Sheet 3 — Regular triangulations and secondary polytopes

1. Prove that all triangulations of one-dimensional point configurations are regular!
2. Prove that all triangulations of n-gons are regular!
3. Recall the coordinates of the secondary polytope of $\mathcal{A} = \{0, 1, 3, 5\} \subset \mathbb{R}^1$:

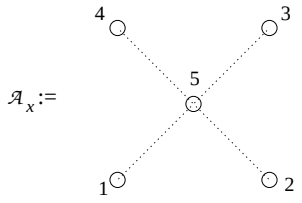
$$\Sigma(\mathcal{A}) = \text{conv}\left\{ \begin{pmatrix} 5 \\ 0 \\ 0 \\ 5 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 0 \\ 4 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 5 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix} \right\}$$

   One relation among those coordinates is:

$$x_1 + x_2 + x_3 + x_4 = 2\text{vol}(\text{conv}(\mathcal{A}))$$

   Find another relation and an interpretation of it!

4. (a) Draw a Gale transform of



   (b) Label the full-dimensional chambers of this Gale transform by the regular triangulations of $\mathcal{A}_x$!
   (c) Sketch the secondary polytope of $\mathcal{A}_x$!

5. (advanced)
   (a) Sketch a Gale transform of the six-gon $\mathcal{A}_6$
   (b) Label the chambers by triangulations of $\mathcal{A}_6$
   (c) Sketch the secondary polytope of $\mathcal{A}_6$
6. (advanced) Prove that all triangulations of $d + 3$ points in dimension d are regular! (Hint: Look at a Gale transform!)

Open Problem: Study the secondary polytopes of:

- Prisms $\mathcal{A} \times I$
- Products of simplices $\Delta_d \times \Delta_n$
- Products $\mathcal{A} \times \mathcal{B}$