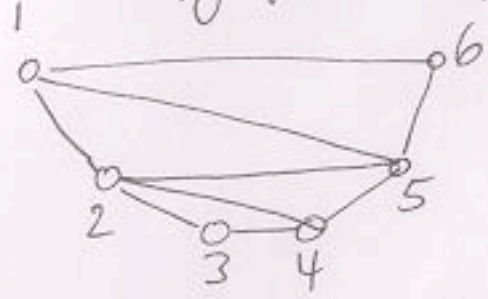
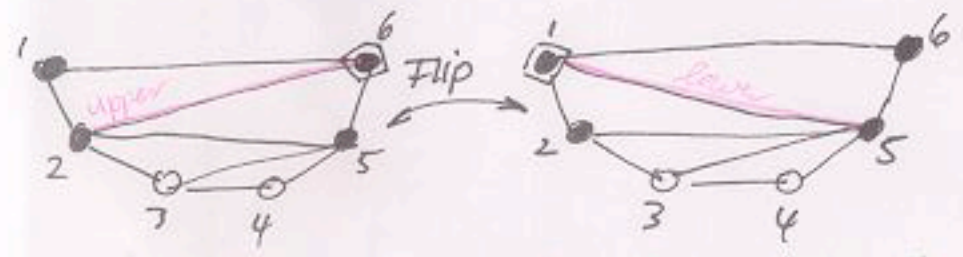


More structure for n -gons $= C(n, 2)$:

Idea: Classify flips in upflips and downflips.

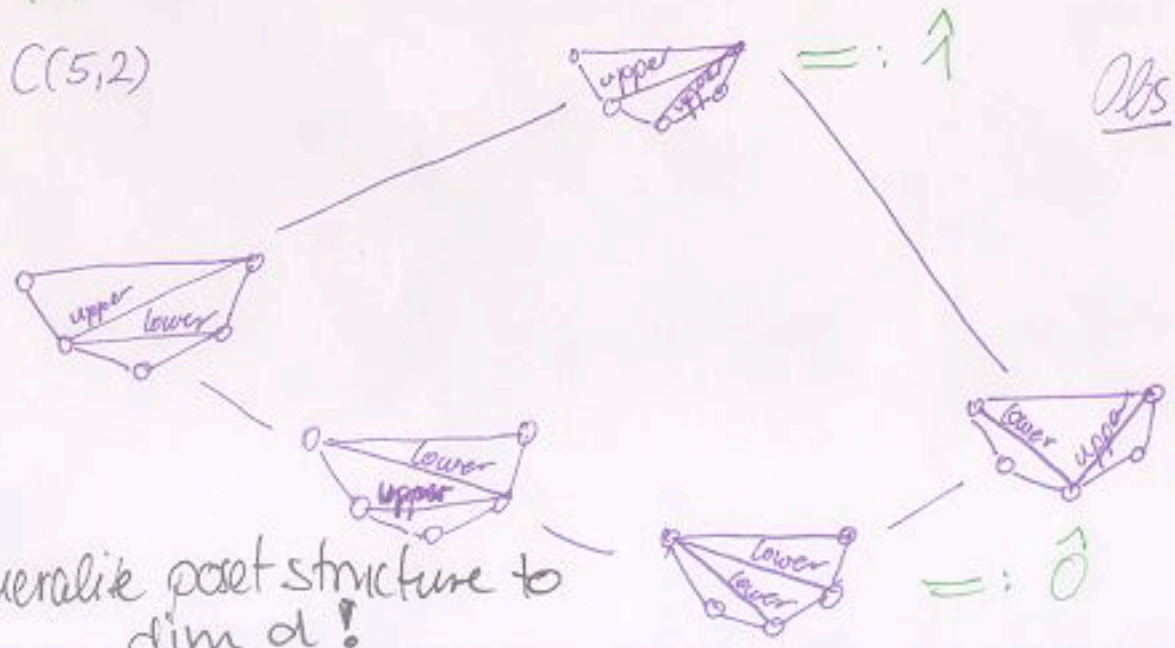


$T = \{156, 125, 245, 234\}$



Def: upflip: replace lower diag. by upper diag.
downflip: vice versa.

Example: $C(5, 2)$



Obs.: (i) bounded poset

(ii) Graph of triang's

= Hasse-diagram of poset

(iii) l.p.: Graph of triang's connected.

Goal: Generalize poset structure to $\dim d$!

posets:

Def: A partially ordered set (P, \leq) is a set P with a binary relation " \leq " s.t.

(i) $x \leq x \quad \forall x \in P$

(ii) $x \leq y \wedge y \leq x \implies x = y$

(iii) $x \leq y, y \leq z \implies x \leq z$

$x \leq y, x \neq y \implies x < y$

A covering relation in (P, \leq) is a relation

$x < y$ s.t. $\nexists z: x < z < y$

A Hasse-diagram of P is a graph $G_P = (P, E)$, directed, with

$(x, y) \in E \iff x < y$
relation " $<$ "

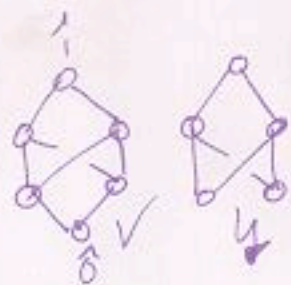
Given an acyclic directed graph G , the transitive closure of G is the poset

$(V(G), \leq)$ with a chain of relations $x < z_1 < \dots < z_k < y$.
 $x < y \iff \exists$ directed paths from x to y in G .

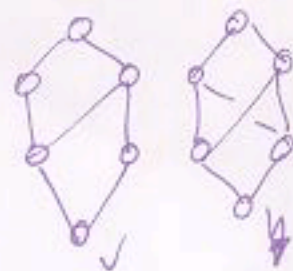
P bounded if $\exists \hat{0}, \hat{1} \in P$ with $\hat{0} \leq x \leq \hat{1} \quad \forall x \in P$.

P is a lattice if there is a unique smallest upper and a unique largest lower bound for any pair of elements x, y .
 $x \vee y$ $x \wedge y$

Example:



Example:

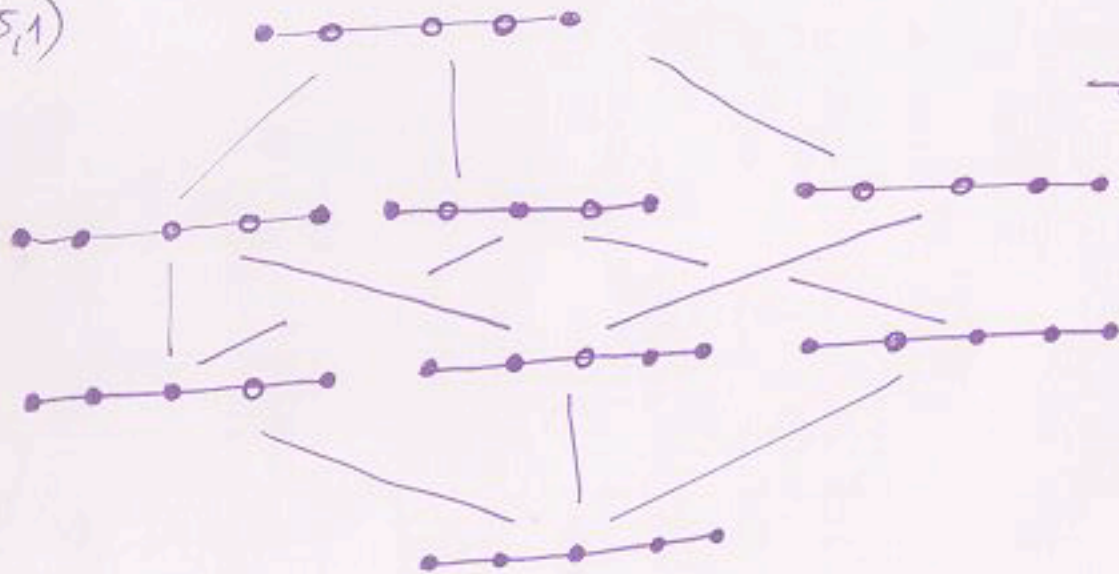


The projective view of up- and downflips:

$C(m, 1)$:  flips $\hat{=}$ add/remove point in interior

Def.: An upflip in $C(n, 1)$ removes a point,
a downflip in $C(n, 1)$ adds a point.

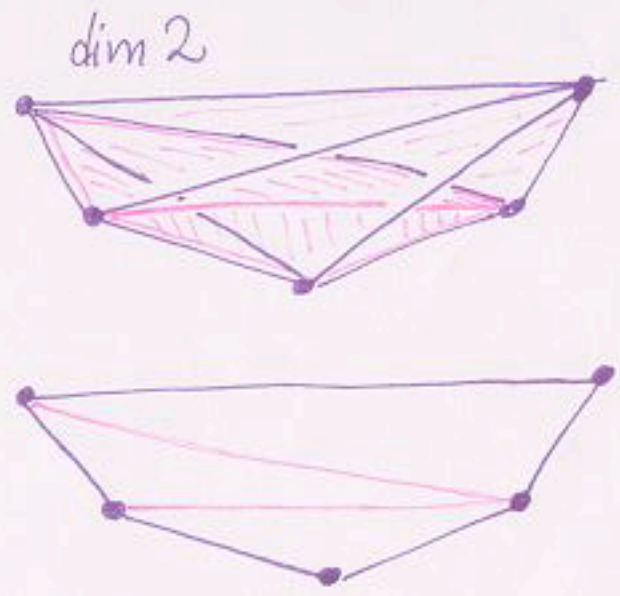
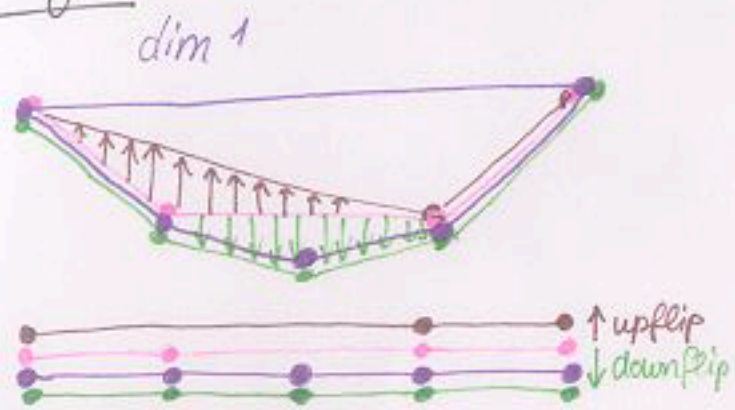
Example: $C(5, 1)$



→ bounded poset!

Q.: Can we unify definitions of up- and downflips in dim 1 and dim 2?

A: Yes!



Obs.: (i) A $T-T'$ flip is an upflip if the lifting of T' is weakly higher than the lifting of T . It is a downflip otherwise (in dim 1 and 2).

(ii) If the triangles T and T' are connected by a flip, then exactly one $(d+1)$ -simplex fits between their liftings.

(iii) If between the liftings of T and T' there is exactly one $(d+1)$ -simplex then

(iv) T and T' differ by a flip. The vertices of this simplex form the circuit in \mathbb{R}^d that supports the flip.

Def.: $P_{nd}: \begin{cases} C(n,d) \rightarrow C(n,d-1) \\ x \mapsto (x_0, x_1, \dots, x_{d-1})^T \end{cases}$ "forget last coordinate"

is the canonical projection of $C(n,d)$.

For T triang of $C(n,d)$ let

$$s_T: \begin{cases} C(n,d) \rightarrow C(n,d+1) \\ v_d(i) \mapsto v_{d+1}(i), i=1, \dots, n, \text{ affinely extended on simplices of } T. \end{cases}$$

This is the characteristic section of T in $C(n,d+1)$

The first Stasheff-Tamari poset $\mathcal{S}_1(n, d)$:

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Def.: $\mathcal{S}_1(n, d)$ is the poset on all triang's of $C(n, d)$ given by the transitive closure of

$T_1 \leq T_2$:iff $T_1 \xrightarrow{\text{upflip}} T_2$, where an upflip is a flip that replaces in T_1 the lower facets of a $(d+1)$ -simplex in $E(n, d+1)$ by its upper facets, yielding T_2 .

[Edelman, Reiner 1995]

Thm.: [TAMARI, HUANG]

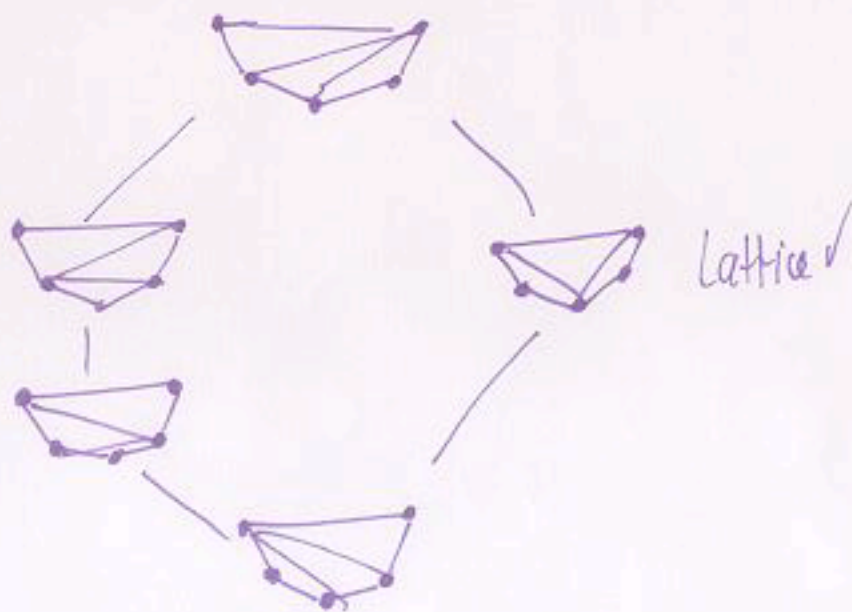
$T_{n-2} = \mathcal{S}_1(n, 2)$ is a lattice $\forall n$. "Tamari lattice"

Thm.: [EDELMAN, REINER]
 $\mathcal{S}_1(n, 3)$ is a lattice $\forall n$.

Cor.: $\mathcal{G}_E(n, 3)$ is connected.

Rem.: > 60 equivalent comb. structures [Stanley] for $\mathcal{S}_1(n, 2) = T_{n-2}$

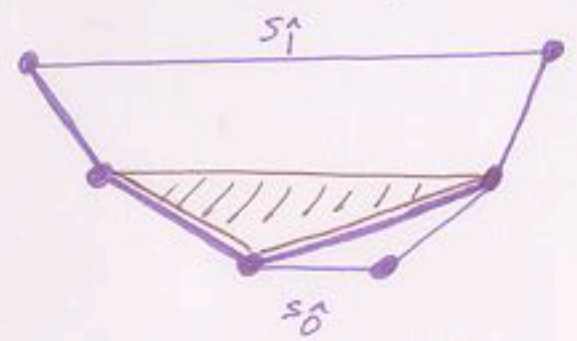
Example:



Proof programme:

$T \neq \uparrow$

1. For every triang T of $C(m, d)$ construct an upflip.
2. For this, find a $(d+1)$ -simplex in $C(m, d+1)$ with all its lower facets in $s_T(C(m, d))$

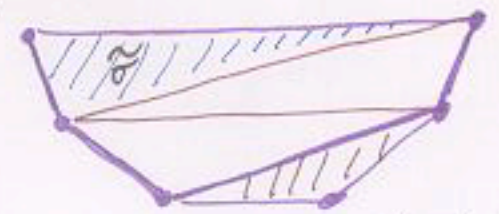


3. For this, triangulate $C(m, d)$ with $s_T(C(m, d))$ as a subcomplex.

!



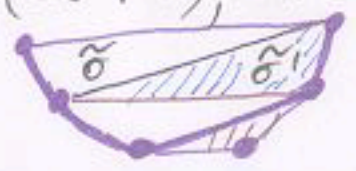
4. Pick a simplex $\tilde{\sigma}$ above $s_T(C(m, d))$.



5. While $\tilde{\sigma}$ has not all lower facets in $s_T(C(m, d))$, set $\tilde{\sigma}'$ to a simplex adjacent at a lower facet of $\tilde{\sigma}$ and set $\tilde{\sigma}$ to $\tilde{\sigma}'$.

!

6. Finitely many simplices \Rightarrow find upflip.



□

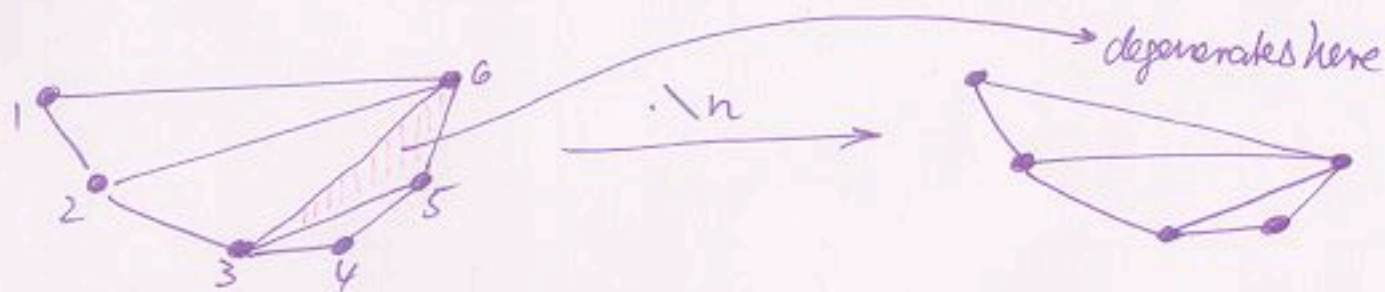
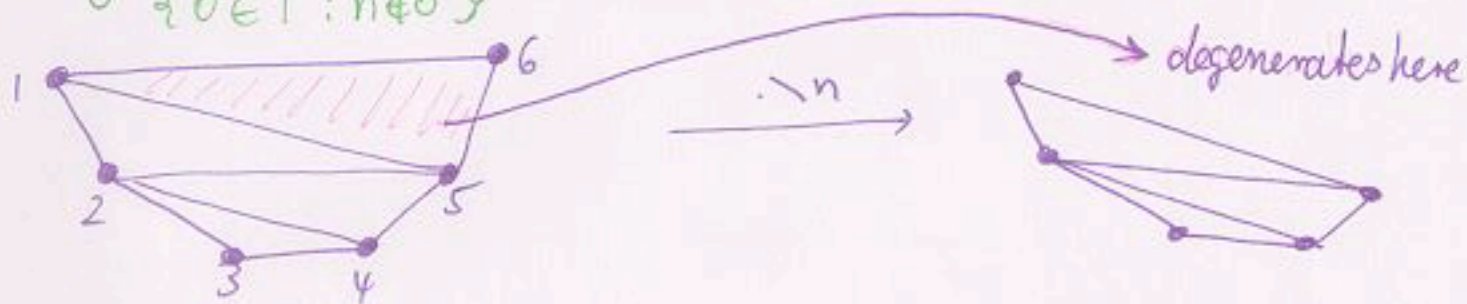
Constructions I:

Def.: For a triangulation T of $C(n, d)$ let

$$T \setminus n := \left\{ \sigma \setminus n \cup \{n-1\} : \begin{array}{l} \sigma \in T \\ n \in \sigma \end{array} \right\} \cup \{ \sigma \in T : n \notin \sigma \}$$

(only full-dim. simplices kept)

Example:



Obs.: $T \setminus n$ can be obtained from T by continuously sliding vertex n to vertex $n-1$ and removing all degenerated simplices.

Prop.: $T \setminus n$ is a triang. of $C(n-1, d)$

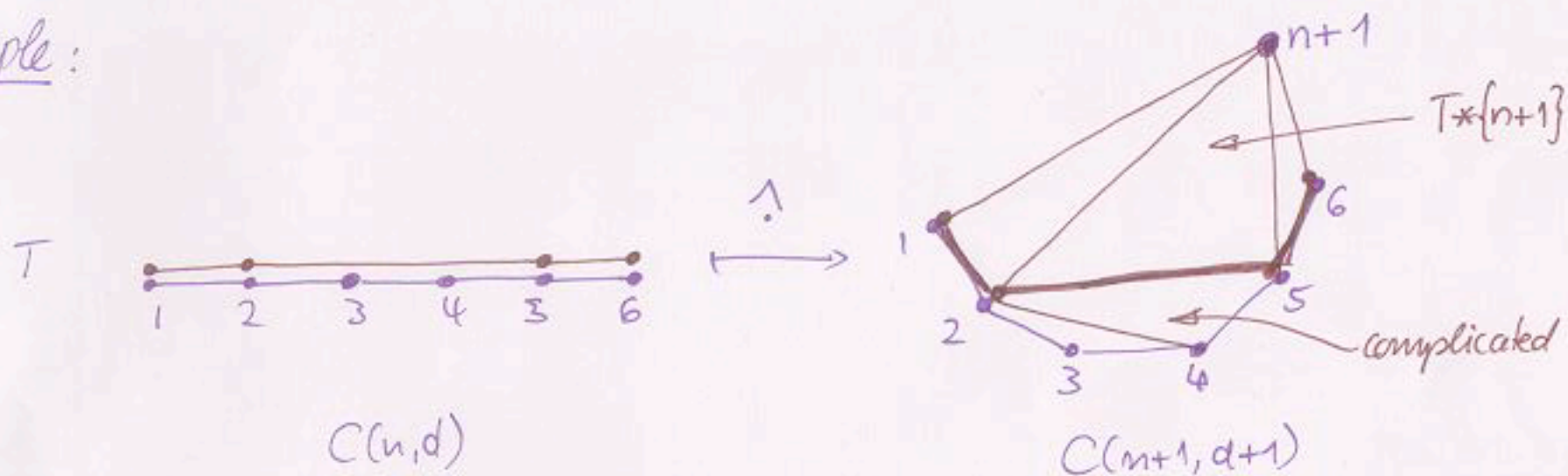
Constructions 2:

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Def.: For a triang. T of $C(m, d)$ let

$$\hat{T} := \{T * \{n+1\}\} \cup \{\text{complicated stuff}\}$$

Example:



Prop.: \hat{T} is a triang. of $C(n+1, d+1)$ containing $S_T(C(n, d))$ as a subcomplex.

Cor.: T triang. of $C(n, d) \implies \hat{T} \setminus \{n+1\}$ is a triang. of $C(n, d+1)$ containing $S_T(C(n, d))$ as a subcomplex.

The full theorem:

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THM: $\mathcal{L}_1(m, d)$ is bounded. Moreover:
The triang's of $C(m, d+1)$ are in one-to-one correspondence with the maximal chains in $\mathcal{L}_1(m, d)$.

Cor: (i) $\# \hat{0} \geq \# T \geq \# \hat{1}$

(ii) for even d , $\# \hat{0} = \# \hat{1}$, so $\# T = \# \hat{0} = \# \hat{1}$.

(iii) $\# \hat{0}$ in dim $d+1 \geq$ length of a max. chain in dim $d \geq \# \hat{1}$ in dim $d+1$

Q: Is $\mathcal{L}_1(m, d)$ a lattice?

A: No: Computer calculated counterexample for $C(9, 5)$ and $C(9, 4)$.

THM: $\Delta(\overline{\mathcal{L}_1(m, d)})$ ($=$ simpl. complex of chains) $\simeq \mathbb{S}^{n-d-3}$
without $\hat{0}, \hat{1}$

Example:

