Counter-examples to the Hirsch conjecture

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Polyhedra and polytopes

Definition

A (convex) polyhedron P is the intersection of a finite family of affine half-spaces in $\mathbb{R}^d$.

The dimension of P is the dimension of its affine hull.

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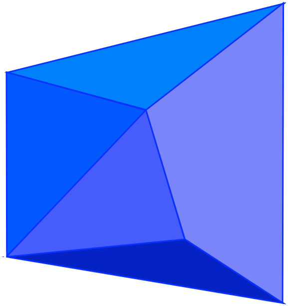
Definition

A (convex) polytope P is the convex hull of a finite set of points in $\mathbb{R}^d$.



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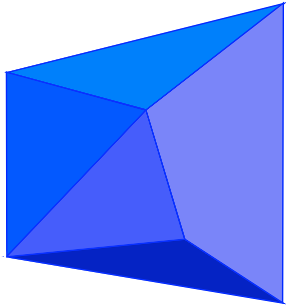
Polyhedra and polytopes

Polytope = bounded polyhedron.

Every polytope is a polyhedron, but not conversely.



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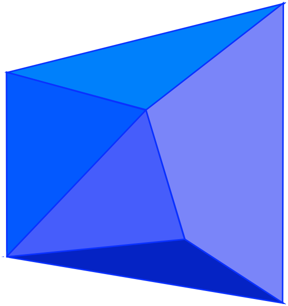

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Faces of P

Let P be a polytope (or polyhedron) and let

$$H = \{x \in \mathbb{R}^d : a_1 x_1 + \cdots a_d x_d \leq a_0\}$$

be an affine half-space.

If $P \subset H$ we say that $\partial H \cap P$ is a face of P.

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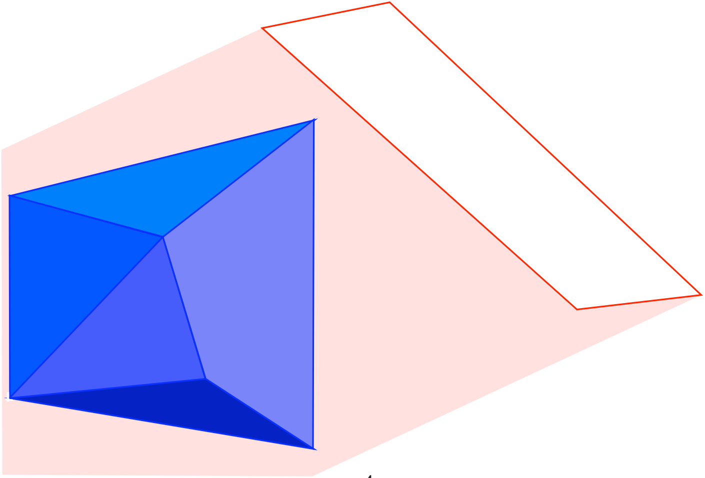
The “empty face” of P.



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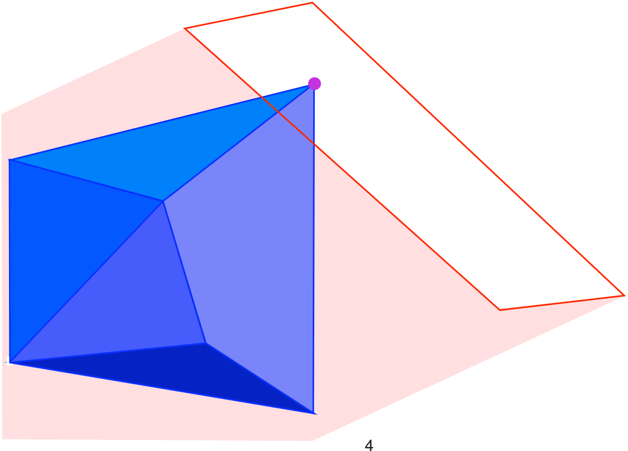
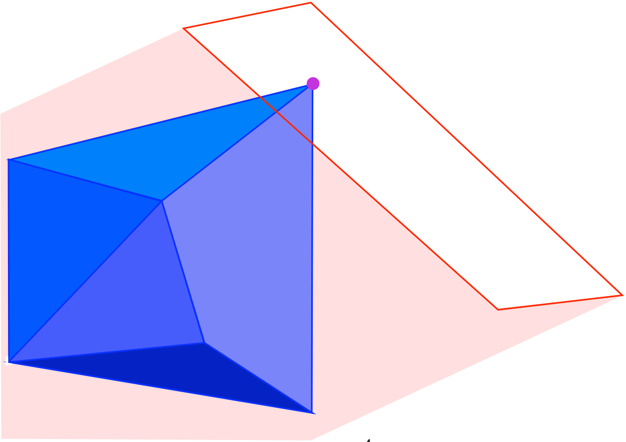


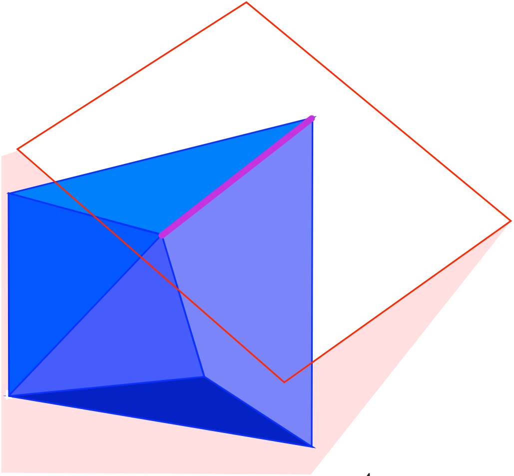
Faces of P

Faces of dimension 0 are called vertices.



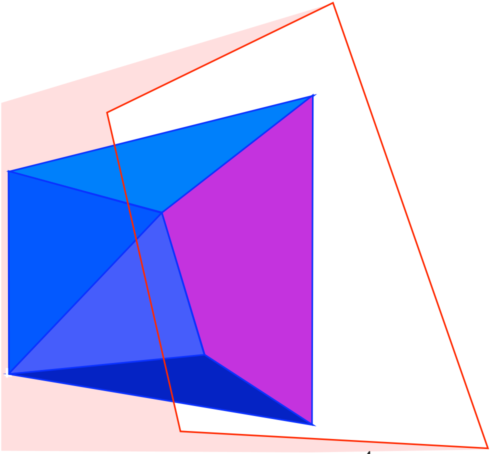
Faces of P

Faces of dimension 1 are called edges.



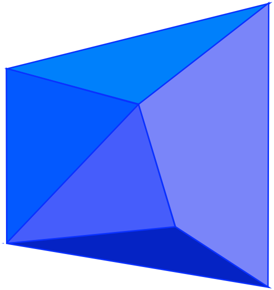
Faces of P

Faces of dimension $d-1$ (codimension 1) are called facets.



The graph of a polytope

Vertices and edges of a polytope P form a graph (finite, undirected)



The distance $d(u, v)$ between vertices u and v is the length (number of edges) of the shortest path from u to v.

For example, $d(u, v) = 2$.

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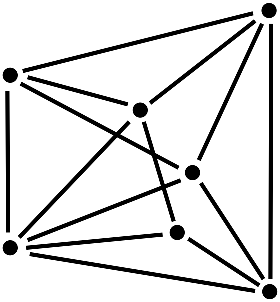

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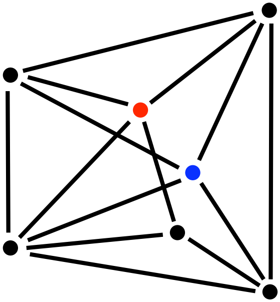

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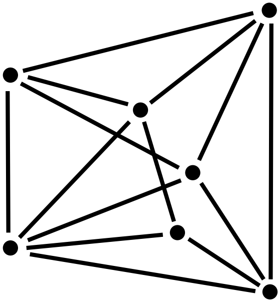

The diameter of $G(P)$ (or of P) is the maximum distance among its vertices:

$$\delta(P) = \max\{d(u,v) : u, v \in V\}.$$

The Hirsch conjecture

Conjecture: Warren M. Hirsch (1957)

For every polytope P with n facets and dimension d,

$$\delta(P) \leq n - d.$$

Fifty three years later...

Theorem (S. 2010+)

There is a 23-dim. polytope with 46 facets and diameter 24.

Corollary (S. 2010+)

There is an infinite family of non-Hirsch polytopes with diameter $\sim (1+\epsilon)n$, even in fixed dimension. (Best so far: $\epsilon = 1/23$).

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A linear program is the problem of maximization / minimization of a linear functional subject to linear inequality constraints. That is:

- Given

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Conection to the Hirsch conjecture

- The set of feasible solutions $P = \{x \in \mathbb{R}^d : Mx \leq b\}$ is a polyhedron P with (at most) n facets.
- The optimal solution (if it exists) is always attained at a vertex.
- The simplex method [Dantzig 1947] solves the linear program by starting at any feasible vertex and moving along the graph of P, in a monotone fashion, until the optimum is attained.
- In particular, the Hirsch conjecture is related to the question of whether the simplex method is a polynomial-time algorithm.

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> The simplex method was chosen one of the “10 algorithms with the greatest influence on the development and practice of science and engineering in the 20th century” in the selection made by the journal *Computing in Science and Engineering* in the year 2000.

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Besides, the polynomial methods for LP known are not *strongly polynomial*. They are polynomial in the “bit model” but not in the “real machine model” [Blum et al. 1989]).

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For every *d*-polytope with *n* facets:

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Given a linear program with *d* variables and *n* restrictions, we consider a random perturbation of the matrix, within a parameter ϵ (normal distribution).

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Proof: Let $H(n, d) = \max\{\delta(P) : P$ is a d-polytope with n facets$\}$. The basic idea is:

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$$d_{P'}(u', v') \geq d_P(u, v).$$

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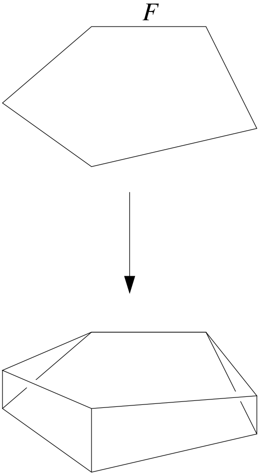
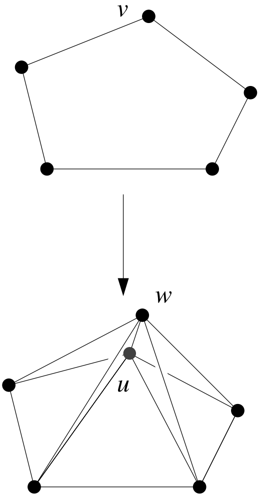
Wedging, a.k.a. one-point-suspension





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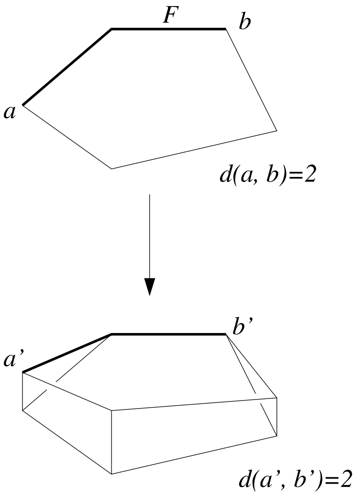

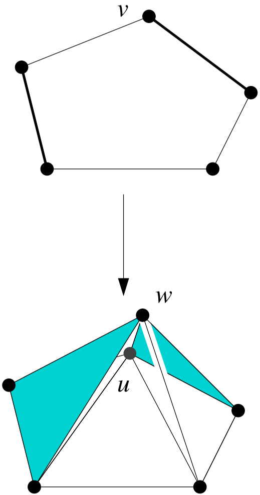

Two ingredients

The construction of our counter-example has two parts:

1. A “strong *d*-step theorem” for spindles/prismatoids.
2. The construction of a prismatoid of dimension 5 and “width” 6.

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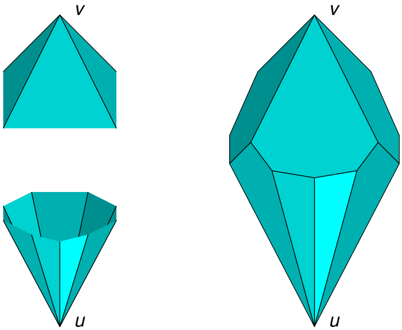
Spindles and prismatoids

Definition

A *spindle* is a polytope P with two distinguished vertices u and v such that every facet contains either u or v.



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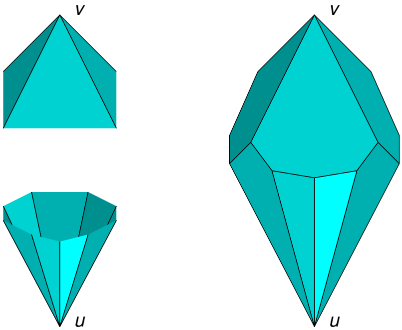

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Spindles and prismatoids

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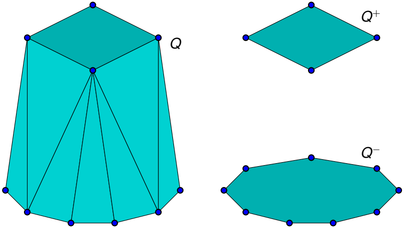

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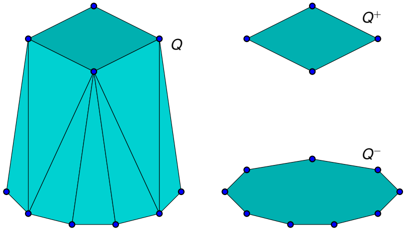

Definition

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The strong d-step Theorem

Theorem (Strong d-step, spindle version)

Let P be a spindle of dimension d, with $n > 2d$ facets, and with length δ.
Then there is another spindle P' of dimension $d+1$, with $n+1$ facets and with length $\delta+1$.

That is: we can increase the dimension, length and number of facets of a spindle, all by one, until $n = 2d$.

Corollary

In particular, if a spindle P has length $> d$ then there is another spindle P' (of dimension $n-d$, with $2n-2d$ facets, and length $\geq \delta + n - 2d > n - d$) that violates the Hirsch conjecture.

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The strong d-step Theorem

Theorem (Strong d-step, prismatoid version)

Let Q be a prismatoid of dimension d, with $n > 2d$ vertices, and with width δ.
Then there is another prismatoid Q' of dimension $d+1$, with $n+1$ vertices and with width $\delta+1$.

That is: we can increase the dimension, width and number of vertices of a prismatoid, all by one, until $n = 2d$.

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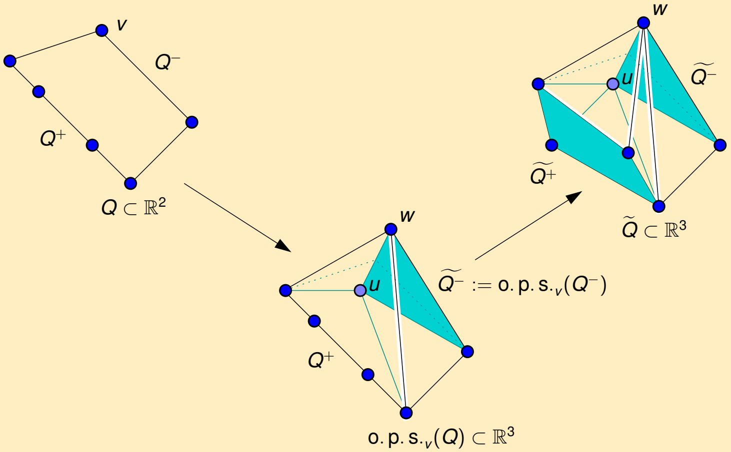
The strong *d*-step Theorem

Proof.



□

The prismatoid

Let Q be the polytope having as vertices the 48 rows of the following matrices:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \left(\begin{array}{c} \pm 18 \\ 0 \\ 0 \\ 0 \\ \pm 15 \\ 0 \\ 0 \\ \pm 10 \end{array}\right. & \begin{array}{c} 0 \\ \pm 18 \\ 0 \\ 0 \\ \pm 15 \\ 0 \\ \pm 10 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ \pm 45 \\ 0 \\ 0 \\ \pm 30 \\ \pm 40 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \pm 45 \\ 0 \\ \pm 30 \\ 0 \\ \pm 40 \end{array} & \left.\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right) \end{array} \qquad \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \left(\begin{array}{c} 0 \\ 0 \\ \pm 45 \\ 0 \\ 0 \\ \pm 30 \\ \pm 40 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 0 \\ 0 \\ \pm 45 \\ 0 \\ \pm 30 \\ 0 \\ \pm 40 \end{array} & \begin{array}{c} 0 \\ \pm 18 \\ 0 \\ 0 \\ \pm 15 \\ 0 \\ \pm 10 \\ 0 \end{array} & \begin{array}{c} \pm 18 \\ 0 \\ 0 \\ 0 \\ \pm 15 \\ 0 \\ 0 \\ \pm 10 \end{array} & \left.\begin{array}{c} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{array}\right) \end{array}$$

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Let Q be the polytope having as vertices the 48 rows of the following matrices:

$$
\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & x_5 \\
\end{array}
\qquad
\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & x_5 \\
\end{array}
$$

$$
\begin{pmatrix}
\pm 18 & 0 & 0 & 0 & 1 \\
0 & \pm 18 & 0 & 0 & 1 \\
0 & 0 & \pm 45 & 0 & 1 \\
0 & 0 & 0 & \pm 45 & 1 \\
\pm 15 & \pm 15 & 0 & 0 & 1 \\
0 & 0 & \pm 30 & \pm 30 & 1 \\
0 & \pm 10 & \pm 40 & 0 & 1 \\
\pm 10 & 0 & 0 & \pm 40 & 1
\end{pmatrix}
\qquad
\begin{pmatrix}
0 & 0 & 0 & \pm 18 & -1 \\
0 & 0 & \pm 18 & 0 & -1 \\
\pm 45 & 0 & 0 & 0 & -1 \\
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The prismatoid

Theorem

The prismatoid Q of the previous slide has width six.

The prismatoid

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Corollary

There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

The prismatoid

Theorem

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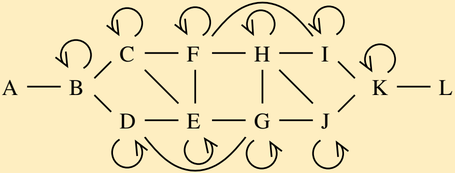
Proof 1 of the Theorem.

It has been verified with `polymake` that the dual graph of Q has the following structure:

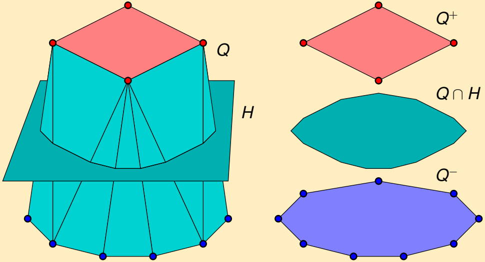


□

The prismatoid

Proof 2 of the Theorem.

Analyzing the combinatorics of a d-prismatoid can be done in a $d-2$-sphere…



…so, the proof is basically 3-dimensional. □

Towards a smaller counter-example

There are two ways in which a smaller non-Hirsch could be obained:

- Find a smaller 5-prismatoid of width > 5 (open), or
- Find a 4-prismatoid of width > 4.

The latter is impossible:

Theorem

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The following prismatoid of dimension 5 has width 6:

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Corollary

There is a 23-polytope with 46 facets violating the Hirsch conjecture.

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Conclusion

- Via glueing and products, the counterexample can be converted into an infinite family that violates the Hirsch conjecture by about 2%.
- This breaks a “psychological barrier”, but for applications it is absolutely irrelevant.

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The end

THANK YOU!