Enumeration in the space of triang's.

Two branches:

① formulas, bounds, asymptotics

[Catalan number (Lecture 1/2)]
[Billera, Filiman, Sturmfels: $\Sigma(A)$]
[Ziegler: 2-dim lattice triang's]
[Santos et al.] 2-dim triang's
[Santos] $\Delta_2 \times \Delta_k$, $C(d+4,d)$

$\boxed{\text{② computer calculations in dim } d}$

[de Loera: PUNTOS]
[R.: HST for $C(n,d)$]
[R.: TOPCOM]
[Imai et al.: codes not distributed]

Tasks:
- ▷ Check triang. prop.
- ▷ Find one triang.
- ▷ Find comp. of flip graph
- ▷ Find all triang's!

Issues:
- ▷ No general position assumption → stability of geometric calculations
- ▷ triang's require condition on intersections of simplices → Linear programming
- ▷ triang's require condition on convex hulls → convex hull computation

$\boxed{\text{Idea: Purely combinatorial calculations}}$

Main actor:

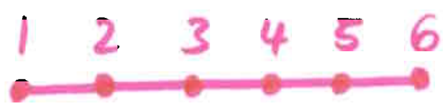
Def.: Let $\mathcal{A}$ be a $\left\{\begin{matrix}\text{point}\\ \text{vector}\end{matrix}\right\}$ configuration in $\left\{\begin{matrix}\mathbb{R}^d\\ \mathbb{R}^r, r=d+1\end{matrix}\right\}$ with $\dim \mathcal{A} = \left\{\begin{matrix} d \\ r \end{matrix}\right\}$.

The <u>chirotope of $\mathcal{A}$</u> is the map

$$\chi : \begin{cases} [n]^r \longrightarrow \{-1, 0, +1\} =: \{-, 0, +\} \\ (i_1, i_2, \dots, i_r) \longmapsto \operatorname{sign} \det \underbrace{\begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_r \\ | & | & & | \end{pmatrix}}_{\text{homogeneous coord's}} \end{cases}$$

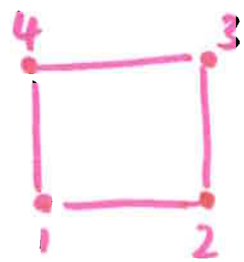
Example:

dim 1 :



$\chi(i,j) = + \iff i<j$

dim 2 :



$\chi(i,j,k) = 1$ for $i<j<k$

Obs.: χ determined by values on $(i_1, i_2, \dots, i_r)$ with $i_1 < i_2 < \dots < i_r$.

Q: How can we get circuit and cocircuit signitures from χ ?

Preview:

THM.: $T \in \binom{\mathcal{A}}{d+1}$, $T \neq \emptyset$, is a triang. of $\mathcal{A} \overset{\text{full-dim}}{\subset} \mathbb{R}^d$ iff

(IP) $\forall \sigma, \sigma' \in T$ $\nexists$ circuit (C_+, C_-) with

$$C_+ \subseteq \sigma, \quad C_- \subseteq \sigma'$$

(UP) Every interior facet of a simplex $\sigma \in T$ is contained in another simplex $\sigma' \in T$ with $\sigma' \neq \sigma$. (Lecture 5)

Obs.: Purely combinatorial.

(IP) ✓

(UP) facet $\tau \subset \sigma$ is interior if the induced cocircuit has "+" and "−" at the same time.

THM.: (IP) and (UP) can be checked by computing determinants of square submatrices of $\mathcal{A}$, no more than $\binom{n}{d+1} = \binom{n}{r}$.

Rem.:
- This is much nicer than LP for every pair of simplices.
- Can be done in exact arithmetics efficiently if $\mathcal{A}$ is rational.

Cocircuits and the chirotope:

Prop.: Assume, $\bar{C}^* \subseteq \mathcal{A}$ spans a $(d-1)$-dim. hyperplane in $\mathbb{R}^d$.

Then

$$C^* : \begin{cases} [n] \longrightarrow \{-,0,+\} & \text{"sign vector"} \\ i \longmapsto \chi(\bar{C}^*, i), & \text{with a fixed ordering on } \bar{C}^*, \end{cases}$$

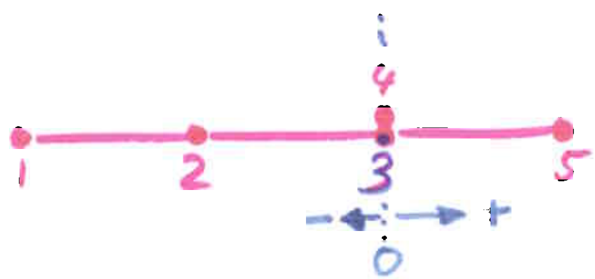
with

$$C^*_+ := \{ i \in [n] : C^*(i) = + \}$$
$$C^*_- := \{ i \in [n] : C^*(i) = - \}$$

is a cocircuit signature on C^*, and also the opposite.

Example:

dim 1:



$\bar{C}^* := \{3\} \Rightarrow$

$C^*(1) = \chi(3,1) = -\chi(1,3) = -$

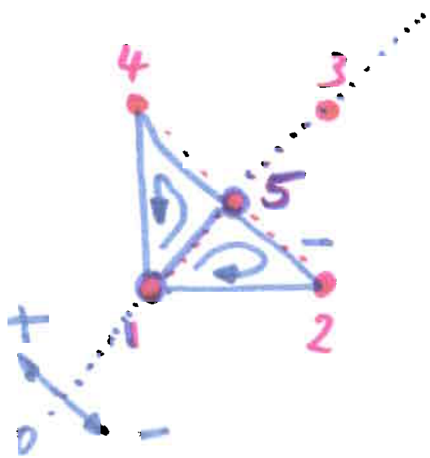
$C^*(2) = \chi(3,2) = -\chi(2,3) = -$

$C^*(3) = \chi(3,3) = 0$

$C^*(4) = \chi(3,4) = +$

$C^*(5) = \chi(3,5) = +$

dim 2:



$\bar{C}^* = \{1\,5\} \Rightarrow$

$C^*(2) = \chi(1,5,2) = -\chi(1,2,5) = -$

Hyperplane equations and determinants:

PROP: The hyperplane H spanned by $a_{i_1}, a_{i_2}, \dots, a_{i_d} \in \mathcal{A}$ in $\mathbb{R}^d$ (affine space) is the zero-set of

$$\psi_H(x) = \det\begin{pmatrix} a_{i_1} & a_{i_2} & \cdots & a_{i_d} & x \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \quad \text{w.l.o.g. } i_1 < i_2 < \cdots < i_d.$$

Moreover, ψ_H is a lin. functional, and its sign yields the corresponding cocircuit signature.

Rem: (i) C^* can have more than d zeros.

(ii) Positive cocircuits correspond to facets of $\operatorname{conv} \mathcal{A}$.

Circuits and the chirotope:

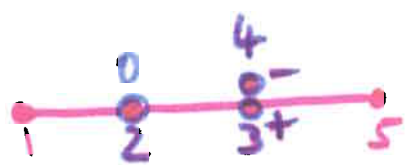
Prop.: Let C be a set $\{i_1 < i_2 < \dots < i_{d+2}\}$ of $d+2$ points in $\mathbb{R}^d$. Then

$$C: \begin{cases} [n] \longrightarrow \{-,0,+\}, \\ i_j \longmapsto (-1)^j \chi(C \setminus i_j), \\ i \longmapsto 0 \text{ if } i \in C, \end{cases}$$

is a circuit signature on C, and the opposite as well.

Example:

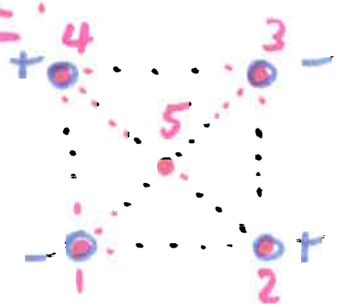
dim 1:



, $C = \{2,3,4\}$:

$$\Rightarrow \quad \begin{aligned} C(1) &= 0 \\ C(2) &= (-1)^1 \chi(3,4) = 0 \\ C(3) &= (-1)^2 \chi(2,4) = + \\ C(4) &= (-1)^3 \chi(2,3) = - \quad ✓ \\ C(5) &= 0 \end{aligned}$$

dim 2:



$C = \{1,2,3,4\} \Rightarrow$

$$\begin{aligned} C(1) &= (-1)^1 \chi(2,3,4) = - \\ C(2) &= (-1)^2 \chi(1,3,4) = + \\ C(3) &= (-1)^3 \chi(1,2,4) = - \\ C(4) &= (-1)^3 \chi(1,2,3) = + \quad ✓ \end{aligned}$$

Q: Why is that?

Circuits and Cramer's rule:

Proof: Assume homogeneous coord's for $a_i \in \mathcal{A}$, $i = 1, \dots, n$; let $C \subset \mathcal{A}$, $|C| = d+2$.

$$\sum_{i \in C} \lambda_i a_i = 0 \iff \sum_{i \in C \setminus j} \underset{\substack{\uparrow \\ \text{variables}}}{\lambda_i} a_i = -\underset{\substack{\uparrow \\ \text{fixed}}}{\lambda_j} a_j$$

Cramer's rule:

$$\lambda_i = \frac{\det(a_1 \dots \overset{-\lambda_j a_j}{\hat{a}_i} \dots \hat{a}_j \dots a_{d+2})}{\det(a_1 \dots a_i \dots \hat{a}_j \dots a_{d+2})}$$

$\hat{}$ 1 elem. missing

$$= -\lambda_j \frac{\det(a_1 \dots a_j \xrightarrow{j-i-1 \text{ exchanges}} \hat{a}_j \dots a_{d+2})}{\det(a_1 \dots a_i \dots \hat{a}_j \dots a_{d+2})}$$

$$= -\lambda_j \frac{(-1)^{j-i-1} \det(a_1 \dots \hat{a}_i \dots a_{d+2})}{\det(a_1 \dots \hat{a}_j \dots a_{d+2})}$$

$$\iff \frac{\lambda_i}{\lambda_j} = \frac{(-1)^j}{(-1)^i} \frac{\chi(C \setminus j)}{\chi(C \setminus i)}$$

$\iff$ λ_i, λ_j have the same sign iff $(-1)^j \chi(C \setminus j)$ and $(-1)^i \chi(C \setminus i)$ have the same sign. $\square$

The oriented matroid of $\mathcal{A}$:

Def.: The oriented matroid of $\mathcal{A}$ is the equivalence class of all point conf's $\mathcal{A}'$ with

$$|\mathcal{A}'| = |\mathcal{A}|$$
$$\dim \mathcal{A}' = \dim \mathcal{A}$$
$$\left.\begin{array}{r} \text{circuits}(\mathcal{A}') \cong \text{circuits}(\mathcal{A}) \\ \text{cocircuits}(\mathcal{A}') \cong \text{cocircuits}(\mathcal{A}) \\ \chi_{\mathcal{A}'} \cong \chi_{\mathcal{A}} \end{array}\right\}$$
isomorphism given by a bijection on labeled points.

Rem.: The oriented matroids of point conf's are just the tip of the iceberg (axiom systems for circuits, cocircuits, χ, vectors, covectors, topes lead to structures not coming from point conf's.)