A counter-example to the Hirsch conjecture

arXiv:1006.2814

Francisco Santos
http://personales.unican.es/santosf/Hirsch

Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria, Spain

EUROPT Workshop, Aveiro, July 10, 2010

Polyhedra and polytopes

Definition

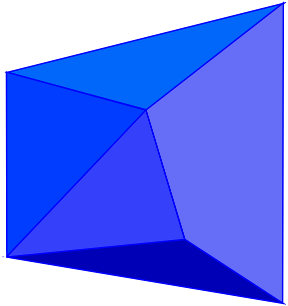
A (convex) polyhedron P is the intersection of a finite family of affine half-spaces in $\mathbb{R}^d$.

The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Definition

A (convex) polytope P is the convex hull of a finite set of points in $\mathbb{R}^d$.

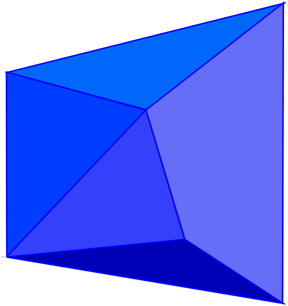


The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

Polytope = bounded polyhedron.

Every polytope is a polyhedron, but not conversely.



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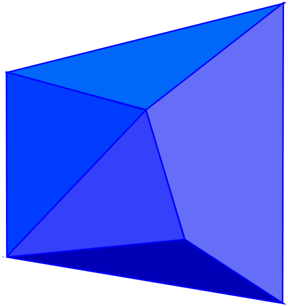

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Faces of P

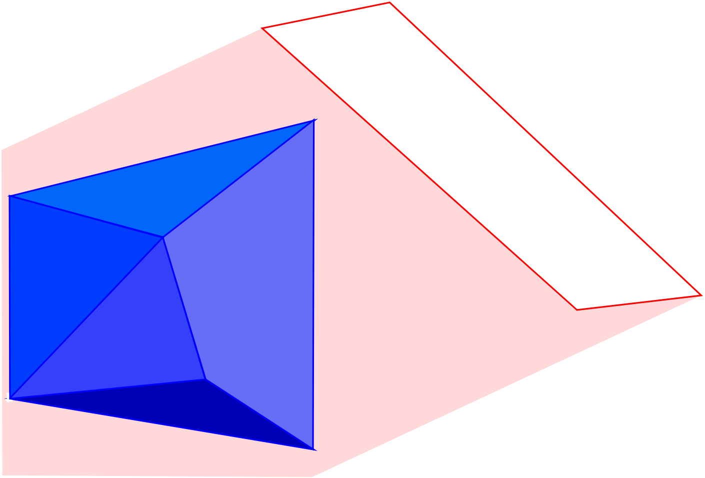
Let P be a polytope (or polyhedron) and let

$$H = \{x \in \mathbb{R}^d : a_1 x_1 + \cdots a_d x_d \leq a_0\}$$

be an affine half-space.

Faces of P

If $P \subset H$ we say that $\partial H \cap P$ is a face of P.



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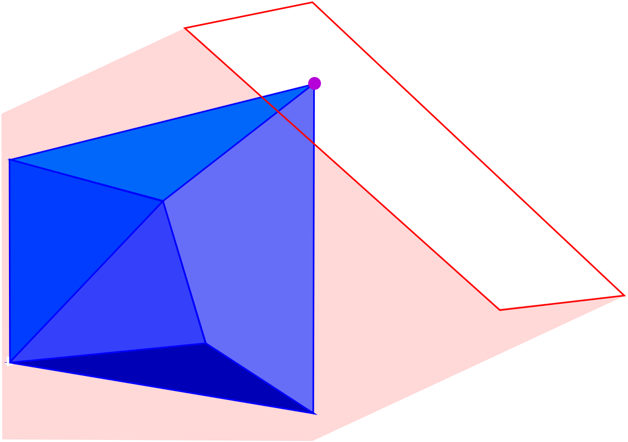
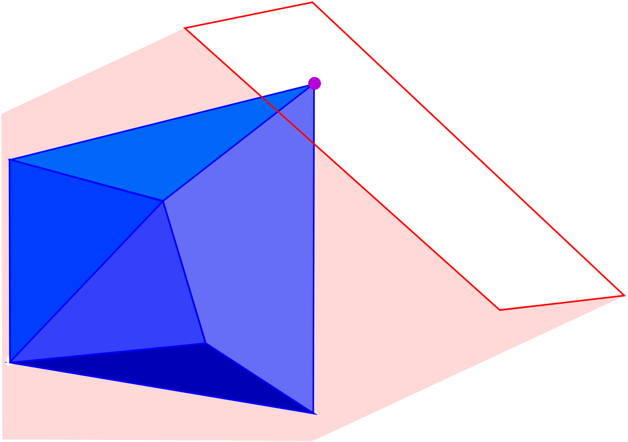


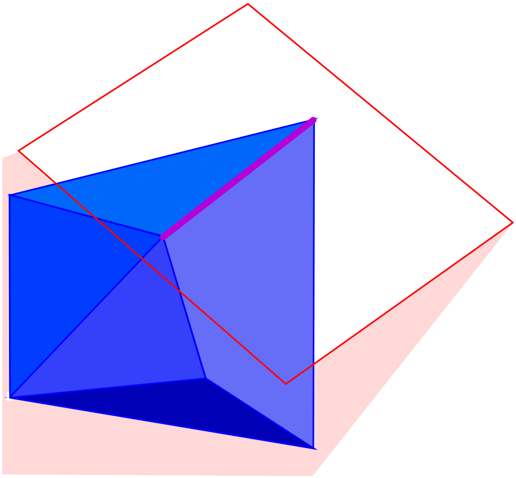
Faces of P

Faces of dimension 0 are called vertices.



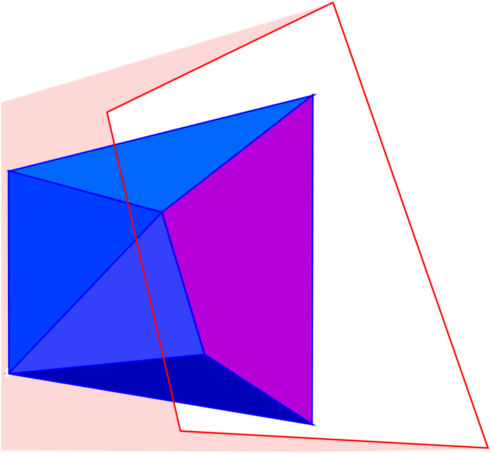
Faces of P

Faces of dimension 1 are called edges.



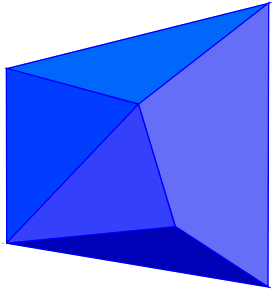
Faces of P

Faces of dimension $d - 1$ (codimension 1) are called facets.



The graph of a polytope

Vertices and edges of a polytope P form a graph (finite, undirected)



The distance $d(u, v)$ between vertices u and v is the length (number of edges) of the shortest path from u to v.

For example, $d(u, v) = 2$.

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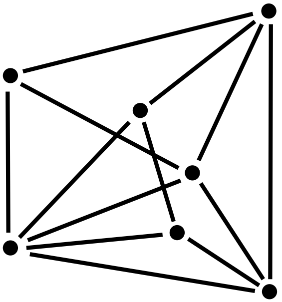

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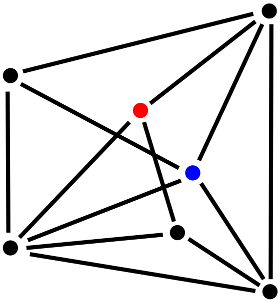

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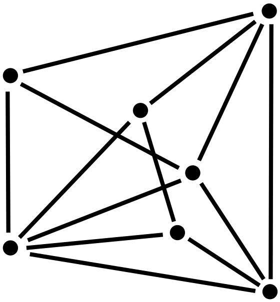

The diameter of $G(P)$ (or of P) is the maximum distance among its vertices:

$$\delta(P) = \max\{d(u, v) : u, v \in V\}.$$

The Hirsch conjecture

Conjecture: Warren M. Hirsch (1957)

For every polytope P with n facets and dimension d,

$$\delta(P) \leq n - d.$$

Fifty three years later...

Theorem (S. 2010+)

There is a 38-dim. polytope with 76 facets and diameter 39.

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There is an infinite family of non-Hirsch polytopes with diameter $\sim (1+\epsilon)n$, even in fixed dimension. (Best so far: $\epsilon = 1/43$).

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Motivation: linear programming

A linear program is the problem of maximization / minimization of a linear functional subject to linear inequality constraints. That is:

- Given

  a matrix M of size $n \times d$,
  a vector $b \in \mathbb{R}^n$
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- Find a $x \in \mathbb{R}^d$ that minimizes $\langle z, x \rangle$
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Motivation: linear programming

"If one would take statistics about which **mathematical problem** *is using up* **most of the computer time in the world***, then (not including database handling problems like sorting and searching) the answer would probably be linear programming."*

(László Lovász, 1980)

Conection to the Hirsch conjecture

- The set of feasible solutions $P = \{x \in \mathbb{R}^d : Mx \leq b\}$ is a polyhedron P with (at most) n facets.
- The optimal solution (if it exists) is always attained at a vertex.
- The simplex method [Dantzig 1947] solves the linear program starting at any feasible vertex and moving along the graph of P, in a monotone fashion, until the optimum is attained.
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- As of today, for every deterministic pivot rule there is an analogue of the Klee-Minty cube, which makes the simplex method take an exponential number of steps:

Importance of $\delta(P)$

The diameter $\delta(P)$ of a polytope is a lower bound for the number of steps of **any** pivot rule in the simplex method.

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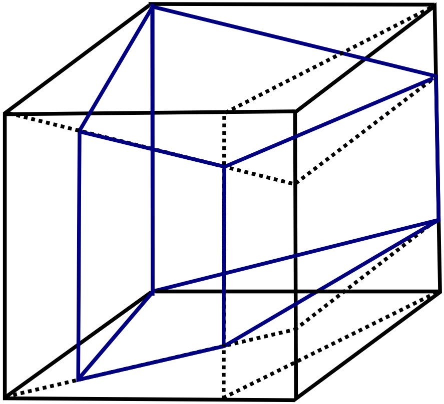

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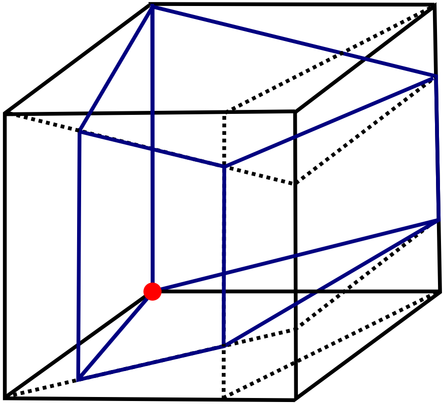

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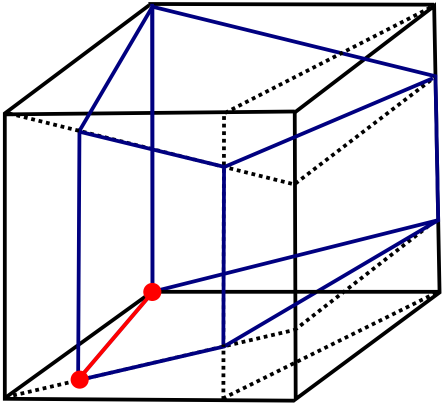

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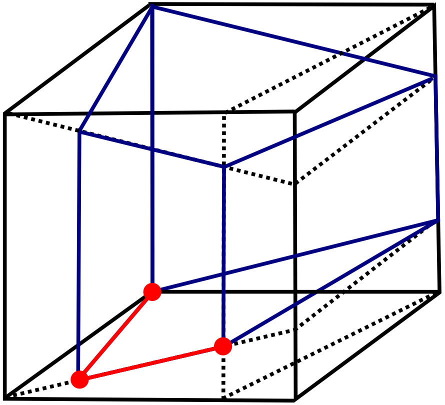

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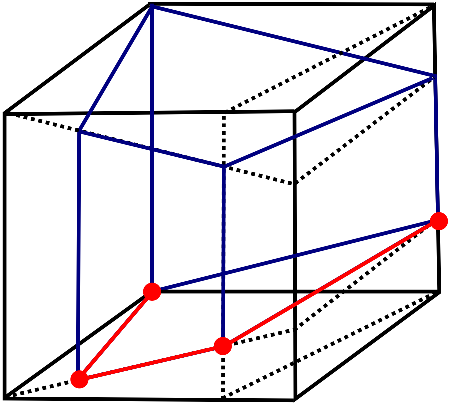

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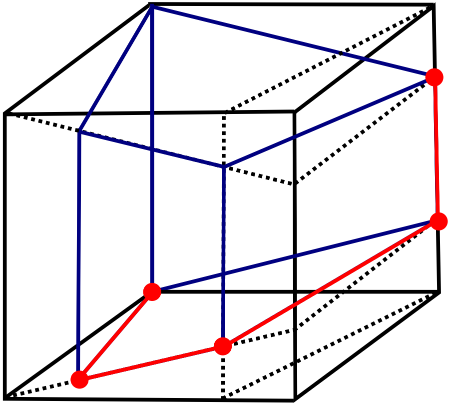

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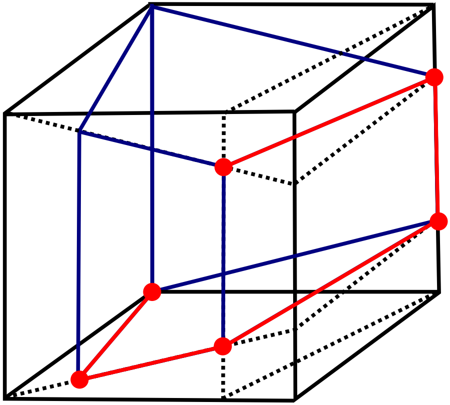

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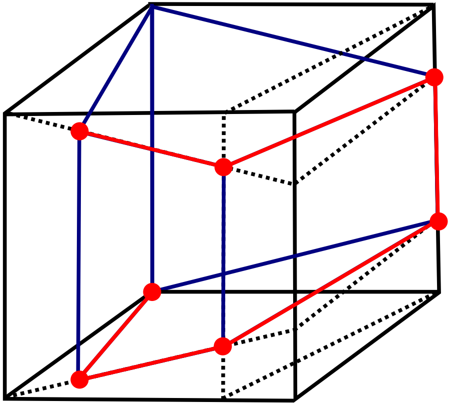

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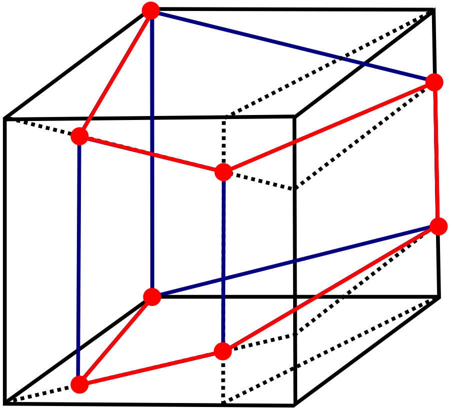

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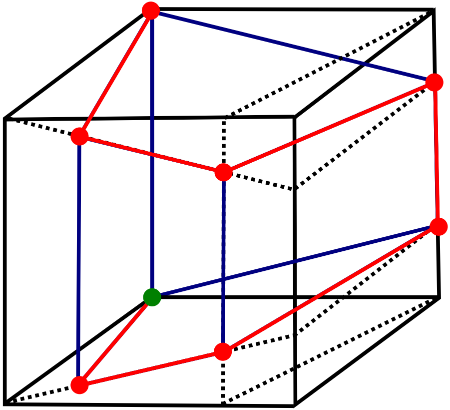

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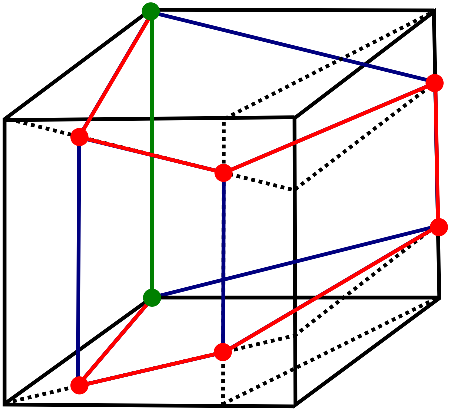

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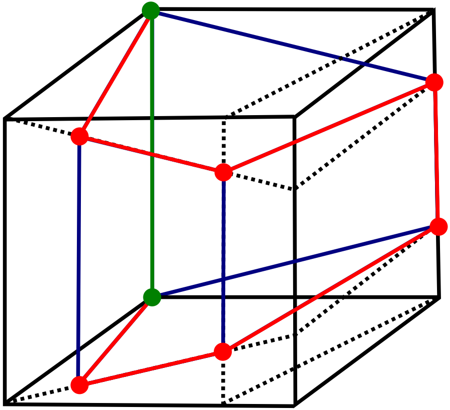

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> The simplex method was chosen one of the “10 algorithms with the greatest influence on the development and practice of science and engineering in the 20th century” in the selection made by the journal *Computing in Science and Engineering* in the year 2000.

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Besides, the methods known are not *strongly polynomial*. They are polynomial in the “bit model” but not in the “real machine model” [Blum et al. 1989]).

Finding strongly polynomial algorithms for linear programming is one of the “mathematical problems for the 21st century” according to [Smale 2000]. A polynomial pivot rule would solve this problem in the affirmative.

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Some known cases

Hirsch conjecture holds for

- $d \leq 3$: [Klee 1966].
- $n-d \leq 6$: [Klee-Walkup, 1967] [Bremner-Schewe, 2008]
- $H(9,4) = H(10,4) = 5$ [Klee-Walkup, 1967]
  $H(11,4) = 6$ [Schuchert, 1995],
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- 0-1 polytopes [Naddef 1989]
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- Polynomial bound for network flow polytopes [Goldfarb 1992, Orlin 1997]

Some known cases

Hirsch conjecture holds for

- $d \leq 3$: [Klee 1966].
- $n - d \leq 6$: [Klee-Walkup, 1967] [Bremner-Schewe, 2008]
- $H(9,4) = H(10,4) = 5$ [Klee-Walkup, 1967]
  $H(11,4) = 6$ [Schuchert, 1995],
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Theorem [Kalai-Kleitman 1992]

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Why is $n - d$ a "reasonable" bound?

- It holds with equality in simplices ($n = d + 1$, $\delta = 1$) and cubes ($n = 2d$, $\delta = d$).
- If P and Q satisfy it, then so does $P \times Q$: $\delta(P \times Q) = \delta(P) + \delta(Q)$. In particular:

  For every $n \leq 2d$, there are polytopes in which the bound is tight (products of simplices).
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Proof: Let $H(n,d) = \max\{\delta(P) : P \text{ is a } d\text{-polytope with } n \text{ facets}\}$. The basic idea is:

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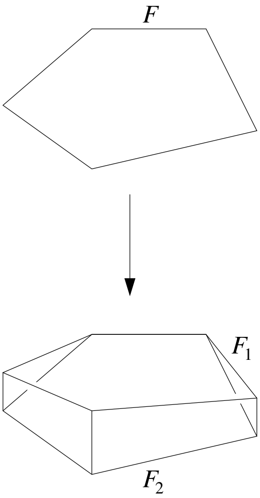

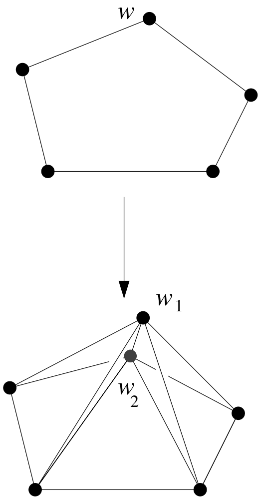

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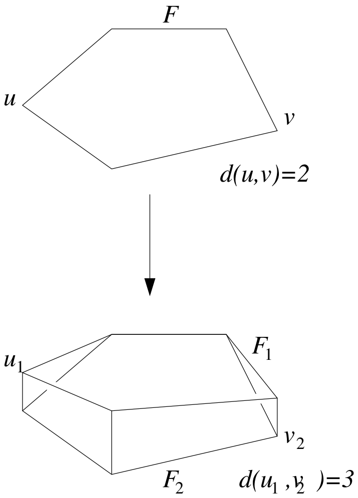

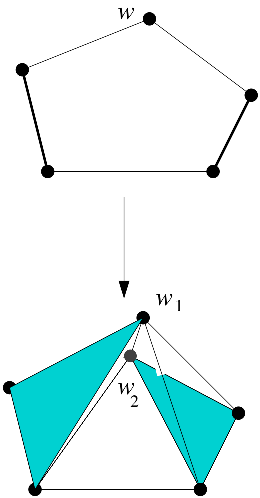

Two ingredients

The construction of our counter-example has two parts:

1. A “generalized d-step theorem” for spindles/prismatoids.
2. The construction of a prismatoid of dimension 5 and “width” 6.

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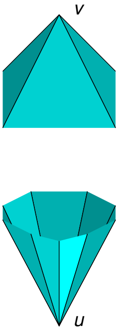
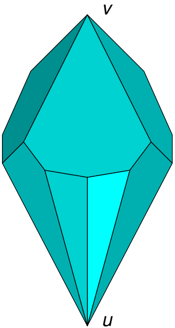
Spindles and prismatoids

Definition

A *spindle* is a polytope P with two distinguished vertices u and v such that every facet contains either u or v.





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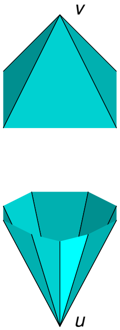

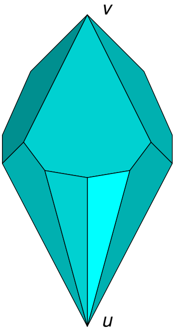

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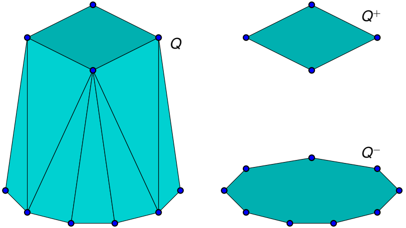

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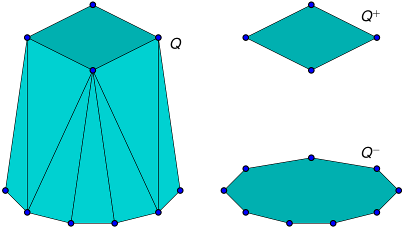

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Theorem (Generalized d-step, spindle version)

Let P be a spindle of dimension d, with $n > 2d$ facets, and with length δ.
Then there is another spindle P' of dimension $d+1$, with $n+1$ facets and with length $\delta+1$.

That is: we can increase the dimension, length and number of facets of a spindle, all by one, until $n = 2d$.

Corollary

In particular, if a spindle P has length $> d$ then there is another spindle P' (of dimension $n-d$, with $2n-2d$ facets, and length $\geq \delta + n - 2d > n - d$) that violates the Hirsch conjecture.

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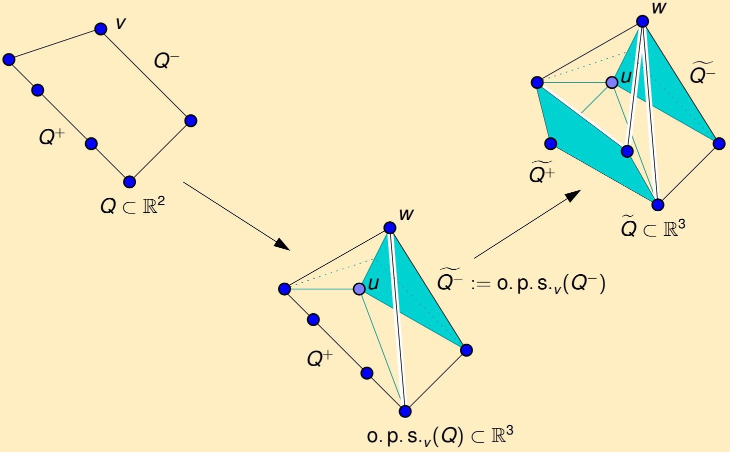
Proof.



□

The prismatoid

$$
Q := \operatorname{conv}\left\{
\begin{array}{cc}
 & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\begin{array}{c} 1^+ \\ 2^+ \\ 3^+ \\ 4^+ \\ 5^+ \\ 6^+ \\ 7^+ \\ 8^+ \\ 9^+ \\ 10^+ \\ 11^+ \\ 12^+ \\ 13^+ \\ 14^+ \\ 15^+ \\ 16^+ \\ 17^+ \\ 18^+ \\ 19^+ \\ 20^+ \\ 21^+ \\ 22^+ \\ 23^+ \\ 24^+ \end{array} &
\begin{pmatrix}
18 & 0 & 0 & 0 & 1 \\
-18 & 0 & 0 & 0 & 1 \\
0 & 18 & 0 & 0 & 1 \\
0 & -18 & 0 & 0 & 1 \\
0 & 0 & 45 & 0 & 1 \\
0 & 0 & -45 & 0 & 1 \\
0 & 0 & 0 & 45 & 1 \\
0 & 0 & 0 & -45 & 1 \\
15 & 15 & 0 & 0 & 1 \\
-15 & 15 & 0 & 0 & 1 \\
15 & -15 & 0 & 0 & 1 \\
-15 & -15 & 0 & 0 & 1 \\
0 & 0 & 30 & 30 & 1 \\
0 & 0 & -30 & 30 & 1 \\
0 & 0 & 30 & -30 & 1 \\
0 & 0 & -30 & -30 & 1 \\
0 & 10 & 40 & 0 & 1 \\
0 & -10 & 40 & 0 & 1 \\
0 & 10 & -40 & 0 & 1 \\
0 & -10 & -40 & 0 & 1 \\
10 & 0 & 0 & 40 & 1 \\
-10 & 0 & 0 & 40 & 1 \\
10 & 0 & 0 & -40 & 1 \\
-10 & 0 & 0 & -40 & 1
\end{pmatrix}
\end{array}
\quad
\begin{array}{cc}
 & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\begin{array}{c} 1^- \\ 2^- \\ 3^- \\ 4^- \\ 5^- \\ 6^- \\ 7^- \\ 8^- \\ 9^- \\ 10^- \\ 11^- \\ 12^- \\ 13^- \\ 14^- \\ 15^- \\ 16^- \\ 17^- \\ 18^- \\ 19^- \\ 20^- \\ 21^- \\ 22^- \\ 23^- \\ 24^- \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 18 & -1 \\
0 & 0 & 0 & -18 & -1 \\
0 & 0 & 18 & 0 & -1 \\
0 & 0 & -18 & 0 & -1 \\
45 & 0 & 0 & 0 & -1 \\
-45 & 0 & 0 & 0 & -1 \\
0 & 45 & 0 & 0 & -1 \\
0 & -45 & 0 & 0 & -1 \\
0 & 0 & 15 & 15 & -1 \\
0 & 0 & 15 & -15 & -1 \\
0 & 0 & -15 & 15 & -1 \\
0 & 0 & -15 & -15 & -1 \\
30 & 30 & 0 & 0 & -1 \\
-30 & 30 & 0 & 0 & -1 \\
30 & -30 & 0 & 0 & -1 \\
-30 & -30 & 0 & 0 & -1 \\
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0 & -40 & 0 & -10 & -1
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\end{array}
\right\}
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The prismatoid

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The prismatoid Q of the previous slide has width six.

The prismatoid

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Corollary

There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

The prismatoid

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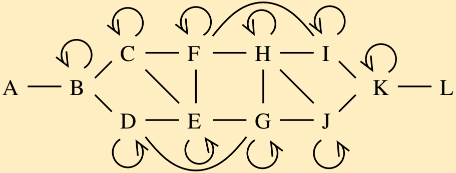
Proof 1 of the Theorem.

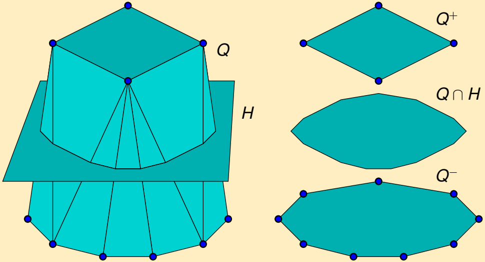
It has been verified with `polymake` that the dual graph of Q has the following structure:



The prismatoid

Proof 2 of the Theorem.

Analyzing the combinatorics of a d-prismatoid can be done in a $d - 2$-sphere…



…so, the proof is basically 3-dimensional. □

Towards a smaller counter-example

There are two ways in which a smaller non-Hirsch could be obained:

- Find a smaller 5-prismatoid of width > 5, or
- Find a 4-prismatoid of width > 4.

The latter is equivalent to

Open Question

Find two (geodesic, polytopal) maps in the 2-sphere such that, when you overlap the two, there is no way of going from one to the other in two steps.

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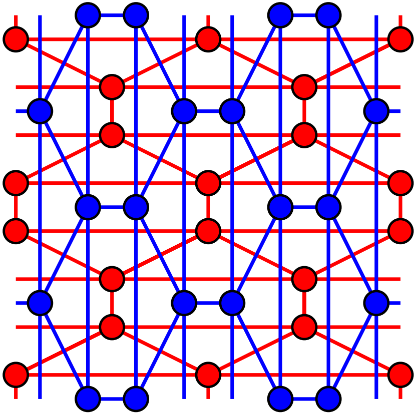
Fact: if the latter does not exist then the reason is “global” and not “local”. Periodic examples in the plane do exist:



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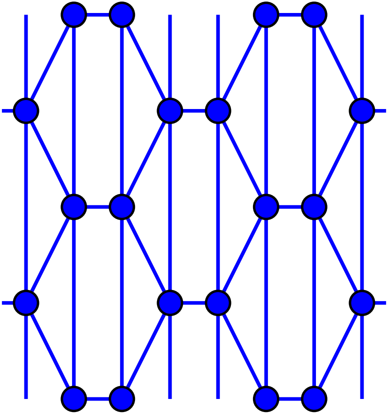

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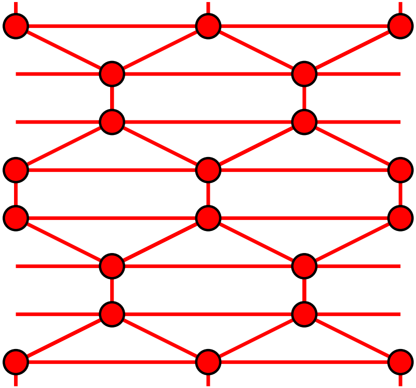

Conclusion

- Via glueing and products, the counterexample can be converted into an infinite family that violates the Hirsch conjecture by about 2%.
- This breaks a "psychological barrier", but for applications it is absolutely irrelevant.

> *Finding a counterexample will be merely a small first step in the line of investigation related to the conjecture.*
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The end

T H A N K   Y O U !