Hirsch Wars Episode II
Attack of the Prismatoids

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A question in topological graph theory

Let G^+ and G^- be a *transversal pair of maps* in the sphere S^2: two graphs (the blue graph and the red graph) embedded in the sphere. Assume that they meet a finite number of times, always transversally, and not twice in the same pair of vertices.

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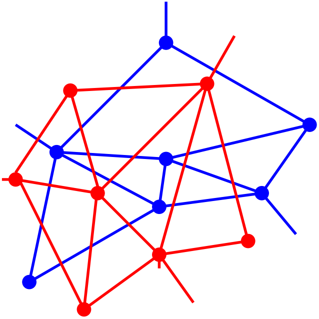

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Merge them into a single graph H (the superposition of the two; their common refinement) that now has blue vertices, red vertices and mixed vertices (former crossings).

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Is there always a path of length two from some blue vertex to some red vertex?

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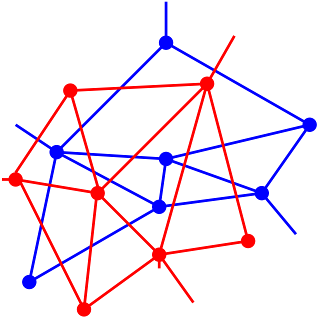

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Theorem (S.-Stephen-Thomas, 2011)

Yes, in every transversal pair of maps in the sphere there is a path of length two from some blue vertex to some red vertex.

Proof: We call an edge of the common refinement of G^+ and G^- *"terminal"* if it is adjacent to a vertex of G^+ or G^-.

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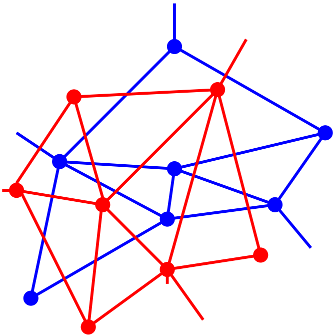

A pair of maps in a surface

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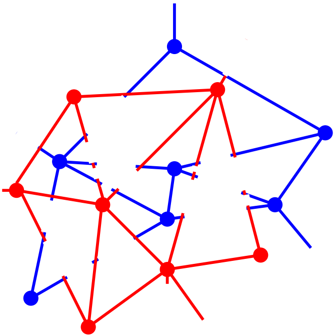

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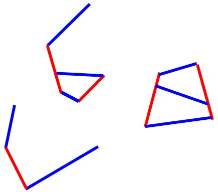

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Assume, to seek a contradiction, that a certain transversal pair of maps (G^+, G^-) in the sphere S^2 does not have a terminal red edge intersect a terminal blue edge.

Then, in the non-terminal part of H every vertex has degree three (a "T" vertex) or four (a crossing).

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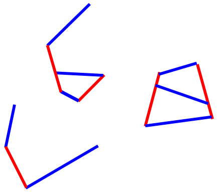

Vertices of degree <3 in the non-terminal graph

$\Updownarrow$

Paths of length two in H

A proof in topological graph theory

Definition

A *zig-zag, color-alternating path* is a path of non-terminal edges such that whenever two consecutive edges have different colors, the path turns right from red to blue and it turns left from blue to red. A *zig-zag, color-alternating loop* is a cycle in which that happens except perhaps at the base point.

Remark 1

Every non-terminal segment can be continued to a zig-zag, color-alternating path until the path crosses itself. At that point it produces a zig-zag, color-alternating loop.

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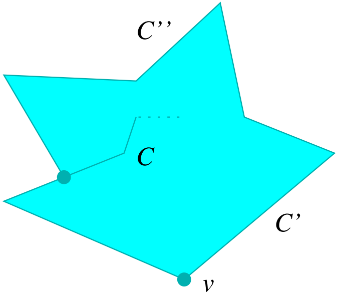
Choose an arbitrary point of S^2 to be “infinity”.

Remark 2

If a zig-zag, color-alternating loop is minimal (i. e., the region it bounds contains no other such loop) then there is no other edge in its interior.



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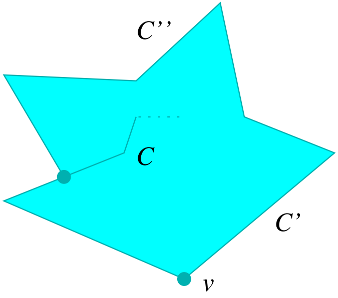

Otherwise, an interior edge can be continued to a zig-zag path, and eventually leads to a smaller alternating loop.

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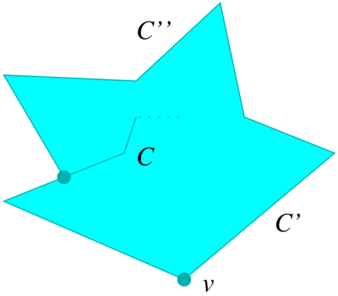

This gives a contradiction, because it implies that the boundary of some face of our pairs of maps is a zig-zag alternating loop, and a zig-zag alternating loop must contain "reflex vertices". □

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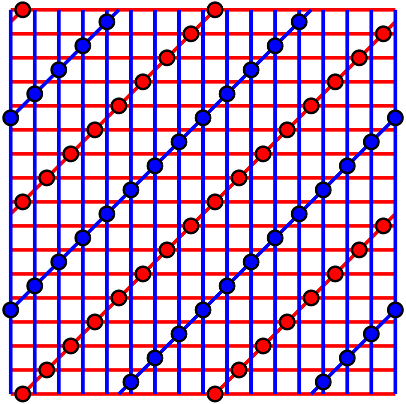

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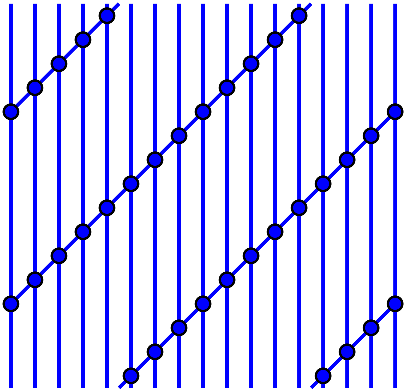

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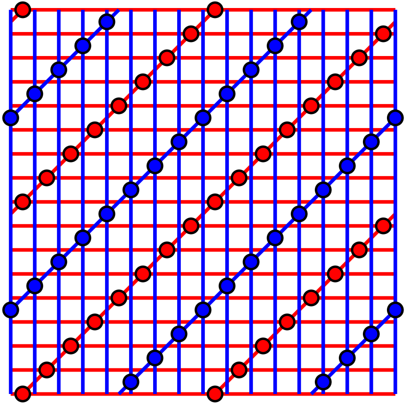

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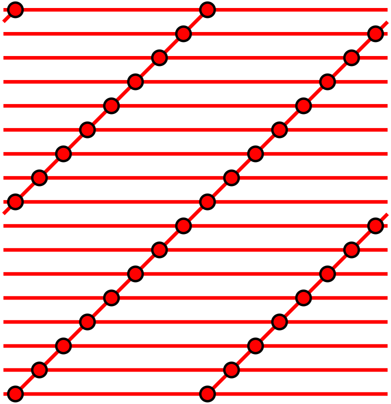

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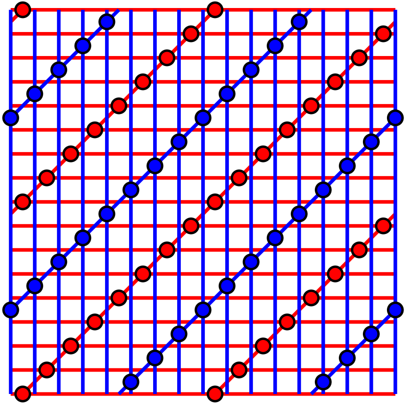

Attack of the prismatoids

The construction of counter-examples to the Hirsch conjecture has two ingredients:

1. A strong d-step theorem for spindles/prismatoids.
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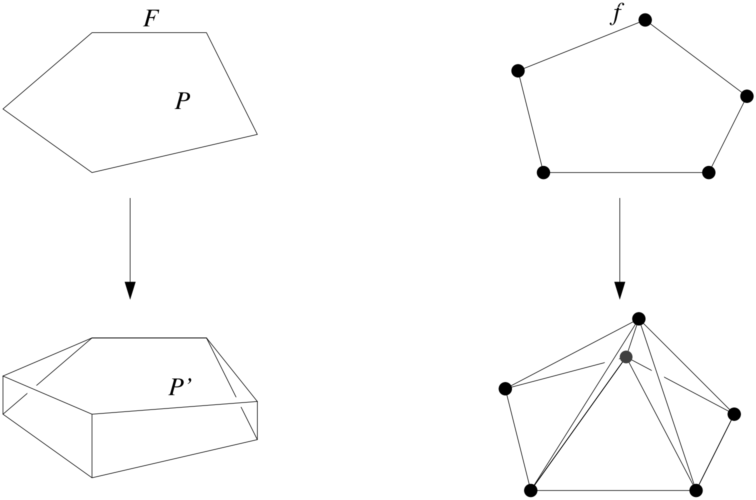

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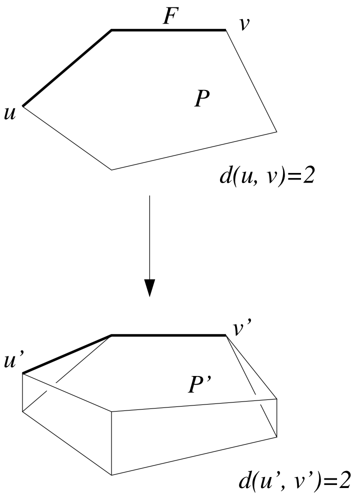

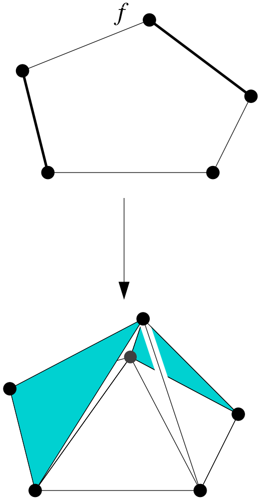

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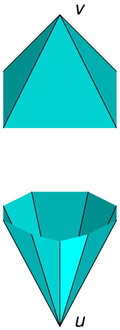
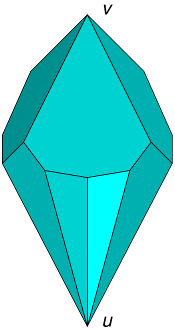
For every d-spindle P with n facets and length λ there is a $d+1$-spindle with one more facet and length $\lambda+1$.

Spindles

Definition

A *spindle* is a polytope P with two distinguished vertices u and v such that every facet contains either u or v (but not both).





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The *length* of a spindle is the graph distance from u to v.

Exercise

3-spindles have length ≤ 3.

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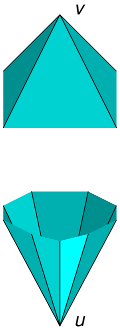

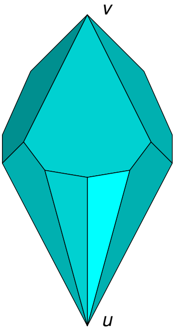

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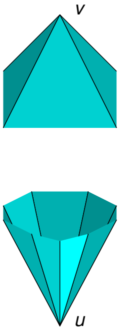

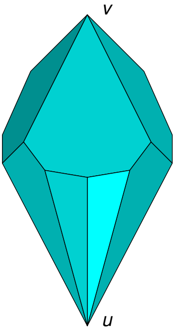

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Theorem (Strong d-step theorem for spindles)

Let P be a spindle of dimension d, with $n > 2d$ facets and length λ. Then there is another spindle P' of dimension $d+1$, with $n+1$ facets and length $\lambda+1$.

That is: we can increase the dimension, length and number of facets of a spindle, all by one, until $n = 2d$.

Corollary

In particular, if a spindle P has length $> d$ then there is another spindle P' (of dimension $n-d$, with $2n-2d$ facets, and length $\geq \lambda + n - 2d > n-d$) that violates the Hirsch conjecture.

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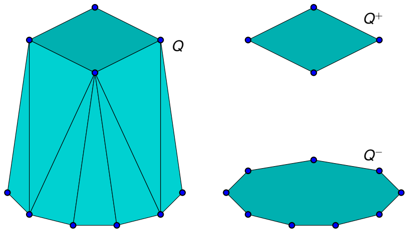

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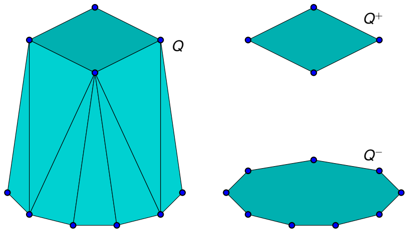

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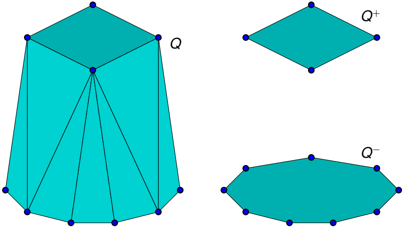

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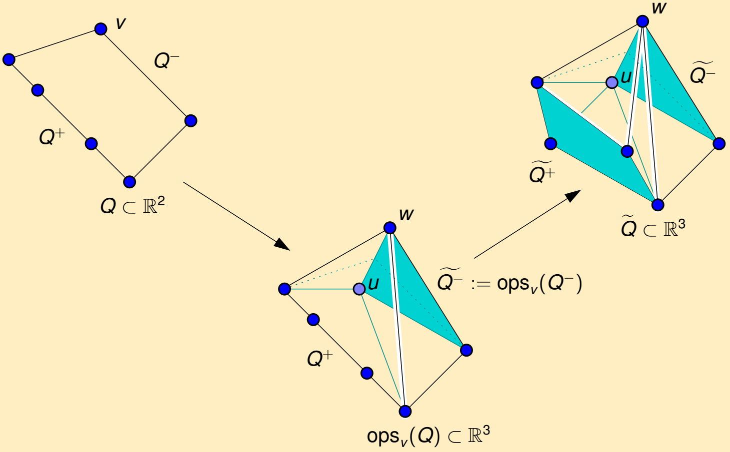
d-step theorem for prismatoids

Proof.



□

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So, to disprove the Hirsch Conjecture we only need to find a prismatoid of dimension d and width larger than d. *Its number of vertices and facets is irrelevant...*

Question

Do they exist?

- 3-prismatoids have width at most 3 (exercise).
- 4-prismatoids have width at most 4 [S.-Stephen-Thomas, 2011].
- 5-prismatoids of width 6 exist [S., 2010] with 25 vertices [Matschke-S.-Weibel 2011].
- 5-prismatoids of arbitrarily large width exist [Matschke-S.-Weibel 2011].

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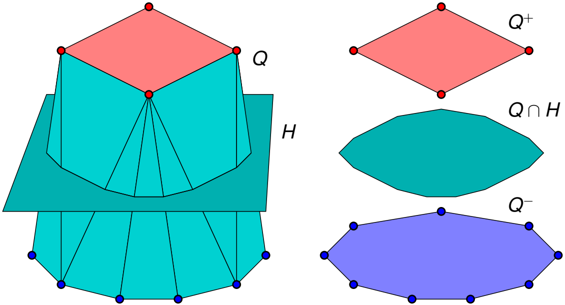
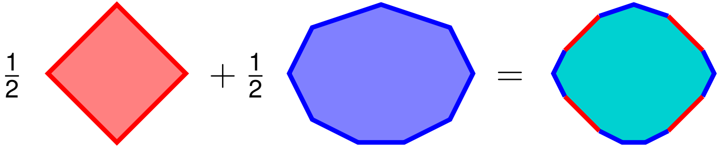
Combinatorics of prismatoids

Analyzing the combinatorics of a d-prismatoid Q can be done via an intermediate slice ...



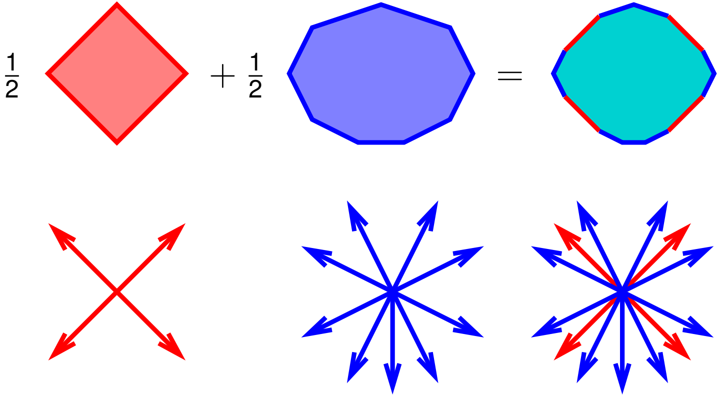
Combinatorics of prismatoids

…which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-.



Combinatorics of prismatoids

...which equals the Minkowski sum $Q^+ + Q^-$ of the two bases Q^+ and Q^-. The normal fan of $Q^+ + Q^-$ equals the "superposition" of those of Q^+ and Q^-.



Combinatorics of prismatoids

So: the combinatorics of Q follows from the superposition of the normal fans of Q^+ and Q^-.

Remark

The normal fan of a $d-1$-polytope can be thought of as a (geodesic, polytopal) cell decomposition ("map") of the $d-2$-sphere.

Theorem

Let Q be a d-prismatoid with bases Q^+ and Q^- and let G^+ and G^- be the corresponding maps in the $(d-2)$-sphere (central projection of the normal fans of Q^+ and Q^-). Then, the width of Q equals 2 plus the minimum number of steps needed to go from a vertex of G^+ to a vertex of G^- in the (graph of) the superposition of the two maps.

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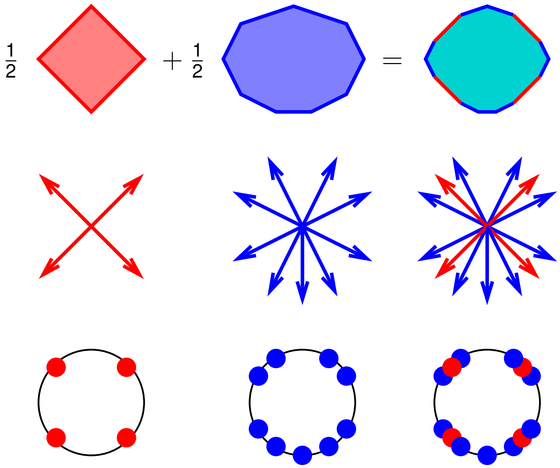

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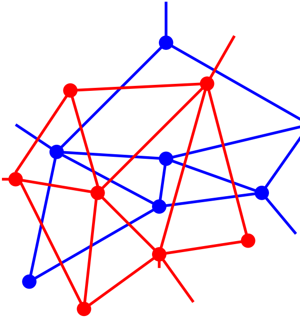

4-prismatoid of width > 4

$\Updownarrow$

pair of (geodesic, polytopal) maps in S^2 so that two steps do not let you go from a blue vertex to a red vertex.

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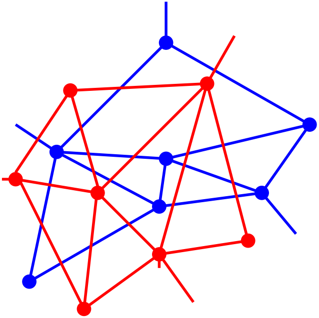

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As we know:

Theorem (S.-Stephen-Thomas, 2011)

In every transversal pair of maps in the sphere there is a path of length two from some blue vertex to some red vertex.

That is to say:

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Every prismatoid of dimension 4 has width four.

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Remember that Klee and Walkup, in 1967, disproved the Hirsch conjecture:

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There is an unbounded 4-polyhedron with 8 facets and diameter 5.

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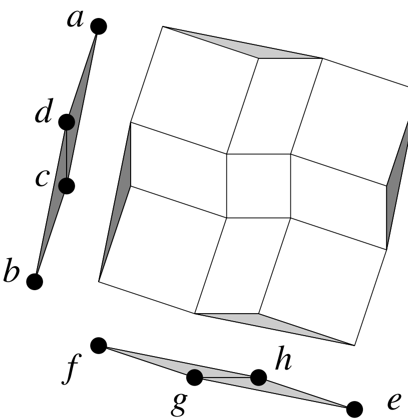

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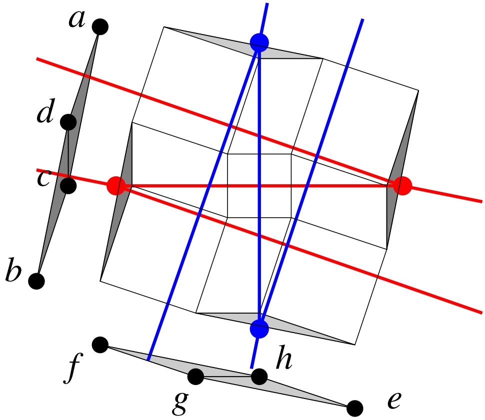

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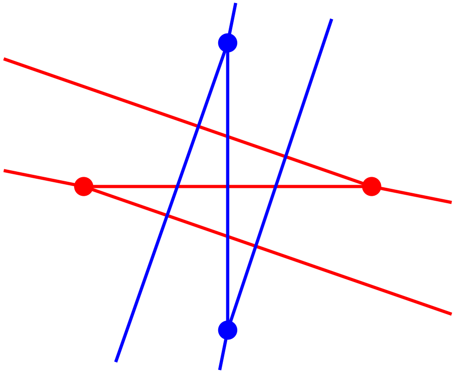
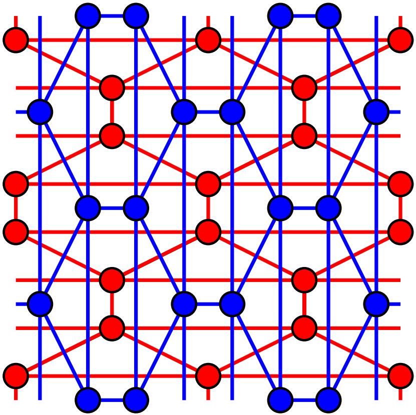


A 4-dimensional prismatoid of width > 4?

Replicating the basic structure of the Klee-Walkup polytope we can get a "non-Hirsch" pair of (infinite) maps in the plane (or a finite one in the torus):



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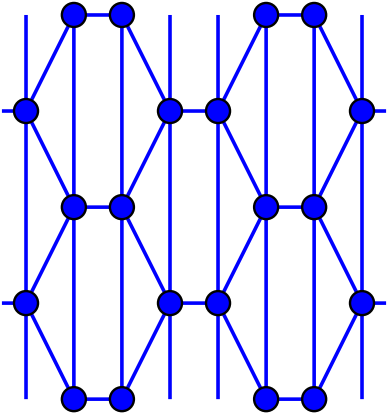

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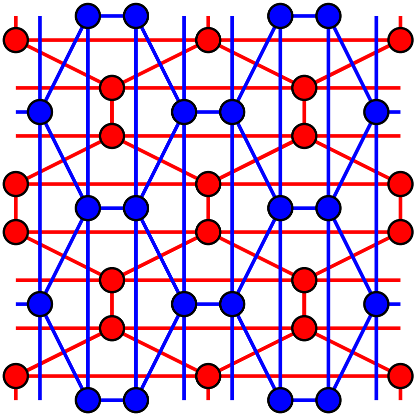

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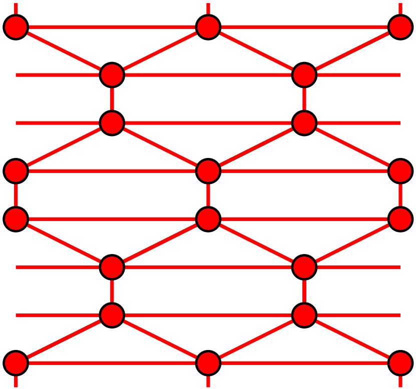

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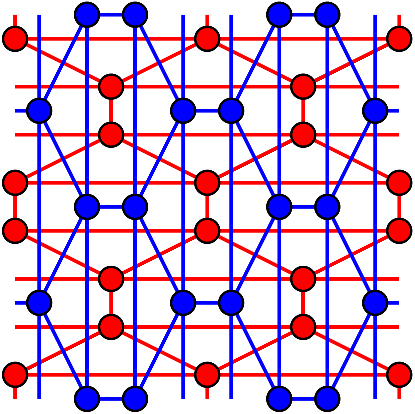

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But, in dimension 5 (that is, with maps in the 3-sphere) we have room enough to construct "non-Hirsch pairs of maps":

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The following prismatoid Q, of dimension 5 and with 48 vertices, has width six.

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There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

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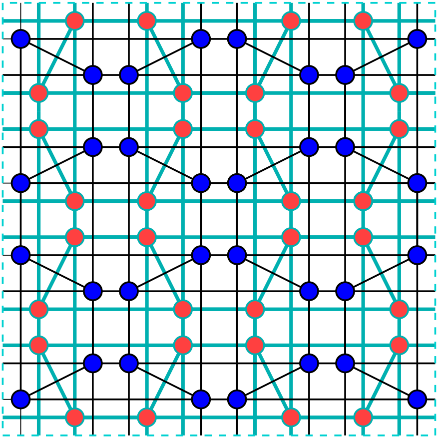

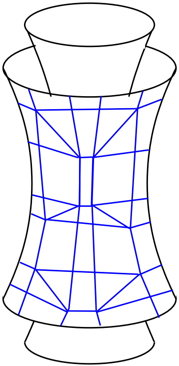

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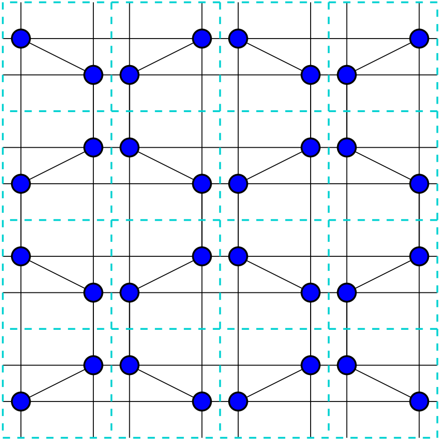

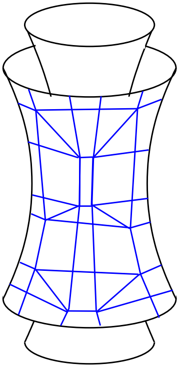

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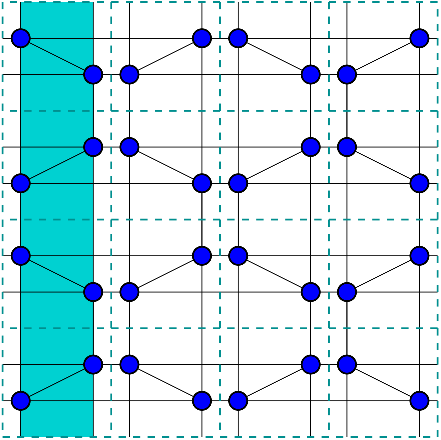

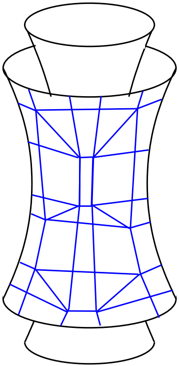

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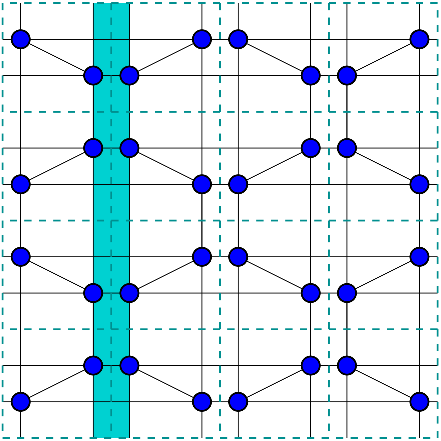

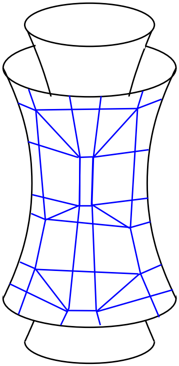

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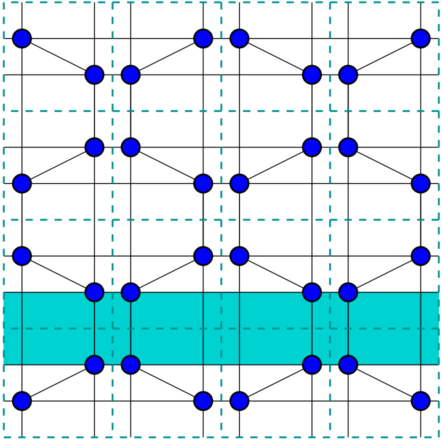

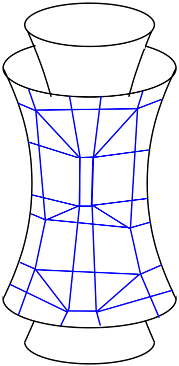

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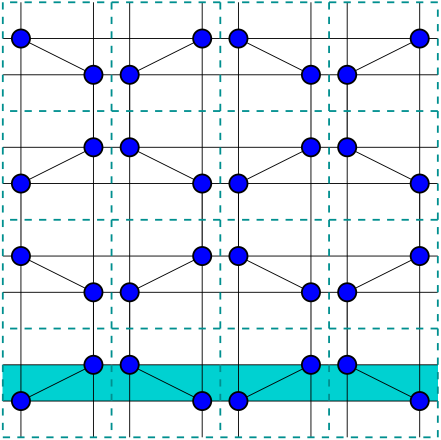

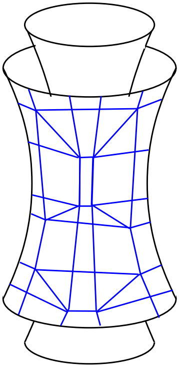

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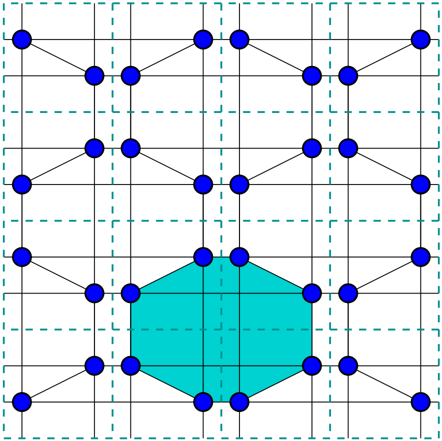
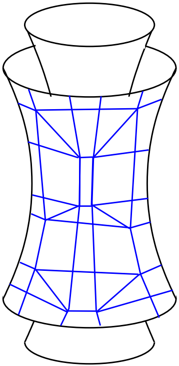




A 5-prismatoid of width > 5

Proof 1.

It has been verified computationally that the dual graph of Q (modulo symmetry) has the following structure:

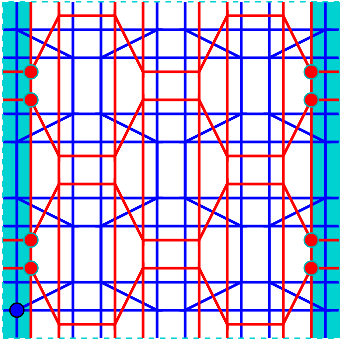
$$
\begin{tikzcd}
 & & C \arrow[rr, dash] \arrow[dr, dash] & & F \arrow[rr, dash] \arrow[dr, dash] & & I \arrow[dr, dash] & & \\
A \arrow[r, dash] & B \arrow[ur, dash] \arrow[dr, dash] & & E \arrow[ur, dash] \arrow[dr, dash] & & H \arrow[ur, dash] \arrow[dr, dash] & & K \arrow[r, dash] & L \\
 & & D \arrow[rr, dash] \arrow[ur, dash] & & G \arrow[rr, dash] \arrow[ur, dash] & & J \arrow[ur, dash] & &
\end{tikzcd}
$$

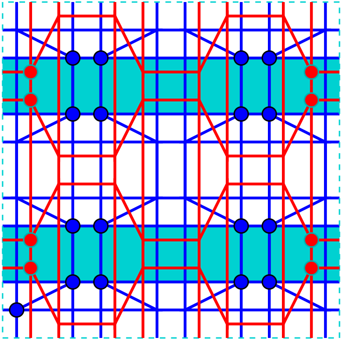
$\square$

A 5-prismatoid of width > 5

Proof 2.

Show that there are no blue vertex a and red vertex b such that a is a vertex of the blue cell containing b and b is a vertex of the red cell containing a. $\square$





Smaller 5-prismatoids of width > 5

With the same ideas

Theorem

The following 5-prismatoid with 28 vertices (and 274 facets) has width 6.

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\end{array}
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0 & 0 & \pm 30 & 0 & 1 \\
0 & 0 & 0 & \pm 30 & 1 \\
0 & \pm 5 & 0 & \pm 25 & 1 \\
0 & 0 & \pm 18 & \pm 18 & 1
\end{pmatrix}
\quad
\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & x_5 \\
\end{array}
\begin{pmatrix}
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Corollary

There is a non-Hirsch polytope of dimension 23 with 46 facets.

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And with some more work:

Theorem (Matschke-Santos-Weibel, 2011)

There is a 5-prismatoid with 25 vertices and of width 6.

Corollary

There is a non-Hirsch polytope of dimension 20 with 40 facets.

This one has been explicitly computed. It has 36,442 vertices, and diameter 21.

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poly20dim21.ext

```
V-representation
begin
 40 21 rational
1 1 0 0 21 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0
1 1 0 0 16 -8 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0
1 1 0 0 0 -25 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0
1 1 0 0 -16 -10 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1
1 1 0 0 -21 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
1 1 0 0 -20 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 0 0 0 39 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 0 0 19 7 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 3/100 -1/50 0 -23 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 -3/100 -1/50 0 37 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 -3/2000 7/2000 0 388/10 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 100000000000 100000000000 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 -10000000 0 0 0 0 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 -10000000 0 0 0 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 -10000000 0 0 0 0 0 1 0 0 0 0 0 0
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1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 100000000000 -100000000000 1 0 0 0 0 0 0
1 -1 30 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0
1 -1 4 -15 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0
1 -1 0 -33/2 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0
1 -1 -1 -16 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0
1 -1 -55/2 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0
1 -1 -17 18 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
1 -1 0 38 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 22 17 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 -10 0 1/5 -1/5 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 2999/100 0 -3/25 -1/5 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0
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1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 -100000 0 0 0 0 0 0
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1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 10000000 100000000 -100000000 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 10000000 100000000 100000000 -1000000000
end
allbases
printcobasis
```

Asymptotic width in dimension five

Theorem

There are 5-dimensional prismatoids with n vertices and width $\Omega(\sqrt{n})$.

Sketch of proof

Start with the "simple, yet more drastic" pair of maps in the torus.

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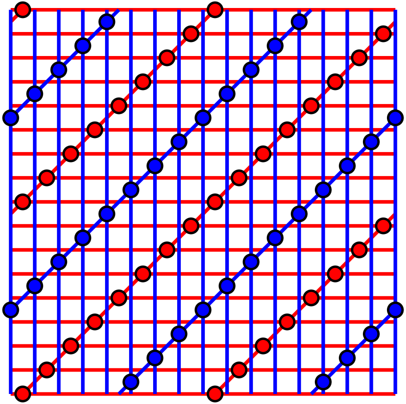
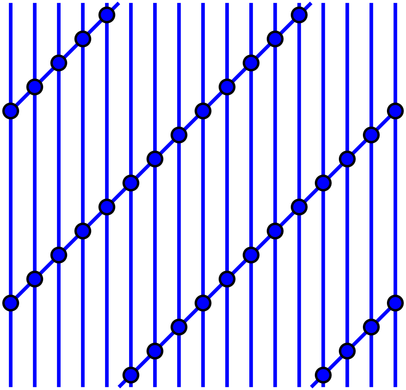
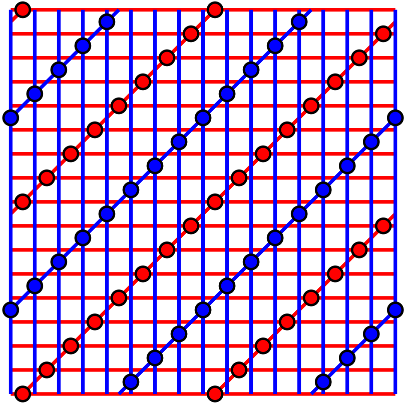


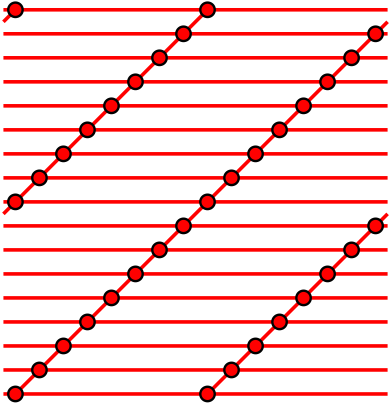
Asymptotic width in dimension five



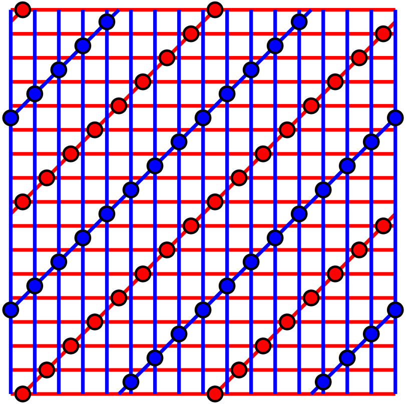
Asymptotic width in dimension five



Asymptotic width in dimension five

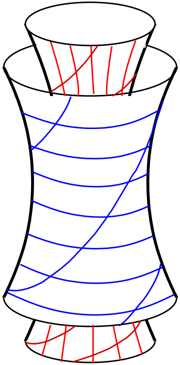


Asymptotic width in dimension five



Asymptotic width in dimension five

Consider the red and blue maps as lying in two parallel tori in the 3-sphere.



Complete the tori maps to the whole 3-sphere (you need quadratically many cells for that).

Between the two tori you basically get the superposition of the two tori maps. □

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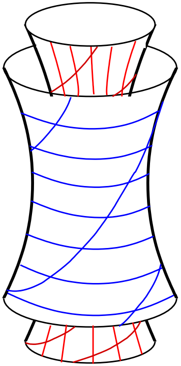

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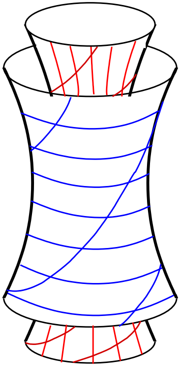

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Thank you

TO BE CONTINUED