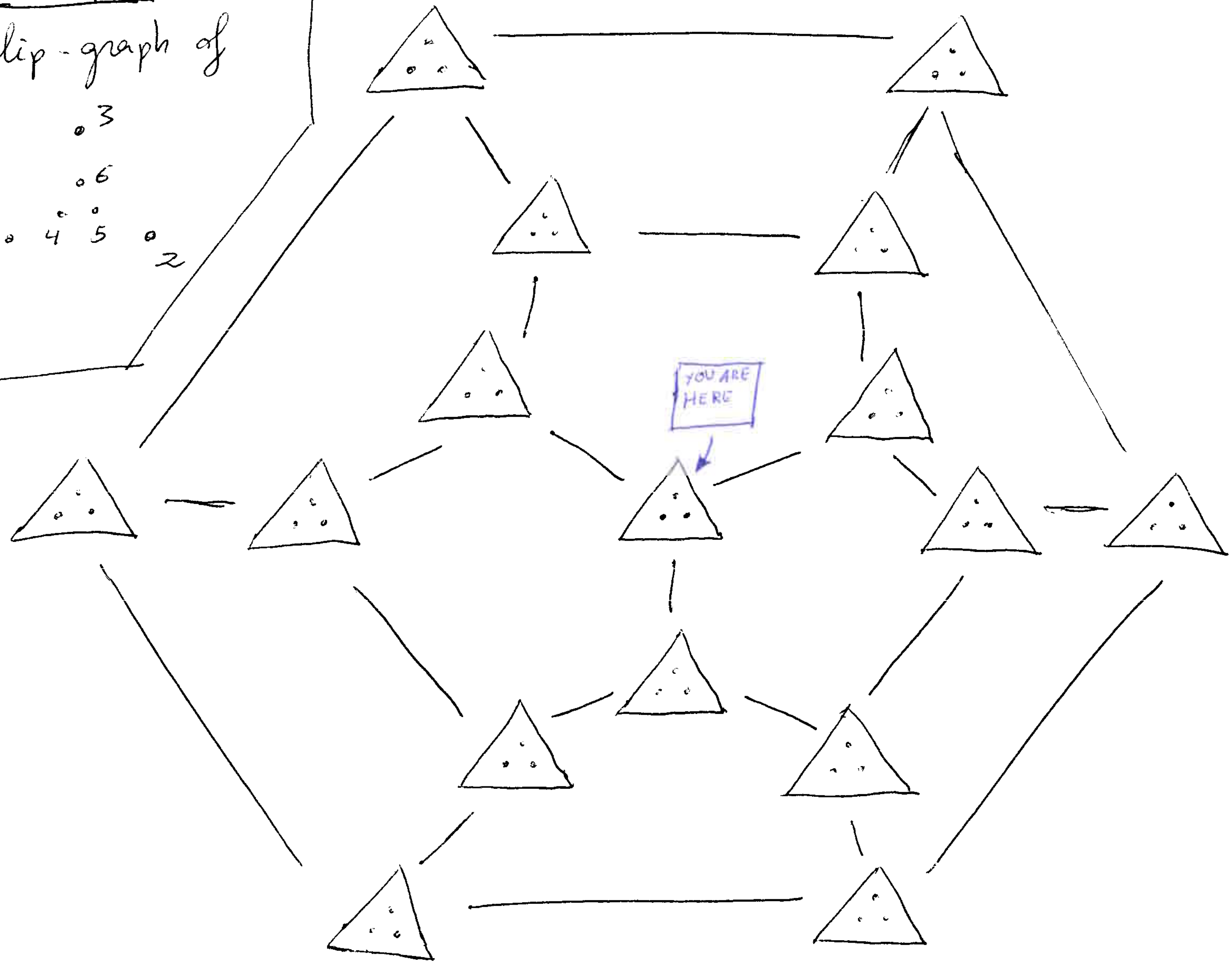
NON-REGULAR TRIANGULATIONS

$\boxed{\text{Example}}$ is the flip-graph of

3
6
5
4
1
2



"THE MOTHER OF ALL EXAMPLES"

How to look for flips in a triangulation.

Prop 1: Any $(d+2)$ points ~~which~~ affinely span $\mathbb{R}^d$ contain a unique circuit.

Pf: The points have a unique affine dependence equation. The subset of points involved in that equation is the unique circuit in the statement $\square$

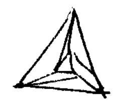
Prop 2: Every flip in a triangulation (other than the ones that insert a point) happens at the unique circuit contained in some pair of adjacent d-simplices

Pf: Let $C=(C^+,C^-)$ be the circuit at which the flip happens. "The flip inserts a point" $\equiv$ "C^+ has one element"

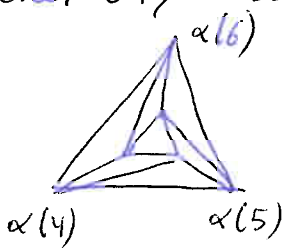
Hence, we assume C^+ to have at least two elements a and b. Then $C\setminus\{a\}$ and $C\setminus\{b\}$ are two adjacent simplices in T_C^+, and the "link condition for flips" implies they can be extended to two adjacent simplices of T. $\square$

Corollary: To look for flips in a given triangulation T you do not need to check all the circuits of A. Only look at pairs of adjacent (full dimensional) simplices. plus the flips that insert points. There is one for each unused pt.

(Remark: these are the "edges" in the adjacency graph or dual graph of the triangulation. Yes, the same graph that Jesús used to prove the "lower bound thm".)

Two proofs that the triangulations  and  are not regular:

(1) No valid heights exist. Without loss of generality assume $\alpha(1)=\alpha(2)=\alpha(3)=0$ and then

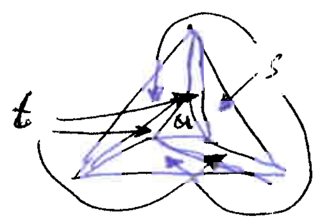


regular would imply $\alpha(6)>\alpha(5)>\alpha(4)>\alpha(6)$

$\square$

(2) Compute the GKZ-vectors of these two triangulations:

$(2s+t,\ 2s+t,\ 2s+t,\ s+2t+u,\ s+2t+u,\ s+2t+u)$

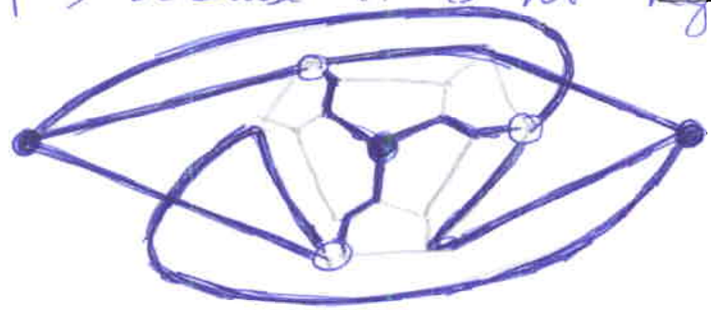
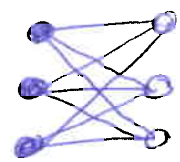


(exactly the same ...)

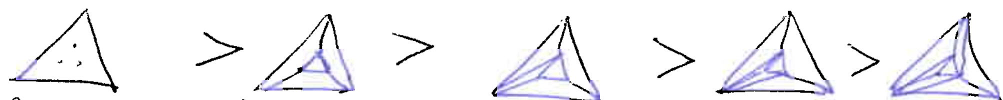
And we know that the regular triangulation of a set of heights $\alpha \in \mathbb{R}^n$ is <u>the</u> one that minimizes scalar product of α with the GKZ-vectors. In particular, if two triangulations have the same GKZ-vector then they are both non-regular.

Two proofs that "the mother" has non-regular triangulations:

(3) The graph of flips is not the graph of any 3-polytope, because it is not ~~regular~~ planar:

 =  $K_{3,3}$

(4) The poset of subdivisions is not the face poset of any 3-polytope, because it has "too long chains":



$\{123456\}$ $\{1245, 2356, 1346\}$ $\{1245, 2356, 146, 136\}$ $\{\ \vdots\ \}$ $\{\ \vdots\ \}$

<u>Remark</u>: strictly speaking, this only shows that it has some non-regular subdivision (one of the subdivisions in the chain must be non-regular, not necessarily the triangulation).

But it is a (non-trivial) fact that every non-regular subdivision can be refined to a non-regular triangulation.

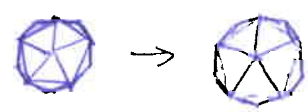
<u>By the way</u>: it is also a (sort of trivial) fact that every subdivision can be refined to a triangulation. That is to say, "triangulations are the minimal elements in the poset of subdivisions".

Other point sets with non-regular triangulations

Prop ④ allows you to derive that the following point sets have non-regular triangulatios "with zero work":

(a) prism + 1 interior point : let n = base of prism.
We have $2n+1$ vertices, secondary has dim $2n-3$.
Cone the interior point to all faces. This gives a non-trivial subdivision of "height" $3n-6 \Rightarrow$ chain of length $3n-5$ in the poset (and $3n-5 > 2n-3$ since $n > 2$).

(b) regular icosahedron + center : divide the bdry. into 10 pairs of adjacent triangles :



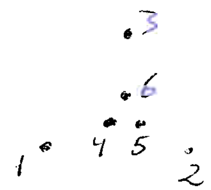
and same on other side...

This gives a non-trivial subdivision of height $10 \Rightarrow$ chain of length 11 in the poset (and secondary polytope has dimension $13-4=9$)

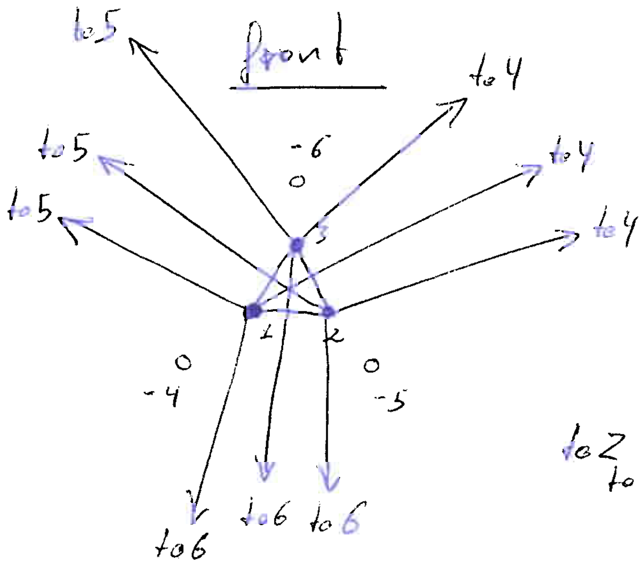
(c) any 3-polytope with # vertices > # facets, together with an interior point (Exercise).
This happens for : cube, dodecahedron, prisms, rhombic dodecahedron, all simple polytopes except tetrahedron, ...

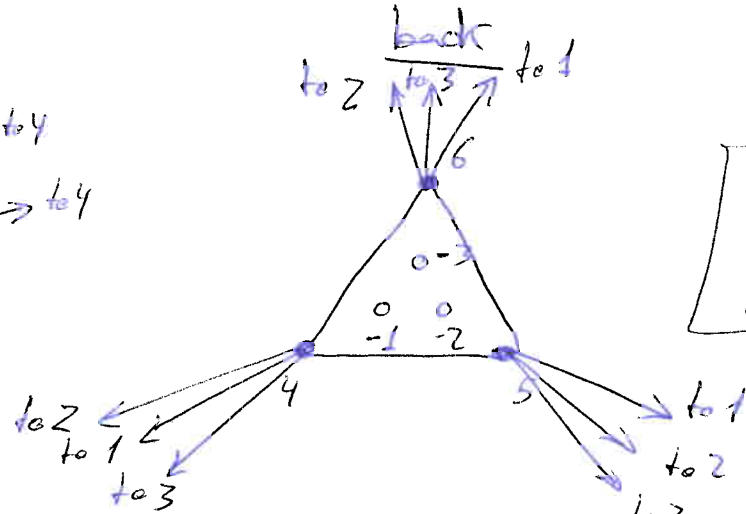
The Gale transform of M.O.A.E.

$$A = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 4 & 0 & 0 & 2 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{pmatrix} \end{array} \qquad B^T = \begin{array}{c} \begin{array}{cccccc} 1^* & 2^* & 3^* & 4^* & 5^* & 6^* \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & -4 & 0 & 0 \\ 1 & 2 & 1 & 0 & -4 & 0 \\ 1 & 1 & 2 & 0 & 0 & -4 \end{pmatrix} \end{array}$$



"affine Gale diagram":



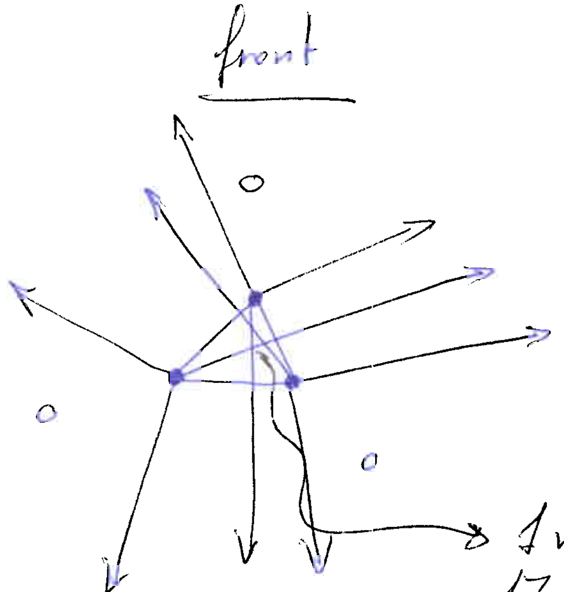


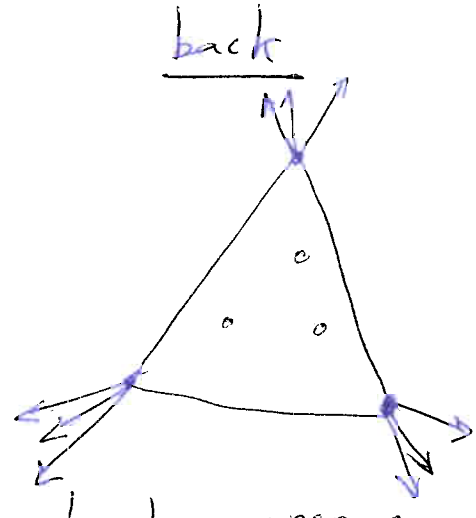
16 chambers

$\Downarrow$

16 regular triangs

What happens if we perturb the coordinates slightly?





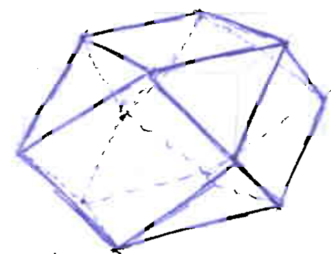
1 new chamber appears.

17 regular triangulations (and 1 non-regular)

Some flip-deficient triangulations $\begin{pmatrix}\text{less than}\\ n-d-1 \text{ flips}\end{pmatrix}$

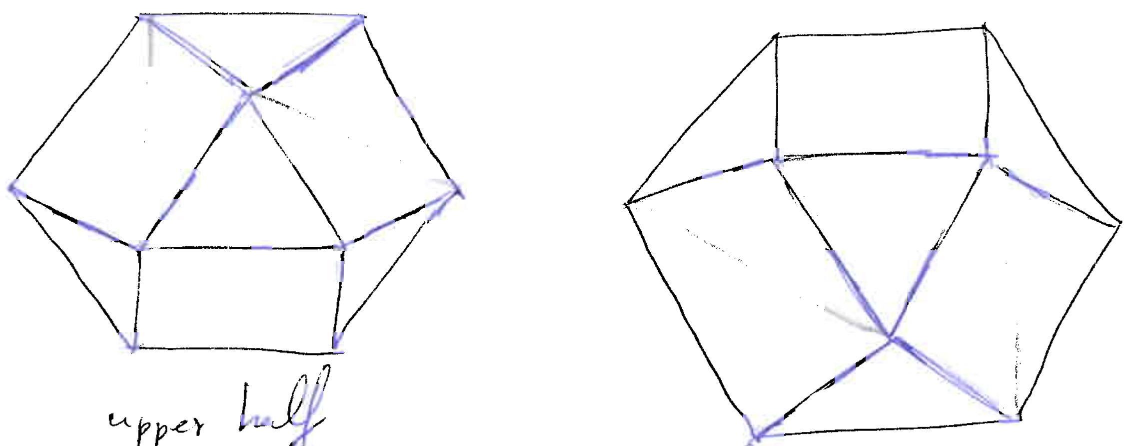
① The cube-octahedron:

obtained by cutting all vertices of a cube.



Let $A = \{\text{vertices of the cubeoctahedron}\} \cup \{O\}$ $\begin{pmatrix}13 \text{ vertices}\\ n-d-1=9\end{pmatrix}$

(i) Triangulate the boundary in the "most skew" way:



upper half

lower half

Remark: every vertex belongs to two squares.
Diagonals are inserted so that every vertex belongs to exactly one diagonal.

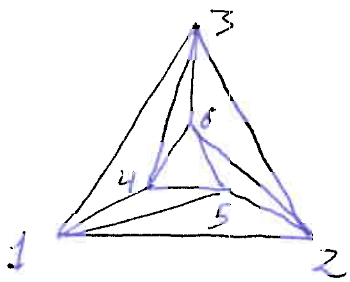
(ii) Cone the triangulation of the boundary to the point in the interior.

$\boxed{\text{This has only 6 flips!!}}$ $\begin{pmatrix}\text{hence it is}\\ \text{non-regular}\end{pmatrix}$

Exercise: give a direct proof that no heights exist giving this triangulation

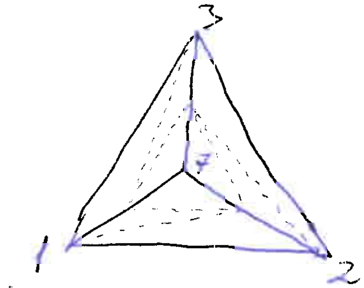
(2) "A Lawrence lift" of the mother of all examples.

(i) Start with MOAE, in a non regular triangulation:



... 3 flips so far

(ii) Cone over a point "7" above that plane.



... same 3 flips; in particular, still not flip-deficient...

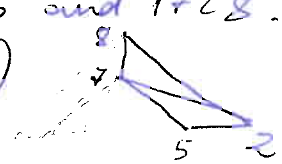
(iii) Cone over a point "8" above point 7:

... same 3 flips!!

Proof: we need to look at pairs of adjacent tetrahedra. That is, to look at interior triangles. Among those incident to MOAE we have already computed the flips, and there are three.

For the others:

- 178 (or 278 or 378): the two adjacent tetrahedra are 1782 and 1783. The circuit is (1238, 7). T_c^+ is not contained in our triangulation: 1237 is missing.
- 172 (or 173 or 273): tetrahedra: 1725 and 1728. circuit: (72, 58)



BUT: 257 is linked to 1 and 6.
278 is linked to 1 and 3

Link condition no satisfied: no flip arises

REMARK: This example can be perturbed into general position.

What is known about triangulations with small dimension and/or number of points?

$\boxed{d=0}$ n triangulations, the graph of flips is the complete graph, the secondary polytope is the $(n-1)$-simplex, all triangulations regular.

$\boxed{d=1}$ all triangulations regular. If there are no repeated points, there are 2^{n-2} triangulations. The secondary polytope is an $(n-2)$-dimensional cube. combinatorially (not metrically...)

$\boxed{d=2\text{, points in convex position}}$ all triangs. regular. There are $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$ of them. Secondary polytope is the associahedron.

$\boxed{d=2}$ Non-regular triangulations appear, but the graph of triangulations is connected (and $(n-3)$-connected??) All triangulations have at least $n-3$ flips.

$\boxed{d=3\text{, points in convex position}}$ all triangulations have at least $n-4$ flips. Graph connected??

$\boxed{n=d+1}$ One triangulation...

$\boxed{n=d+2}$ Two triangulations (a unique circuit). Both regular.

$\boxed{n=d+3}$ Graph of triangulations is a polygon. All triang's regular.

$\boxed{n=d+4}$ Graph is connected and 3-connected. Non-regular triangulations exist (M.O.A.E.)

Deletion and contraction in point/vector configurations

Deletion: if $p \in A$, the <u>deletion of p in A</u> is the point/vector configuration $A \setminus \{p\}$.

Contraction: if $p \in A$, the <u>contraction of A at p</u> (or of p in A) is the ~~point~~/vector configuration:

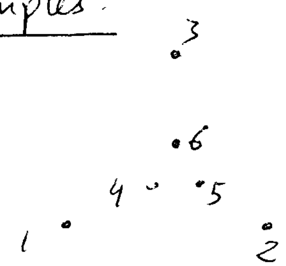
- $\{ q - p \mid q \in A \setminus \{p\} \}$ if A is a point conf.

- $\{ \underbrace{\vec{q} - \frac{\vec{p} \cdot \vec{q}}{\|\vec{p}\| \, \|\vec{q}\|} \vec{p}} \mid \vec{q} \in A \setminus \{\vec{p}\} \}$ if A is a vector conf.

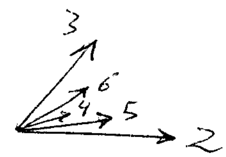
  ↳ orthogonal projection of $\vec{q}$ to the linear hyperplane orthogonal to $\vec{p}$.

Examples:

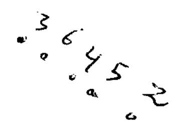
$A =$



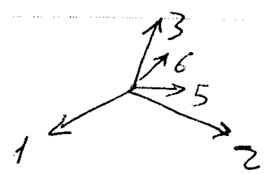
$A/\{1\} =$



$\sim$



$A/\{4\} =$



→ a non-pointed vector configuration
⇒ not equivalent to a point configuration.

$\boxed{\begin{array}{l} \textbf{Basic property: } \text{if } T \text{ is a triangulation of } A \text{ and } p \in A \text{ is used in } T \text{ then:} \\ \qquad \text{link}_T(p) \text{ is a triangulation of } A/\{p\} \end{array}}$

Remarks:

① If we are given a triangulation of A, that induces a triangulation of $A/\{p\}$ (the link) but it may or may not induce (be extendable to) a triangulation of $A \setminus \{p\}$.

Example: let A = "twisted prism" $\cup$ {interior point}

If we use the three "almost flat" tetrahedra in the three "almost quadrilateral faces" of the prism, the interior space is a Schönhardt polyhedron.

If we cone the boundary of that Schönhardt to the interior point, we have a triangulation of A in which the deletion of A cannot be completed to a triangulation of $A \setminus \{p\}$

② If we are given a triangulation of $A \setminus \{p\}$, that induces a triangulation of A $\left(\begin{smallmatrix}\text{cone } p \text{ to} \\ \text{what it "sees"}\end{smallmatrix}\right)$ but if we are given a triangulation of $A/\{p\}$, that may or may not induce (be extendable to) a triangulation of A.

Example: Let $A = \{1, 2, 3\} \cup \{4, 5, 6\} \cup \{7\}$

at height 0 at height 1 at height 1000

Then $A/7$ = MOAE. The triangulation

cannot be extended to a triangulation of A. (Essentially, it produces a non-triangulable Schönhardt).

But: regular triangulations behave well:

Lemma 1: Given a regular triangulation T of A:

((a) $lk_T(p)$ is a regular triangulation of $A/\{p\}$)

(b) deleting p in T can be extended to a regular triangulation of $A \setminus \{p\}$

Lemma 2:

(a) Given a regular triangulation t of $A/\{p\}$, there is a regular triangulation T of A with $t = lk_T(p)$.

((b) Given a regular triangulation t of $A \setminus \{p\}$, there is a regular triangulation T of A with $t \subseteq T$.)

Proof(1): restrict to $A/\{p\}$ and $A \setminus \{p\}$ the heights used in A.

Proof(2): (a) use ~~the~~ in A the same heights as in $A/\{p\}$, with height 0 for p itself.

(b) use in A the same heights as in $A \setminus \{p\}$, with very high height for p itself.