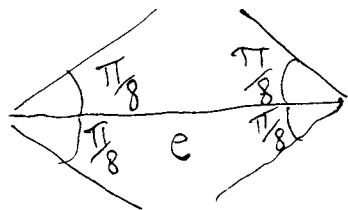


DEF: LMT-skeleton (set of edges belonging to ALL LMT's) ⑤

Lemma: The edge e belongs to MWT then it is necessary that at least one of the regions contains no other vertex.



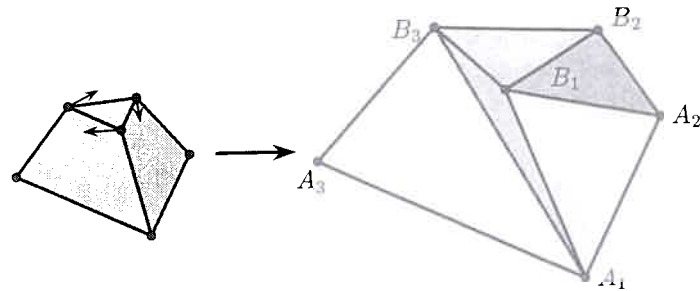
Lemma: Each candidate edge e to belong to ^{an} LMT must be crossed by at least one longer edge. If e is a candidate, and it does not intersect any other candidate edge, then e is in MWT

Using LMT-skeleton people have been to compute MWT for point sets with 1000 points !!! (Snoeyink et al. 1997) BUT it fails sometimes

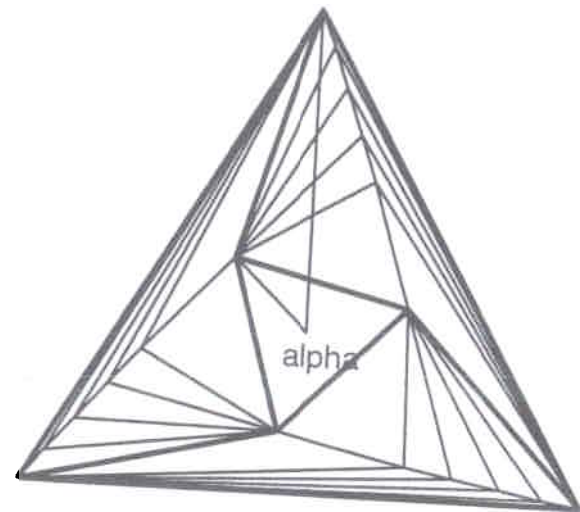
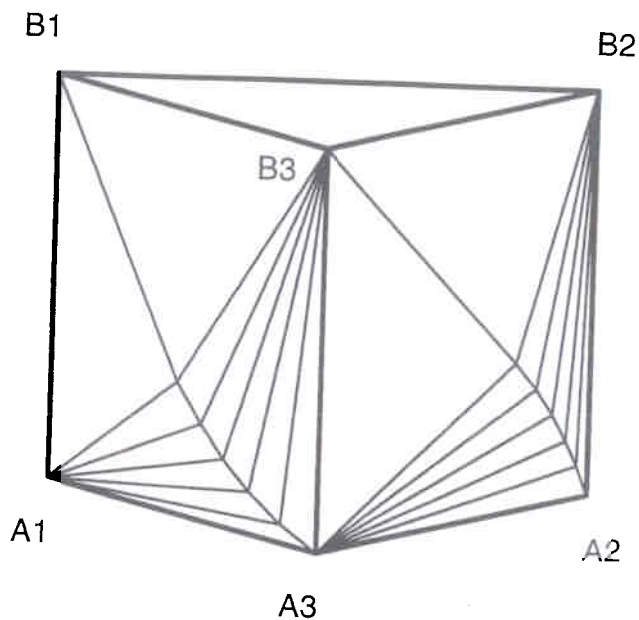
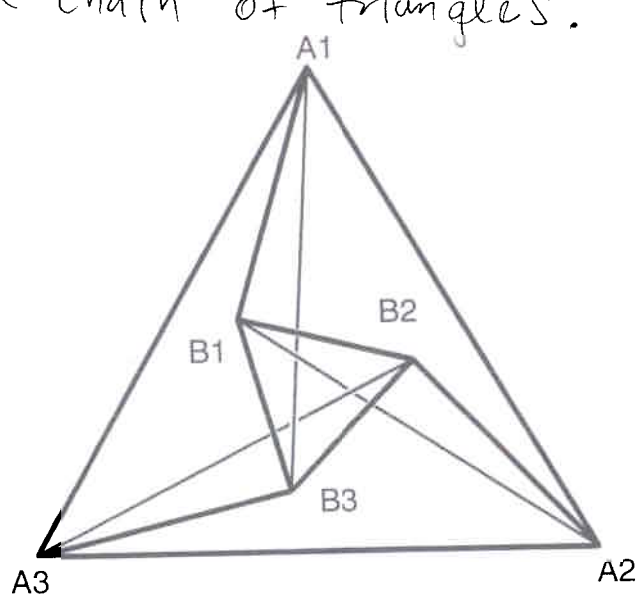
MINIMAL SIZE TRIANGULATION

Theorem: It is an NP-hard to find the smallest size triangulation (Below, Richter-Gebert, J) for 3-d polytopes.

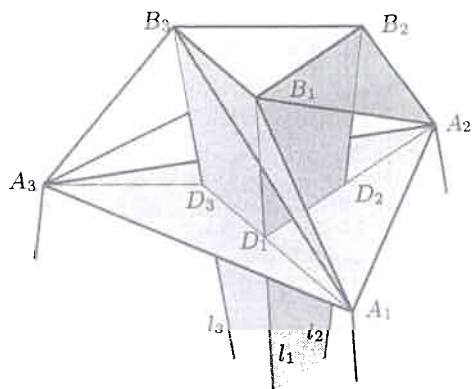
One of the main "switches" is built using Schönhardt's polytope. You need this to construct a polytope that BIG triangulations only if it does not use a point, but triangulations can be SMALL if one uses the extra point:



To construct the "switch polytope" we glue to the sides of Schönhardt ⑥ a chain of triangles.



If an extra point is added to the configuration A , such that it lies in the cone defined by planes $l_1 = A_1 B_1 B_3$, $l_2 = A_2 B_1 B_2$, $l_3 = A_3 B_2 B_3$ then we can do SMALL triangulations, smaller than without the point.



Region is bounded by planes l_1, l_2, l_3

Corollary:

For any fixed dimension, it is NP-hard to compute the smallest triangulation of a point set A .

How about concrete families of polytopes?
What can be done?

Suppose we are given a concrete point set, weights for each simplex ⑦
how can we find an optimal triangulation

a) minimizing sum of weights of d -simplices present

b) minimizing maximal weight among d -simplices present

EXAMPLES: Consider an m -prism. The smallest number of tetrahedra in a triangulation is $2m + \lfloor \frac{m}{2} \rfloor - 5$.

WARNING: Size of maximal triangulation depends on the coordinates (more precisely on the order type).

Do picture of cube vs



EXAMPLE: Regular d -cube. Maximal triangulation $d!$ simplices but we do not know minimal triangulation for $d \leq 7$.

For $d=8$ optimal is between 5522 and 13136 (more this afternoon)

How to compute, calculate concrete examples? We model the problem as **LINEAR & INTEGER PROGRAMMING PROBLEMS !!**

min Cx		Simplex method offers practical solution
subject to		
$Ax \leq b$		

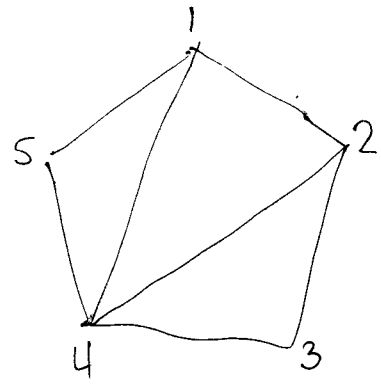
when asking about integral solutions, we have to use Exhaustive enumeration techniques.

THINK of triangulations as LONG vectors of ZEROS/ONES.

Let $A \subseteq \mathbb{R}^d$ be a point set with n points. Each triangulation is a vector with $\binom{n}{d+1}$ entries (one per possible d -simplex)

$$(V_T)_\sigma = \begin{cases} 0 & \text{if } \sigma \text{ not in } T \\ 1 & \text{if } \sigma \text{ in } T \end{cases}$$

EXAMPLE: Pentagon, each triangulation is represented by a vector in \mathbb{R}^{10}



$$(0, 1, 0, 0, 0, 1, 1, 0, 0, 0)$$

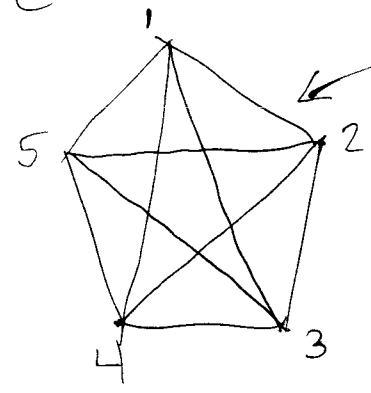
There will be ONE VARIABLE PER d -SIMPLEX

Definition: Let $U_A = \text{convex hull}$ (0-1 vectors of triangulations)
NOTE: $U_A \subseteq \mathbb{R}^{\binom{n}{d+1}}$ (it lives in high dimension)

GOAL: PROVIDE, equations and inequalities describing U_A

A) Equalities

Chambers are regions that result from "overlapping" ALL triangulations



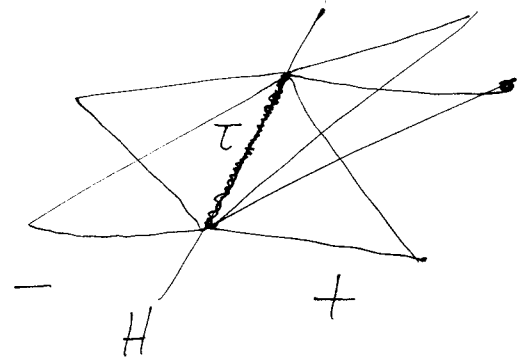
11 chambers

$$X_{245} + X_{134} + X_{133} + X_{124} + X_{235} = 1$$

in general $\sum_{\sigma \text{ contains chamber } \gamma} X_\sigma = 1$

Second kind of equalities

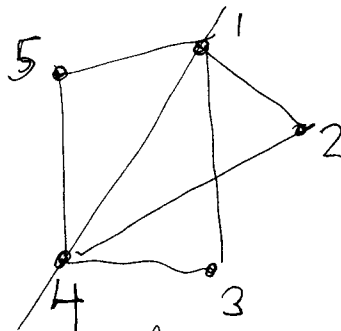
Consider a hyperplane H spanned by a $(d-1)$ -simplex τ



$$\sum_{\substack{\sigma \subseteq H^+ \\ \tau \subset \sigma}} x_\sigma - \sum_{\substack{\sigma' \subseteq H^- \\ \tau \subset \sigma'}} x_{\sigma'} = 0$$

(Both sides of τ should be covered or NONE)

Example:



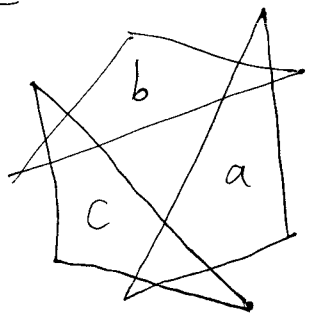
$$z_{145} - z_{124} - z_{134} = 0$$

Theorem: The 0/1 solutions of the above system of equations are EXACTLY the vectors of triangulations.

Observation: If you wish to ~~opt~~ optimize over triangulations that use ALL points of A , how would you do it?

$$\sum_{\substack{\sigma \text{ contains} \\ \text{vertex } p}} x_\sigma \geq 1$$

INEQUALITIES: Suppose you have a bunch of triangle intersecting pairwise $x_a + x_b + x_c \leq 1$ is an inequality satisfied



What are the edges of the universal polytope?

Theorem 9.1.10. Let T_1 and T_2 be two distinct triangulations of A . The following statements are equivalent:

1. v_{T_1} and v_{T_2} are not neighbors in Q_A .
2. v_{T_1} and v_{T_2} are not neighbors in U_A .
3. There exist two triangulations T_3 and T_4 of A (different from T_1 and T_2) such that $v_{T_1} + v_{T_2} = v_{T_3} + v_{T_4}$.
4. There exist partitions $T_1 = R_1 \cup L_1$ and $T_2 = R_2 \cup L_2$ such that $L_1 \cup R_2$ and $R_1 \cup L_2$ are two other triangulations of A , different from T_1 and T_2 .

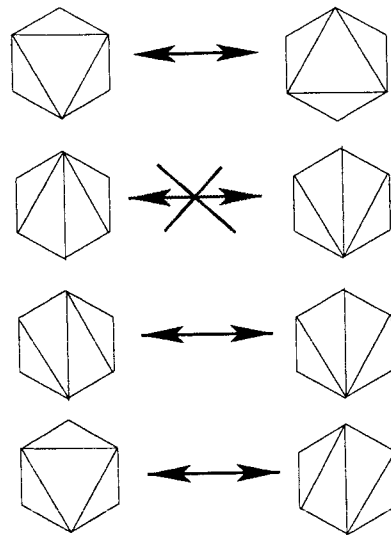


Figure 9.6: adjacencies in the universal of a hexagon

Flips are particular cases of "edges of the universal."

SECONDARY POLYTOPE is a projection of the universal polytope.

There is an interesting connection between MINIMAL SIZE TRIANGULATIONS and SMALLEST NUMBER OF FLIPS NEEDED to transform a triangulation T_1 into T_2 :

Each flip corresponds to stacking up another simplex inside a triangulation of a ball or a polytope.

Theorem minimal size triangulations of convex 3-polytopes are between $n-3$ and $2n-10$ for large values of n .

implies diameter of the graph of triangulations of n -gon is bounded by $2n-10$.