Preview:

Three phenomena

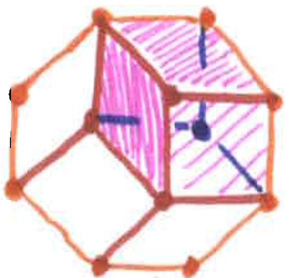
- ▷ Zonotopal tilings of zonotopes
- ▷ Monotone paths of polytopes
- ▷ Triang's of point conf's

⇝ similar structure

⇝ generalization

Zonotopal tilings:

Example: Z, $d = \dim Z$



$n = 2^{d'}$ vertices plus int. points

- ▷ Zonotope Z: Minkowski sum of line segments $\cong$ Projections of a hyper-cube
  - dim 2: centrally symmetric n-gons
- ▷ Zonotopal tiling of Z: Polyhedral subdivision of Z whose elements are zonotopes
- ▷ coherent: the subdivision is regular
- ▷ Flip from $\boxed{T}$ to $\boxed{T'}$: replace $\boxed{\text{upper facets}}$ of a proj. of a (d+1)-cube by $\boxed{\text{lower facets}}$

dim 2:

THM: The graph of all coherent zonotopal tilings of Z is the edge graph of a $(d'-d)$-polytope.

⇝ graph

Example:

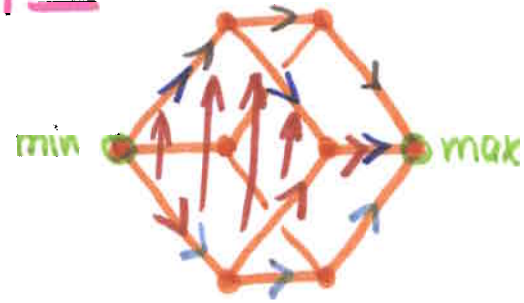
All tilings coherent, $d'-d=3$ (4 line segments, proj. of 4-cube)

Remark:

Not all tilings coherent:

Monotone paths on polytopes:

Example: P, $d := \dim P$



$\vec{c}$ in gen. pos.

▷ Monotone path in (P,c) :=
sequence of vertices $v_1, v_2, \dots, v_k$ of P
with
- $\{v_i, v_{i+1}\}$ is an edge of P,
- $c(v_i) < c(v_j)$ $\forall i < j$,
- $v_1 = \min$, $v_k = \max$

▷ $\boxed{p}$ coherent :⟺ $\boxed{p}$ is the $\boxed{\text{edge path}}$ in a 2-dim proj. of P.

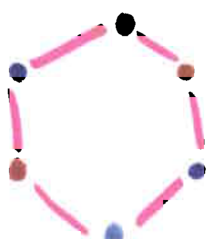
▷ $\boxed{\text{Flip}}$ from $\boxed{p}$ to $\boxed{q}$: "homotopy" wiping out a $\boxed{\text{2-face}}$

⇝ Graph of monotone paths

THM: The graph of all coherent monotone paths in (P,c) is the edge graph of a $(d-1)$-dim. polytope, the monotone path polytope of (P,c)

Example:

All mon. paths coherent.



"Permutahedron" P_2

Faces ≙ "cellular strings"

Remark: In general, there are non-coherent mon. paths

Triangulations of point conf's:

Example:

$\mathcal{A}$

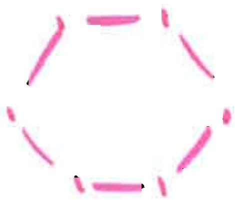
$d := \dim \mathcal{A}$

$n := |\mathcal{A}|$

triang's $\hat{=}$ permutations

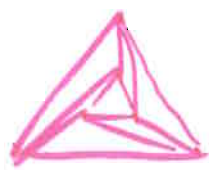
Thm: The graph of triangulations that are regular is the edge graph of an $(n-d-1)$-polytope $\Sigma(\mathcal{A})$, the secondary polytope of $\mathcal{A}$. (Lecture 3)

Example:



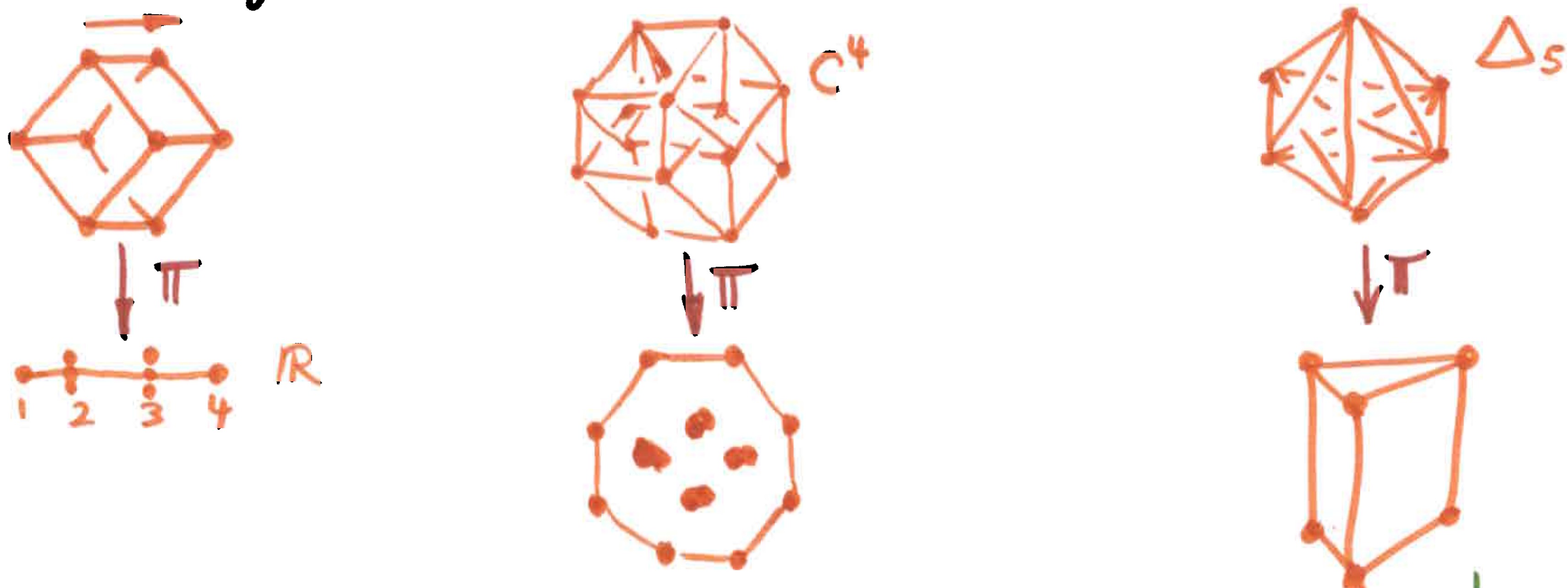
"permutahedron"

Remark: Not all triang's are regular (Lecture 4)



Question: Is there any more general construction in the background?

Answer: Yes: π-induced subdivisions.



Def: • Let $\pi : P \to \mathcal{A}$ be an affine projection of a polytope P, $d' = \dim P$, $d = \dim \mathcal{A}$.
$\mathcal{A}$ polyhedral subdivision T of $\mathcal{A}$ is $\underline{\pi\text{-induced}}$ if every cell $\sigma \in T$ is a projection of a face of P. T $\underline{\text{coherent}}$ $:\Leftrightarrow \exists w \in \mathbb{R}^{d'-d} : \pi^{-1}(T) \cap \pi^{-1}(x) = \pi^{-1}(x)^w \; \forall x$

• T $\underline{\text{refines}}$ T' if every cell of T is contained in a cell of T'.

$\{y \in \mathbb{R}^{d'-d} : \langle w, y \rangle = \max\}$
"face of fiber in dir. w"

THM.: [Billera, Sturmfels]
The refinement poset of all coherent π-induced subdivisions of $\mathcal{A}$ is isomorphic to the face poset of an $(d'-d)$-polytope $\Sigma(P \xrightarrow{\pi} \mathcal{A}) := \int_{x \in \operatorname{conv} \mathcal{A}} \frac{\pi^{-1}(x)\,dx}{\operatorname{vol}(\mathcal{A})}$

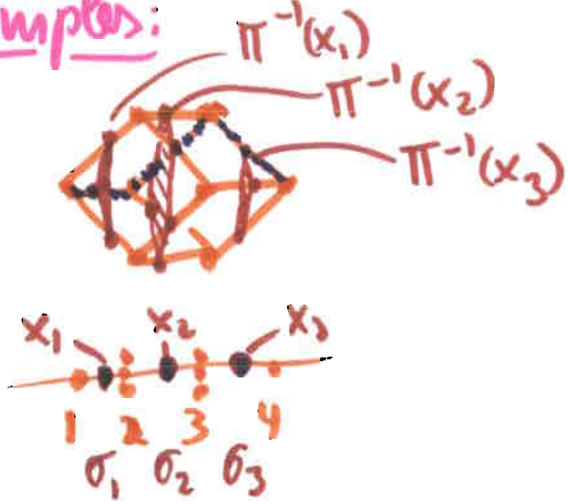
THM.: [Billera, Kapranov, Sturmfels]
For $d = 1$, the refinement poset of all π-induced subdivisions of $\mathcal{A}$ is homotopy equivalent to a $(d'-2)$-sphere.

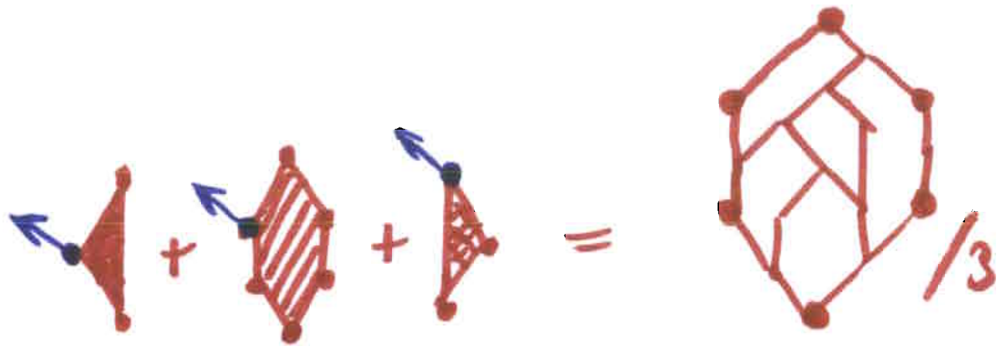
Finite def. of $\Sigma(P \xrightarrow{\pi} \mathcal{A})$:

REM:

$$\frac{1}{\mathrm{vol}\,\mathcal{A}} \int_{x \in \mathrm{conv}\,\mathcal{A}} \pi^{-1}(x)\, dx = \sum_{\substack{\sigma \text{ chamber} \\ \text{of } \mathcal{A} \\ \dim\sigma = \dim\mathcal{A}}} \pi^{-1}(\mathrm{barycenter}(\sigma)) \cdot \frac{\mathrm{vol}\,\sigma}{\mathrm{vol}\,\mathcal{A}} \qquad (\text{Minkowski sum})$$

Examples:





$\leftrightarrow$ $\omega \rightsquigarrow$ coh. mon. path "..."

Another application:

Def.: Let $P_1, \dots, P_k$ be d-polytopes in $\mathbb{R}^d$.

$$P_1 + \dots + P_k := \{p_1 + \dots + p_k : p_i \in P_i,\ i = 1, \dots, k\}$$

is the <u>Minkowski sum</u> of the P_i, $i = 1, \dots, k$.

Def.:

$$\pi_M : \begin{cases} P_1 \times \dots \times P_k \longrightarrow P_1 + \dots + P_k \\ (p_1, \dots, p_k) \longmapsto p_1 + \dots + p_k \end{cases}$$

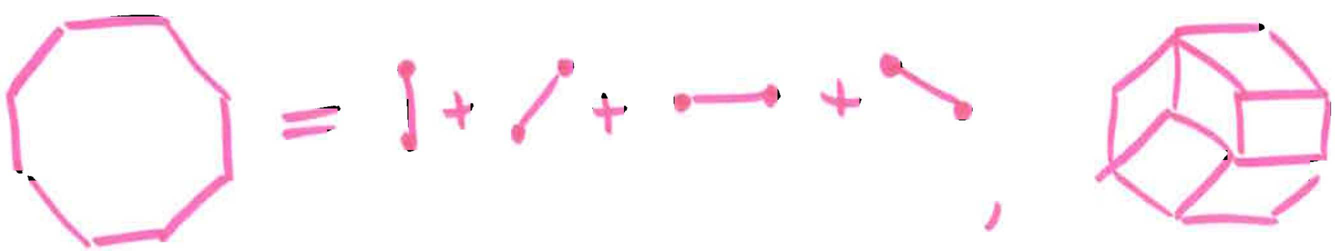
"Minkowski projection"

Def.:

- A polyhedral subdivision of $P_1 + \dots + P_k$ is a <u>mixed subdivision</u> if it is π_M-induced.
- It is <u>coherent</u> if it is π_M-coherent.
- It is <u>fine</u> if it is a minimal elem. in the refinement poset of all π_M-induced subdiv's.

<u>Remark</u>: Important for solving sparse polyn. systems (Bernstein bound)

<u>Example</u>:



mixed

The generalized Baues Problem:

Def.: Is the poset of <u>all</u> π-induced polyh. subdiv's for a given $\pi: P \to \mathcal{A}$ homotopy equivalent to a $(\dim P - \dim \mathcal{A}) - 1$ sphere?

"<u>Generalized Baues Problem for $\pi: P \to \mathcal{A}$</u>"

THM.:

- Yes, if $\dim \mathcal{A} \leq 1$ [Billera, Kapranov, Sturmfels '91]
- Yes, if $\dim P - \dim \mathcal{A} \leq 2$ [R., Ziegler '96]
- No in general if $\dim P - \dim \mathcal{A} > 2$ and $\dim \mathcal{A} \geq 2$ (counterexample in $\dim 5 \xrightarrow{\pi} \dim 2$, gen. pos., 10 verts., 42 faces) [R., Ziegler '96]
- Yes, for $\pi: C(n,d) \to C(n,d')$, $d' \leq d$ [Athanasiadis, R., Santos 1998]
- Yes, for $\dim \mathcal{A} \leq 2$ [Reiner '97]

Open:

- Cubes
- Hypersimplices
- $\Delta_k \times \Delta_l$
- ⋮