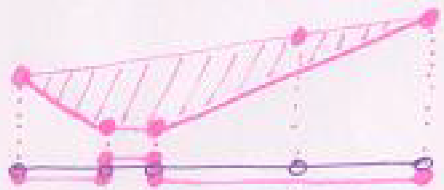
Main actor:

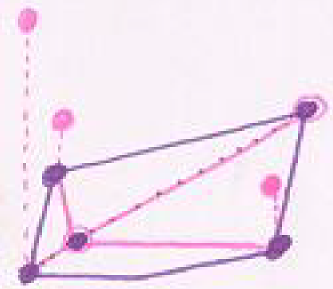
Def.: A $\left\{\begin{matrix}\text{subdiv.}\\ \text{triang.}\end{matrix}\right\}$ T of $\mathcal{A}$ is regular if there exist heights $\alpha(a)$ for all $a \in \mathcal{A} \subset \mathbb{R}^d$ such that T is the set of lower facets of $\mathcal{A}^\alpha := \left\{ \begin{pmatrix} a \\ \alpha(a) \end{pmatrix} \in \mathbb{R}^{d+1} : a \in \mathcal{A} \right\}$

Example:

dim 1:



dim 2:



Notation: $T(\mathcal{A}, \alpha) :=$ regular subdivision induced by $\alpha \in \mathbb{R}^{\mathcal{A}}$

$n := |\mathcal{A}|$

W.l.o.g: $\alpha(a) \geq 0 \quad \forall\, \alpha, a$ considered in this lecture.

Preview:

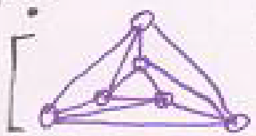
What is nice about subdivisions in dim ≤ 2 ?

1. Easy to visualize
2. Graph of triang's is connected (Lecture 2)

Even better for dim $= 1$ and n-gon:

3. All triang's are regular (exercise)
4. Graph of triang's is the graph of a polytope (dim 1: $(n-2)$-cube, n-gon: associahedron) [Haiman, Lee]

What structure is responsible for 4. ?

dim 2 ? No.! [] [tomorrow, Lecture 4]

convex pos.? No.!

regularity ? Yes! [today]

THM.: [Gelfand, Kapranov, Zelevinsky 1989]
The graph of all regular triang's of a d-dim. point configuration with n points $\mathcal{A}$ is the graph of an $(n-d-1)$-dim. polytope, the secondary polytope of $\mathcal{A}$.

COR.: (i) Graph of all reg. triang's is $(n-d-1)$-connected [Balinsky's thm., Ziegler book]
(ii) Can do linear programming to optimize lin. functionals for reg. triang's.

How does the secondary polytope look like ?

A strange polytope:

Def.: Let T be a triang. of $\mathcal{A}$. Then

$$\phi_{\mathcal{A}}(T) := \sum_{a\in\mathcal{A}} \sum_{\sigma\in T : a\in\sigma} \mathrm{vol}(\sigma)\, e_a \in \mathbb{R}^{\mathcal{A}},$$

e_a: coord. unit vector in dir. a,

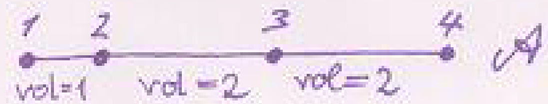
is the GKZ-vector of T.

Def.: $\Sigma(\mathcal{A}) := \mathrm{conv}\{\phi_{\mathcal{A}}(T) : T \text{ triang of } \mathcal{A}\}$ is the secondary polytope of $\mathcal{A}$.

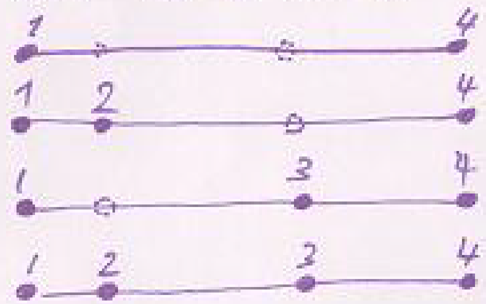
Rem.: Regularity not required in definition.

Example:

dim 1 :



triang's:



$$\phi_{\mathcal{A}}(T_1) = \begin{pmatrix}5\\0\\0\\5\end{pmatrix}$$

$$\phi_{\mathcal{A}}(T_2) = \begin{pmatrix}1\\1+4\\0\\4\end{pmatrix} = \begin{pmatrix}1\\5\\0\\4\end{pmatrix}$$

$$\phi_{\mathcal{A}}(T_3) = \begin{pmatrix}3\\0\\3+2\\2\end{pmatrix} = \begin{pmatrix}3\\0\\5\\2\end{pmatrix}$$

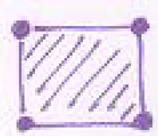
$$\phi_{\mathcal{A}}(T_4) = \begin{pmatrix}1\\1+2\\2+2\\2\end{pmatrix} = \begin{pmatrix}1\\3\\4\\2\end{pmatrix}$$

$$\Rightarrow \quad \Sigma(\mathcal{A}) = \mathrm{conv}\left\{\begin{pmatrix}5\\0\\0\\5\end{pmatrix}, \begin{pmatrix}1\\5\\0\\4\end{pmatrix}, \begin{pmatrix}3\\0\\5\\2\end{pmatrix}, \begin{pmatrix}1\\3\\4\\2\end{pmatrix}\right\}$$

Obs.: (i) $x_1 + x_2 + x_3 + x_4 = 10 \quad \forall \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \in \Sigma(\mathcal{A})$. Why? $10 = 2\cdot \mathrm{vol}(\mathrm{conv}\,\mathcal{A})$! $\Rightarrow \mathrm{rank}\,\Sigma(\mathcal{A}) \le 3$

(ii) $\mathrm{rank}\,\Sigma(\mathcal{A}) = 2$ (exercise).

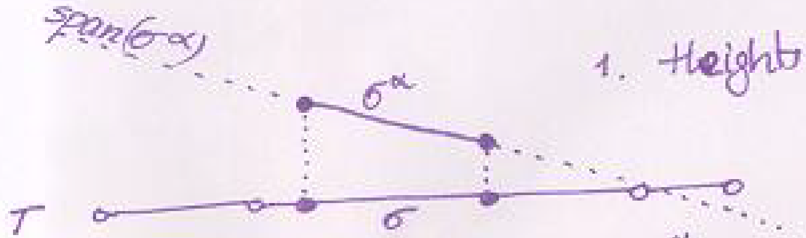
(iii) GKZ-thm. $\Rightarrow$ $\Sigma(\mathcal{A})$ =



Why does $\Sigma(\mathcal{A})$ look like this?

The secondary fan:

When is $T = T(\mathcal{A}, \alpha)$? When does T refine $T(\mathcal{A}, \alpha)$?
Idea: Find conditions on α regarded as set of n variables!



1. Heights of σ must be in a hyperplane.
2. Given the heights on σ, all other heights must be strictly above this hyperplane for "$T = T(\mathcal{A}, \alpha)$" "Strict Convexity"
weakly above this hyperplane for "T refines $T(\mathcal{A}, \alpha)$". "Convexity"
i.e. $\alpha(a) >$ affine function of $a, \alpha(s), s \in \sigma \ \forall a \in \mathcal{A} \setminus \sigma$.
resp. $\alpha(a) \geq$ —"— —"—"— —"— —"— —"—.

Def.:
$C \subseteq \mathbb{R}^n$ is a <u>convex polyhedral cone</u> if it is the intersection of finitely many halfspaces.
<u>relint C</u> is the intersection of the corresponding open halfspaces.

Prop.: $\mathring{\mathcal{C}}_{\mathcal{A}}(T) := \{\alpha \in \mathbb{R}^n : T = T(\mathcal{A}, \alpha)\}$ is the relint of a convex polyhedral cone
$\mathcal{C}_{\mathcal{A}}(T) := \{\alpha \in \mathbb{R}^n : T \text{ refines } T(\mathcal{A}, \alpha)\}$.

Def.:
$\mathcal{C}_{\mathcal{A}}(T)$ is the <u>secondary cone</u> of T. $\mathring{\mathcal{C}}_{\mathcal{A}}(T)$ is the <u>open secondary cone</u> of T.

Def.:
A <u>polyhedral fan</u> in $\mathbb{R}^n$ is a collection $\mathcal{C}$ of polyhedral cones with
(i) $C \in \mathcal{C} \Rightarrow$ all faces of C are in $\mathcal{C}$
(ii) $C, C' \in \mathcal{C} \Rightarrow C \cap C' \in \mathcal{C}$
(iii) $C \cap C'$ is a face of both C and C'!

Def.:
$\mathcal{C}_{\mathcal{A}} := \{ C_{\mathcal{A}}(T) \mid T \text{ polyhedral subd. of } \mathcal{A} \}$ is the <u>secondary fan</u> of $\mathcal{A}$.

Def.: A polyhedral fan $\mathcal{C}$ in $\mathbb{R}^n$ is complete if
$$\bigcup_{C \in \mathcal{C}} C = \mathbb{R}^n .$$

Prop.:
$\mathcal{C}_{\mathcal{A}}$ is a complete polyhedral fan in $\mathbb{R}^n$. $\square$

What has this fan to do with $\Sigma(\mathcal{A})$?

Normal fans of polytopes:

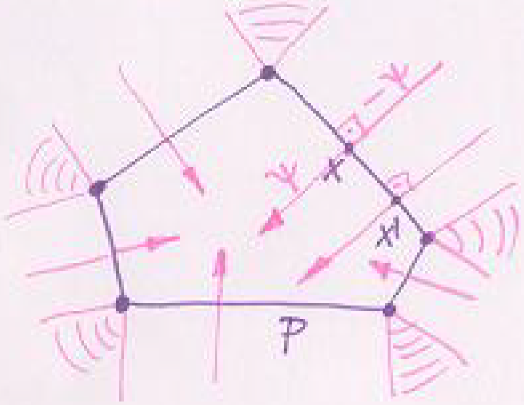
Def.: Let P be a polytope in $\mathbb{R}^n$ and $x \in P$.
The <u>inner normal cone</u> of x in P is

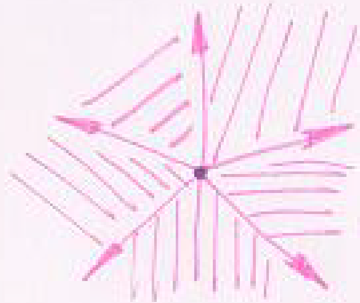
$$\mathcal{N}_P(x) := \{\psi \in \mathbb{R}^n : \langle \psi, x\rangle \leq \langle \psi, y\rangle \ \forall y \in P\}$$

The <u>inner normal fan</u> of P is

$$\mathcal{N}_P := \{\mathcal{N}_P(x) : x \in P\}$$

Example:





$\mathcal{N}_P$

Hope: $\mathcal{N}_{\Sigma(\mathcal{A})} = \mathcal{C}_{\mathcal{A}}$.

Obs.: (i) vertices in P $\longleftrightarrow$ full-dim cones in $\mathcal{N}_P$.
(ii) P determined by its vertices $\Leftrightarrow$ $\mathcal{N}_P$ determined by full-dim cones in $\mathcal{N}_P$.
(iii) vertices of $\Sigma(\mathcal{A})$ are of the form $\phi_{\mathcal{A}}(T)$ for triang's T of $\mathcal{A}$.

Need to show:

$$\langle \alpha, \phi_{\mathcal{A}}(T)\rangle \leq \langle \alpha, \phi_{\mathcal{A}}(T')\rangle \quad \forall T' \text{ triang. of } \mathcal{A} \quad \forall \alpha \in \mathcal{C}_{\mathcal{A}}(T).$$

Proof: For all α and all T we have:

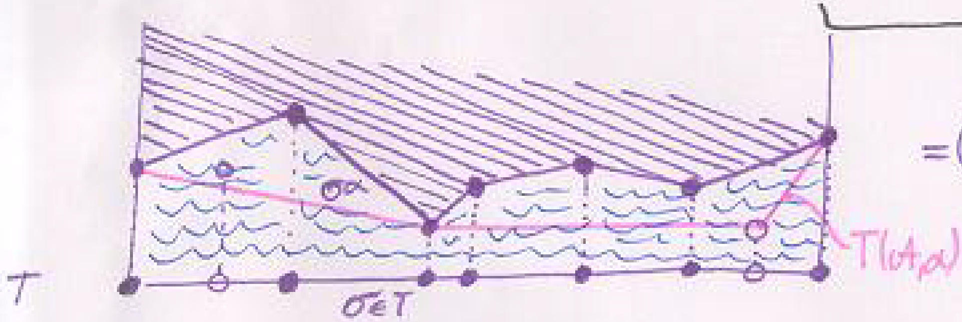
$$\langle \alpha, \phi_{\mathcal{A}}(T) \rangle = \langle \alpha, \sum_{a\in\mathcal{A}} \sum_{\substack{\sigma\in T\\ a\in\sigma}} vol(\sigma) e_a \rangle$$

$$= \sum_{a\in\mathcal{A}} \sum_{\substack{\sigma\in T\\ a\in\sigma}} vol(\sigma)\alpha(a)$$

$$= \sum_{\sigma\in T} vol(\sigma) \sum_{a\in\sigma} \alpha(a)$$

$$= (d+1) \sum_{\sigma\in T} \underbrace{\Big(\underbrace{vol(\sigma)}_{\text{volume of } \sigma} \cdot \underbrace{\frac{1}{d+1}\sum_{a\in\sigma}\alpha(a)}_{\text{barycenter of } \sigma^\alpha} \Big)}_{\text{volume below } \sigma^\alpha}$$

$= (d+1) \cdot$ amount of water below $\{\sigma^\alpha : \sigma\in T\}$



Which T has the smallest amount of water below?

$T(\mathcal{A},\alpha)$!

Remark: (i) vertices in $\Sigma(\mathcal{A}) \longleftrightarrow$ regular triang's of $\mathcal{A}$

(ii) $\Sigma(\mathcal{A})$ not full-dim.

(iii) $\mathcal{C}_{\mathcal{A}}(T)$ contain linear subspaces (not pointed):

$\Rightarrow$ adding affine functions to α does not change $T(\mathcal{A},\alpha)$,

$\Rightarrow$ $d+1$ degrees of freedom $\Rightarrow$ $\Sigma(\mathcal{A})$ is in $\mathbb{R}^{n-d-1}$.

How can we mod this out?

Gale transforms: Regard $\mathcal{A}$ as a vector configuration $\{\binom{a}{1} : a \in \mathcal{A}\} \in \mathbb{R}^{d+1}$.

Associate to $\mathcal{A}$ the matrix

$$A := \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 1 & & 1 \end{pmatrix}, \quad a_i \in \mathcal{A},\ i = 1, \dots, n, \quad A \in \mathbb{R}^{(d+1)\times n} = \mathbb{R}^{r \times n}$$

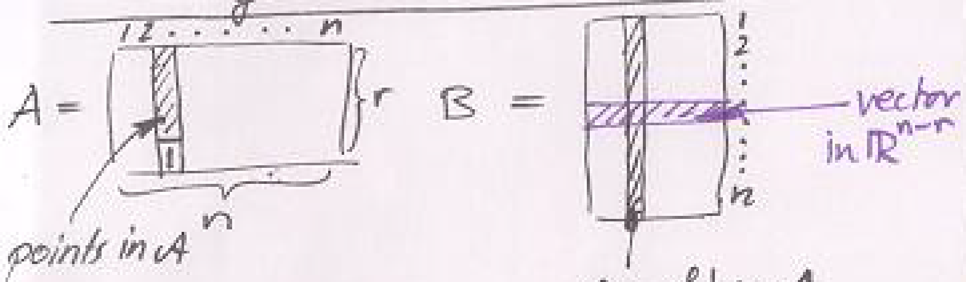
Assumption:

$$\operatorname{rank} A = d+1 =: r.$$

Consider $\mathcal{B} \in \mathbb{R}^{n \times (n-r)}$ with

$$A\mathcal{B} = 0 \in \mathbb{R}^{r \times (n-r)}$$

General



Interprete the rows of $\mathcal{B}$ as vectors in $\mathbb{R}^{n-r}$

$\rightsquigarrow$ Vector configuration $\mathcal{B}$.

Def.: Every $\mathcal{B}$ as above is a Gale transform of $\mathcal{A}$, $\mathcal{B} \in \text{Gale}(\mathcal{A})$.

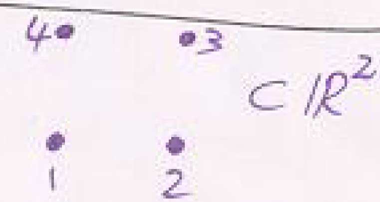
Obs.: $\mathcal{B} \in \text{Gale}(\mathcal{A}) \Rightarrow \mathcal{A} \in \text{Gale}(\mathcal{B})$

because

$$AB = 0 \Leftrightarrow B^T A^T = 0.$$

Example

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$



$r = 3,\ n = 4,\ n - r = 1$

Choose $\mathcal{B} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \begin{matrix} \to b_1 \\ \to b_2 \\ \to b_3 \\ \to b_4 \end{matrix} \Big\} \in \mathbb{R}^1$

But:
- tedious to calculate
- lots of choices

Idea: Look at properties of a Gale transform!

Circuits and cocircuits:

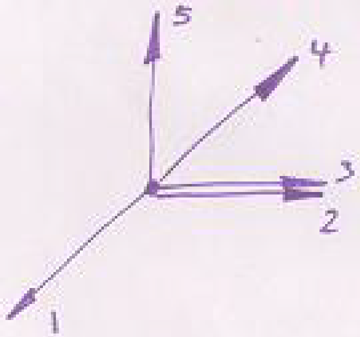
Def.: A circuit signature on a minimally $\left\{\begin{matrix}\text{affinely}\\ \text{linearly}\end{matrix}\right\}$ depend. subset $C \subseteq \mathcal{A}$ of a $\left\{\begin{matrix}\text{point}\\ \text{vector}\end{matrix}\right\}$ configuration $\mathcal{A}$ is a partition

$$C = C_+ \cup C_-$$

with

$$\sum_{a_i \in C_+} \lambda_i a_i = \sum_{a_j \in C_-} \lambda_j a_j \quad , \begin{cases} \lambda_i \geq 0 , \sum_{a_i \in C_+} \lambda_i = \sum_{a_j \in C_-} \lambda_j \\ \lambda_i \geq 0 \ \forall a_i \in C . \end{cases}$$

Example:



Circuits

C_+	C_-
2	3
1,4	
1,2,5	
1,3,5	
2,5	4
3,5	4

also written as

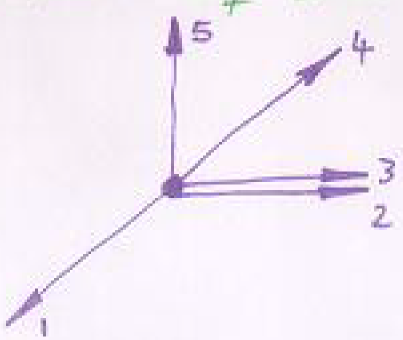
$2\bar{3}$
14
125
135
$25\bar{4}$ or $2\bar{4}5$
$3\bar{4}5$

Def.: A cocircuit signature on the complement of a maximal subset $\bar{C}^* \subseteq \mathcal{A}$ of points on a hyperplane spanned by $\bar{C}^*$ is a partition

$$C^* = \mathcal{A} \setminus \bar{C}^* = C^*_+ \cup C^*_-$$

such that C^*_+ and C^*_- lie strictly on opposite sides of this hyperplane.

Example:



Cocircuits

H	C^*_+	C^*_-
1	2,3	5
2	1	4,5
3	1	4,5
4	2,3	5
5	1	2,3,4

also written as

$23\bar{5}$
$1\bar{4}\bar{5}$
$1\bar{2}\bar{3}\bar{4}$

Circuits and cocircuits in the Gale transform

Prop.: $B \in \mathrm{Gale}(A) \Rightarrow$ circuits of A are the cocircuits of B, cocircuits of A are the circuits of B. $\square$

Pf: (Lin. Alg).

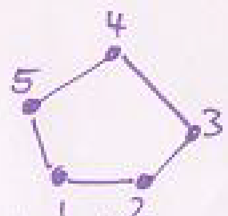
Example:

Circuits:	Cocircuits:	Circuits:	Cocircuits:
$1\bar{2}3\bar{4}$	34	12	$1\bar{2}3\bar{4}$
	14	14	
	12	23	
	23	34	
	$1\bar{3}$	$1\bar{3}$	
	$2\bar{4}$	$2\bar{4}$	

✓

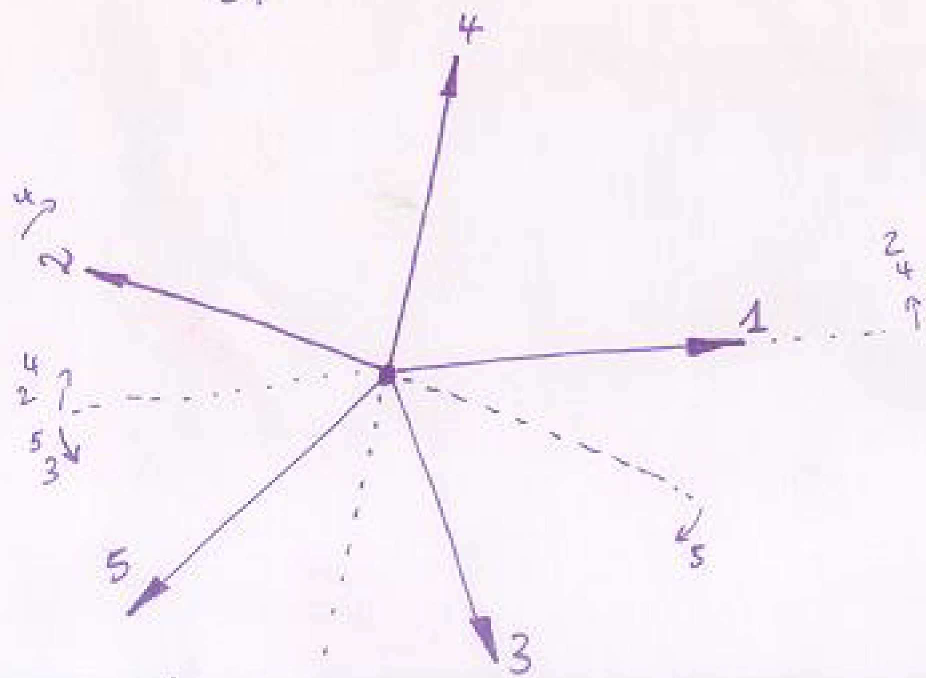
Application

Draw Gale transform of



!

Circuits:	Cocircuits:
$1\bar{2}3\bar{4}$	123
$1\bar{2}3\bar{5}$	234
$1\bar{2}4\bar{5}$	345
$1\bar{3}4\bar{5}$	$12\bar{4}$
$2\bar{3}4\bar{5}$	$23\bar{5}$
	$\bar{1}34$
	$\bar{2}45$
	$1\bar{3}5$



The chamber fan:

Def.: Let $\mathcal{B} \in Gale(\mathcal{A})$. Then

$$B(\mathcal{C}_{\mathcal{A}}) := \{ B(\mathcal{C}_{\mathcal{A}}(T)) : T \text{ pol. subdiv. of } \mathcal{A} \}$$

$$:= \{ \{ \alpha B : \alpha \in \mathcal{C}_{\mathcal{A}}(T) \} : T \text{ pol. subdiv. of } \mathcal{A} \}$$

is the chamber fan of $\mathcal{A}$. $B(\mathcal{C}_{\mathcal{A}}(T))$ is the chamber of T.

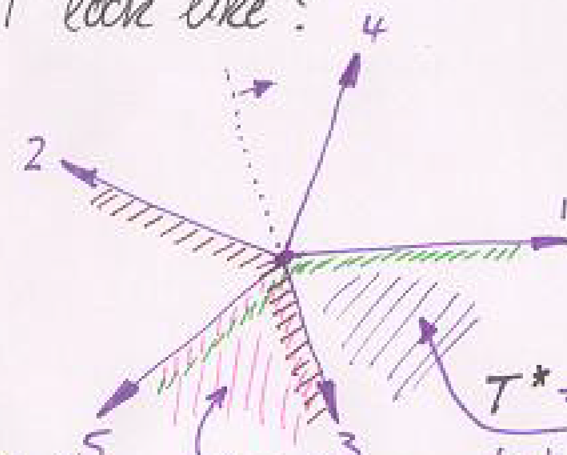
Remark: (i) αB is a lin. comb. of vectors in $\mathcal{B}$ (rows of $\mathcal{B}$) with coeff. $\alpha_i = \alpha(a_i)$ f. b_i.

(ii) $\alpha \mathcal{B} \in \mathbb{R}^{n-r}$.

What does the chamber of T look like?

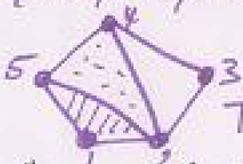
Magic example:

$T = \{124, 145, 234\}$



$T^* = \{35^*, 23^*, 15^*\} = \{\sigma^* := \mathcal{A} \setminus \sigma : \sigma \in T\}$

$B(\mathcal{C}_{\mathcal{A}}(T)) = \text{cone}\{b_1, b_5\} \cap \text{cone}\{b_2 b_3\} \cap \text{cone}\{b_3, b_5\}$



$T' = \{125, 245, 234\}$ $\quad B(\mathcal{C}_{\mathcal{A}}(T')) = \text{cone}\{b_3, b_4\} \cap \text{cone}\{b_1, b_3\} \cap \text{cone}\{b_1, b_5\}$

Why is this always true?

Flip from T to T' at circuit $1\bar{2}4\bar{5}$

Same as crossing cocircuit $1\bar{2}4\bar{5}$ in $\mathcal{B}$!

Equivalent heights leveling off $\sigma \in T$:

Obs.: For all heights α there is for all $\sigma \in T(\mathcal{A},\alpha)$ another height α_σ with

$$\alpha_\sigma(a) \begin{cases} = 0 & \text{for all } a \in \sigma \\ > 0 & \text{for all } a \in \mathcal{A} \setminus \sigma \end{cases}$$

s.t. $T(\mathcal{A},\alpha) = T(\mathcal{A},\alpha_\sigma)$.

$\Rightarrow$ αB and $\alpha_\sigma B$ lie in the same chamber for all $\sigma \in T$

Moreover, $\alpha_\sigma B$ is a positiv comb. of $b_i \in \mathcal{B}$ with $a_i \in \mathcal{A} \setminus \sigma = \sigma^*$, for all $\sigma \in T$.

$\Rightarrow$ αB is in the intersection of all cones spanned by σ^*, $\sigma \in T$.

$\Rightarrow$ $\alpha B \in \bigcap_{\sigma \in T} \operatorname{cone} \sigma^*$, $|\sigma^*| = n-r$.

$\Rightarrow$ Chambers are intersections of simplicial cones in the Gale transform.

THM: [Gelfand, Kapranov, Zelevinsky 1989]

(i) The chamber fan of $\mathcal{A}$ is the normal fan of a polytope in dim $n-r$.

(ii) Every full-dim. chamber corresponds to a regular triang of $\mathcal{A}$

(iii) Adjacent full-dim. chambers correspond to bistellar flips.

(iv) The face poset of the chamber fan is the opposite of the refinement poset of regular subdivisions of $\mathcal{A}$.

COR: The Graph of regular triangs of $\mathcal{A}$ is the edge graph of the secondary polytope.