Bernd Sturmfels: STATE POLYTOPES

MSRI
07/30/03

1. Review of Triangulations and Secondary Polytopes

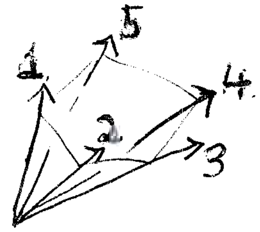
$A = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^{d \times n}$ a matrix of rank d

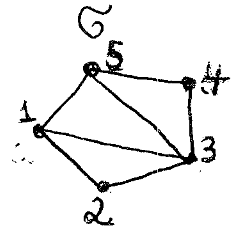
Map of cones: $\pi : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0} A$, $u \mapsto Au$

Def: A *triangulation of A* is a section σ of π such that $\mathrm{image}(\sigma)$ is an order ideal in the poset $\mathbb{R}^n_{\geq 0}$.
We identify σ with the simplicial complex $\{\mathrm{supp}(\sigma(b)) : b \in \mathbb{R}_{\geq 0} A\}$

Example 1
$n=5, d=3$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 9 & 16 \end{bmatrix}$$



σ


$$\sigma\begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

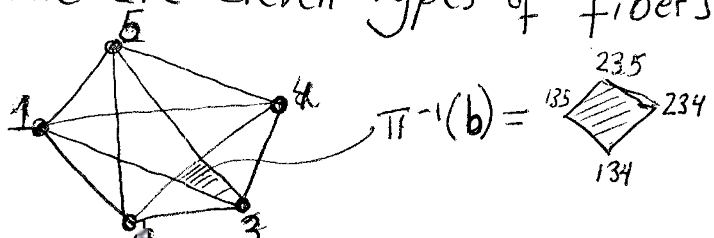
Every generic $w \in \mathbb{R}^n$ defines a *regular triangulation* σ_w as follows:

$\sigma_w(b) =$ the unique point in $\pi^{-1}(b) = \{u \in \mathbb{R}^n_{\geq 0} : Au = b\}$ at which $u \mapsto w \cdot u$ is minimized

The *secondary polytope* is $\Sigma(A) = \int_b \pi^{-1}(b)\,db \subset \mathbb{R}^n$

Its vertices correspond to the regular triangulations of A.

Example 1 There are eleven types of fibers



Their Minkowski sum $\Sigma(A)$ is also a pentagon, corresponding to the five triangulations of A.

Example 2 $A = (1, 2, 3)$
$n=3, d=1$

What is $\Sigma(A)$?

2. Replace the Real Numbers by the Integers

Map of semigroups $\quad p: \mathbb{N}^n \to \mathbb{N}A, \ u \mapsto Au$

We call $b \in \mathbb{N}A$ <u>nice</u> if $\pi^{-1}(b) = \mathrm{conv}(p^{-1}(b))$

<u>Def.</u> A <u>staircase for A</u> is a section s of p such that $\mathrm{image}(s)$ is an order ideal in the poset $\mathbb{N}^n$.

<u>Theorem</u>

(1) Every staircase s determines a unique triangulation σ by the rule: $\quad \sigma(b) = s(b)$ for all nice b.

(2) The map $s \mapsto \sigma$ is generally neither injective nor surjective

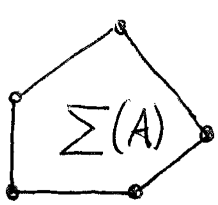
Every generic vector $w \in \mathbb{R}^n$ defines a <u>regular staircase</u> s_w as follows.

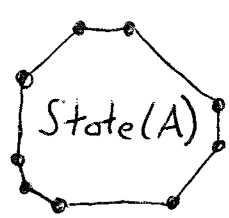
$s_w(b) =$ the unique point in $p^{-1}(b) = \{u \in \mathbb{N}^n : Au = b\}$ at which $u \mapsto w \cdot u$ is minimized.

<u>Theorem</u> There is a polytope $\mathrm{State}(A) = \int_b p^{-1}(b)\,db$ in $\mathbb{R}^n$ whose vertices correspond to the regular staircases for A.

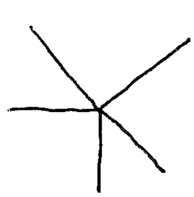
<u>Corollary</u>: The restriction of $s \mapsto \sigma$ to regular staircases is surjective. The secondary polytope $\Sigma(A)$ is a Minkowski summand of $\mathrm{State}(A)$.

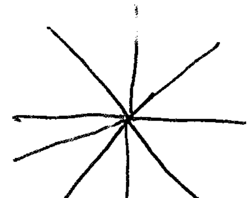
polytope





fan





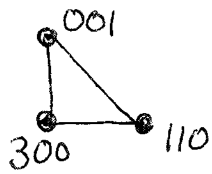
3. A Simple Example

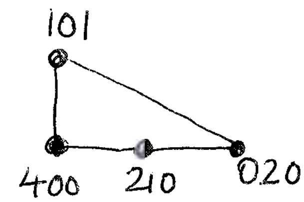
$n = 3, \quad d = 1 \qquad A = [1\ 2\ 3] \qquad \mathbb{N}A = \mathbb{N}$

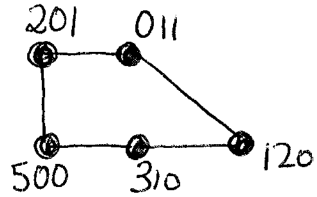
$p : \mathbb{N}^3 \longrightarrow \mathbb{N}, \quad (u_1, u_2, u_3) \mapsto u_1 + 2u_2 + 3u_3$

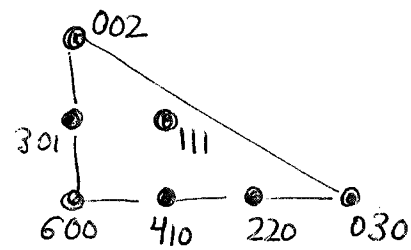
Q: Which integers $b \in \mathbb{N}$ are nice?

Fibers

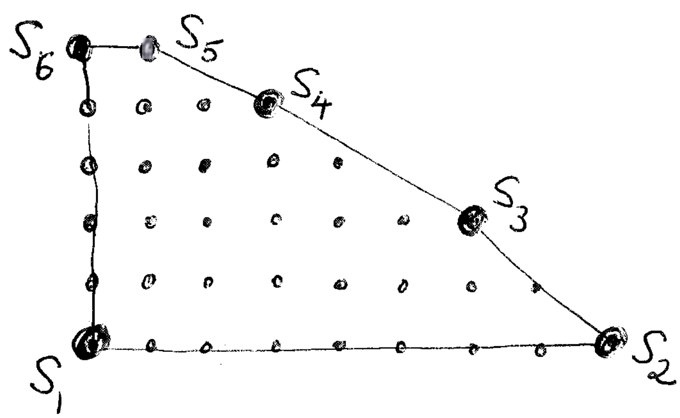
$p^{-1}(3) =$ 

$p^{-1}(4) =$ 

$p^{-1}(5) =$ 

$p^{-1}(6) =$ 

State(A) = the Minkowski sum of the polygons



The six sections S_i are determined by their values on $3, 4, 5, 6$ and the requirement that $\mathrm{image}(S_i) \subseteq \mathbb{N}^3$ is an order ideal

Ex. $\quad S_3(27) = (0, 12, 1)$

4. Commutative Algebra, Finally

Lattice points $u = (u_1, \dots, u_n)$ $\leftrightarrow$ Monomials $x^u = x_1^{u_1} \cdots x_n^{u_n}$

The toric ideal of A is

$$I_A = \langle x^u - x^v : Au = Av \rangle \subset k[x_1, \dots, x_n]$$

Proposition: For $w \in \mathbb{R}^n$ generic, the initial monomial ideal is

$$in_w(I_A) = \langle x^u : u \notin image(S_w) \rangle \quad \text{and}$$

$Rad(in_w(I_A))$ is the Stanley-Reisner ideal of the regular triangulation σ_w

Example 1 $\quad I_A = \langle x_1x_3^3 - x_2^3x_4, \ x_1x_4^2 - x_2^2x_5, \ x_2x_4^3 - x_3^3x_5 \rangle$

For $w = (13, 11, 7, 5, 3)$ we have

$$in_w(I_A) = \langle x_2^3x_4, \ x_2^2x_5, \ x_2x_4^3, \ x_1x_4^5 \rangle \quad \text{and}$$

$$Rad(in_w(I_A)) = \langle x_2x_4, \ x_2x_5, \ x_1x_4 \rangle = \langle x_1, x_2 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_4, x_5 \rangle$$

Corollary The vertices of State(A) correspond to the initial monomial ideals of I_A

Example 2 $\quad I_A = \langle x^2 - y, \ x^3 - z \rangle$

$S_1 = \langle y, z \rangle \quad S_2 = \langle x^2, z \rangle \quad S_3 = \langle x^2, xy, xz, z^2 \rangle$

$S_4 = \langle x^2, xy, z^2 \rangle \quad S_5 = \langle x^2, y^3, xy, xz \rangle \quad S_6 = \langle y, x^3 \rangle$

Q: If s is an arbitrary staircase for A, what is the meaning of $\langle x^u : u \notin image(s) \rangle$?

A: Torus-fixed points on the toric Hilbert scheme