GMV respectivamente) trbajando respectivamente en proyectos relacionados con Gráficos por ordenador y realidad virtual, y con sistemas de satélites para geoposicionamiento. El doctor Orden es **Profesor** Titular en la Universidad de Alcalá (y forma parte de este proyecto). El doctor Pilaud es ***Chargé de Recherche*** **del CNRS** francés, destinado en la Ecole Polytechnique. La doctora Sabariego es profesora en un Instituto de Enseñanza Secundaria. Aunque ha abandonado la investigación como tal, participa frecuentemente con sus alumnos en programas de iniciación a la investigación, en los que ha obtenido **varios premios** (por ejemplo, un segundo premio y un accésit en el concurso convocado en 2014 por el Instituto Cántabro de Estadística http://www.icane.es/icane/school-contest/fifth).

Pedro Ramos has advised one thesis:

- Raquel Viaña (2005), A Multiresolution Model for Plane Graphs Representing Geographic Maps, U. de Alcalá.

Doctor Viaña is currently Profesora Contratado Doctor at the University of Alcalá.

The group is currently directing the Ph D thesis of Mónica Blanco (expected date: December 2016) via an FPI scholarship from the Ministry of Science and Innovation.

The University of Cantabria has a Doctorate programme in "Mathematics and Computer Science" that was awarded a "**Mención hacia la excelencia**" by the MICINN in 2011. Francisco Santos is the coordinator of it, and also the coordinator-to-be of the new doctorate programme in Science and Technology which is about to start (October 2014) under the regulations of RD99/2011.

C.4. IMPLICACIONES ÉTICAS Y/O DE BIOSEGURIDAD

None