Main Actor:

Def.: The d-dim. moment curve is defined as

$$\nu_d : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R}^d \\ t \longmapsto (1;t;t^2;\dots;t^d)^T \end{cases}$$ homogeneous coord's.

The d-dim. cyclic point configuration with n vertices is $\mathcal{C}(n,d) := \{\nu_d(1), \nu_d(2), \dots, \nu_d(n)\}$.

The d-dim. cyclic polytope with n vertices is

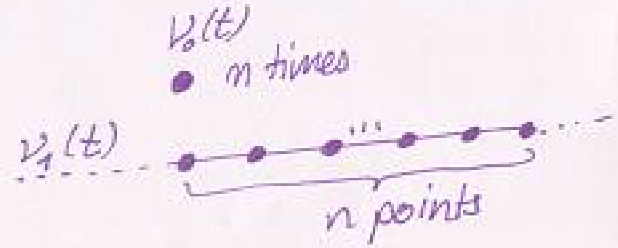
$$C(n,d) := \operatorname{conv}\{\nu_d(1), \nu_d(2), \dots, \nu_d(n)\}$$, i-th point in this list is labeled "i".

Notation: Vertex $\nu_d(i)$ is labeled i.

Example:

dim = 0: $C(n,0) =$ n copies of the same point

dim = 1: $C(n,1) =$ n distinct points on a line

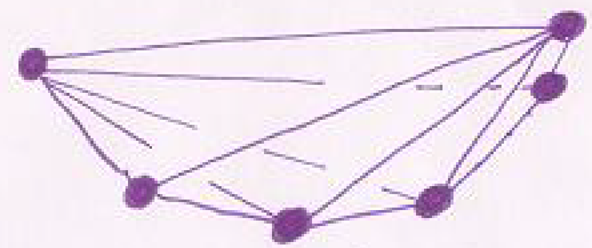


dim = 2: $C(n,2) =$ 

= n-gon

dim = 3: $C(n,3) =$ 

expand $\longleftrightarrow$ drawing:



Example: $C(6,3)$

Fact: (i) $C(10,6)$, $C(9,4)$, $C(9,5)$ has non-regular triang's!

(ii) Therefore, the secondary polytope of $C(n,d)$ does in general not describe all triang's of $C(n,d)$.

Preview:

Q.: Does the set of all triang's of a point conf. have a friendly structure ?
A.: Stay tuned (Lecture 6)
Q.: Does the set of all friendly triangs of a point conf. have a friendly structure ?
A.: If friendly = regular then yes: secondary polytope (Lecture 3).
Q.: Does the set of all triang's of a friendly point conf. have a friendly structure ?
A.: If friendly = cyclic then yes: bounded poset (today's lecture)

THM. [R. 1996]:

The graph $\mathcal{G}_{C(n,d)}$ of all triang's of $C(n,d)$ is the Hasse-diagram of a bounded poset.

COR.: (i) $\mathcal{G}_{C(n,d)}$ is connected

(ii) Reverse search [Avis, Fukuda] possible for enumeration.

Facts about $C(n,d)$:

→ Exerc.

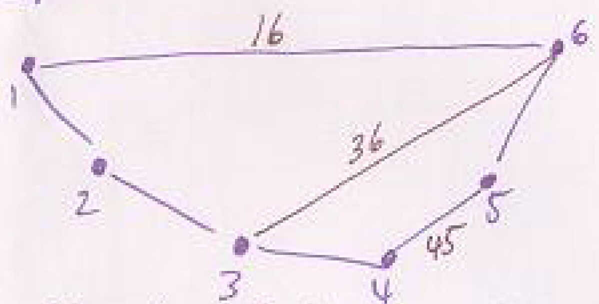
Lemma: (i) $\mathcal{C}(n,d)$ is in general position.
(ii) $\mathcal{C}(n,d)$ has only full-dim. circuits, i.e., every circuit has $d+2$ elements
(iii) $\mathcal{C}(n,d)$ has only simplicial faces.

Proof: (i) (Exercise), (ii) and (iii) follow from (i).

Thm [Gale's Evenness Criterion]
$F \subset \mathcal{C}(n,d)$ is a facet of $C(n,d)$ iff
F contains only $\underbrace{\text{odd}}_{\text{"upper facets"}}$ or only $\underbrace{\text{even gaps}}_{\text{"lower facets"}}$:

Def.:
Odd gap: $a \in \mathcal{C}(n,d) \setminus F$ s.t. $\#\{a' \in F : a' > a\}$ odd
Even gap: $a \in \mathcal{C}(n,d) \setminus F$ s.t. $\#\{a' \in F : a' > a\}$ even

Example:



36 not a facet because $\begin{cases} \text{5 is an odd gap} \\ \text{2 is an even gap} \end{cases}$

16 an upper facet because 2,3,4,5 are odd gaps

45 a lower facet because 6, 1,2,3 are even gaps

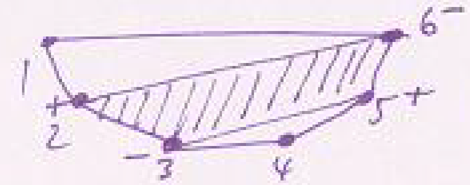
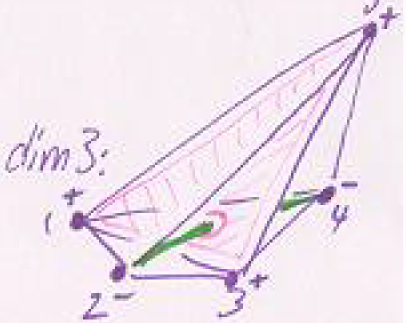
$C(9,5)$:

1	2	3	4	5	6	7	8	9	
*	e	*	*	e	*	*	e	e	✓ lower
*	*	o	o	o	*	*	o	*	✓ upper
*	*	o	*	*	*	e	e	e	no facet.

More facts about $C(n,d)$:

Lemma: The circuits of $C(n,d)$ are alternating, i.e., for $1 \le i_1 < \dots < i_{d+2} \le n$ there is a circuit with

$$C_+ = \{i_1, i_3, \dots\}$$
$$C_- = \{i_2, i_4, \dots\}$$ and vice versa.

Example: dim 1:  dim 2:  dim 3: 

Def.: A d-polytope / point configuration is *cyclic* if it has the same set of circuits as $C(n,d)$ for some labeling of its vertices.

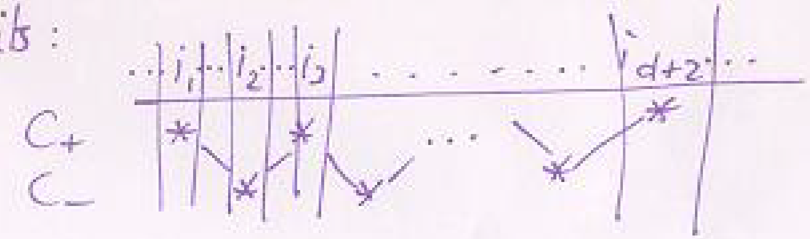
Example:

(i) $\{\nu_d(t_1), \nu_d(t_2), \dots, \nu_d(t_n)\}$ is cyclic for all $t_1 < t_2 < \dots < t_n \in \mathbb{R}$.

(ii) Δ_d is cyclic: same circuits as $C(d+1,d)$.

(iii) A $\left\{\begin{array}{l}\text{full-dim.}\\ \text{balanced}\end{array}\right\}$ circuit is cyclic: same circuits as $C(d+2,d)$
(balanced: $|C_+| - |C_-| \in \{-1,0,1\}$)

Table notation for circuits:



A combinatorial characterization of triangs

<u>THM</u>: $\emptyset \neq T \subseteq \binom{\mathcal{A}}{d+1}$ is a triang of $\mathcal{A}$ = point conf. in $\mathbb{R}^d$ iff

(IP) $\forall \sigma_1, \sigma_2 \in T \not\exists$ circuit C with $C_+ \subseteq \sigma_1$, $C_- \subseteq \sigma_2$.

(UP) Every interior facet of a simplex $\sigma \in T$ lies in at least one other simplex $\sigma' \in T$.

<u>Rem</u>: We know everything about $\left\{ \begin{matrix} \text{circuits} \\ \text{facets} \end{matrix} \right\}$ of $C(n,d)$ $\Rightarrow$ can study triang's more easily.

<u>Example</u>: Check (IP) for σ_1, σ_2 simplices in $C(n,d)$.

<u>$n = 9$, $d = 5$</u>:

		1	2	3	4	5	6	7	8	9
124578	σ_1	*	*		*	*		*	*	
245789	σ_2		*		*	*		*	*	*
123469		*	*	*	*		*			*
234579			*	*	*	*		*		*

zig-zag with 6 elements only $\Rightarrow$ (IP) ✓

7-elem zig-zag $\Rightarrow$ (IP) ↯