MONOTONE PATHS ON ZONOTOPES AND ORIENTED MATROIDS

CHRISTOS A. ATHANASIADIS AND FRANCISCO SANTOS

Abstract. Monotone paths on zonotopes and the natural generalization to maximal chains in the poset of topes of an oriented matroid or arrangement of pseudo-hyperplanes are studied with respect to a kind of local move, called polygon move or flip. It is proved that any monotone path on a d-dimensional zonotope with n generators admits at least $\lceil 2n/(n-d+2) \rceil - 1$ flips for all $n \geq d+2 \geq 4$ and that for any fixed value of $n-d$, this lower bound is sharp for infinitely many values of n. In particular, monotone paths on zonotopes which admit only three flips are constructed in each dimension $d \geq 3$. Furthermore, the previously known 2-connectivity of the graph of monotone paths on a polytope is extended to the 2-connectivity of the graph of maximal chains of topes of an oriented matroid. An application in the context of Coxeter groups of a result known to be valid for monotone paths on simple zonotopes is included.

1. Introduction

Let P be a d-dimensional polytope in $\mathbb{R}^d$ and f be a generic linear functional on $\mathbb{R}^d$, meaning that f is nonconstant on every edge of P. Such a functional f takes its minimum and maximum over P at unique vertices s and t, respectively. An *f-monotone path* on P is a sequence $s = a_0, a_1, \ldots, a_n = t$ of vertices of P such that for each i with $1 \leq i \leq n$, vertices a_{i-1} and a_i are joined by an edge of P and $f(a_{i-1}) < f(a_i)$. A *flip* for such a path γ is a 2-dimensional face F of P such that one of the two f-monotone paths on F, considered as a polytope on its own, forms a subsequence of consecutive vertices in γ. Two distinct f-monotone paths on P are said to *differ by a polygon move*, or *flip*, across F [1, 14] if they have F as a common flip and agree on vertices not in F.

The polygon moves give rise to a natural graph structure $\mathcal{G}(P, f)$ on the vertex set of f-monotone paths on P, which is the analogue of the graph of triangulations of a given point configuration and geometric bistellar operations and that of cubical tilings of a zonotope and cube-flips, in the context of the generalized Baues problem [4]; see [14, Section 2]. In the case of triangulations and tilings and for a fixed dimension, the problem to determine the level of connectivity of such a graph and, in particular, the minimum possible number of neighbors of a vertex, has attracted considerable attention in recent years; see e.g. [14, 15]. For the graph $\mathcal{G}(P, f)$ these questions were raised in [14, Section 6] and studied in [1]. Recall that a graph G is *k-connected* if any subgraph of G obtained by removing a set of at most $k-1$

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vertices and their incident edges is connected and contains at least two vertices. The relevant results of [1] can be summarized as follows.

Theorem 1.1 ([1]). (i) *If P is any polytope of dimension $d \geq 3$ then $\mathcal{G}(P, f)$ is 2-connected. In particular, any f-monotone path on P has at least two flips.*

(ii) *For any $d \geq 4$ there exists a d-polytope P, a linear functional f and an f-monotone path on P with as few as two flips. Furthermore, for $d = 4$ the polytope in the last statement can be chosen to be a zonotope with six generators.*

(iii) *If P is a simple d-polytope then the graph $\mathcal{G}(P, f)$ is $(d-1)$-connected. In particular, any f-monotone path on P has at least $d-1$ flips.*

An f-monotone path on P is called *coherent* if it can be obtained by minimizing some fixed and sufficiently generic linear functional on each slice $P \cap f^{-1}(c)$ of P or, in other words, if the path projects to the boundary of some 2-dimensional and sufficiently generic projection of P. This notion agrees with the definition of coherency in the more general framework introduced in [3]. The results there imply that there exists a polytope of dimension $d-1$, the *monotone path polytope* of P and f, whose vertices are in bijection with coherent f-monotone paths and whose edges represent flips among them (see also [14], [16, Chapter 9] and Remark 3.2). In particular, coherent monotone paths have at least $d-1$ flips. For this reason, monotone paths with less than $d-1$ flips are called *flip-deficient*.

We will be interested in the special case in which P is a zonotope (see Section 2 for definitions). In this case the graph $\mathcal{G}(P, f)$ has an alternative description in terms of the arrangement $\mathcal{A}$ of linear hyperplanes polar to P (see [5, Section 1.2]), which we briefly outline next. If B is the region of $\mathcal{A}$ which corresponds to the f-minimizing vertex of P then the f-monotone paths on P biject to the maximal chains of the poset of regions of $\mathcal{A}$ (see [10]) with basis B. The flips correspond to the "elementary homotopies" connecting these maximal chains (see Section 6), a concept which originated in the work of Deligne [9], and reduce to the "Coxeter moves" in the important special case of reflection arrangements (see [5, Section 2.3] and Section 7). Moreover, this graph of maximal chains and elementary homotopies can be defined for an arbitrary oriented matroid $\mathcal{L}$ with fixed tope B and was shown to be connected in [8]. We will consider the problem to determine the minimum number of flips possible for monotone paths on zonotopes of fixed dimension as well as number of generators. Our first result gives a lower bound on this number.

Theorem 1.2. *Let Z be a zonotope of dimension $d \geq 2$ with n generators and f be a generic linear functional on Z. Any f-monotone path on Z admits at least $m(n, d)$ flips, where*

$$m(n,d) = \begin{cases} d-1, & \text{if } n \leq d+1; \\ \left\lceil \dfrac{2n}{n-d+2} \right\rceil - 1, & \text{if } n \geq d+2. \end{cases}$$

The proof of Theorem 1.2, given in Section 4, is in fact valid in the setting of arbitrary oriented matroids. Note that $m(n, d) \geq 2$ for all $d \geq 3$, which is in agreement with Theorem 1.1 (i).

It is tempting to conjecture that the graph of monotone paths on a zonotope and flips (or its oriented matroid analogue) is always $m(n, d)$-connected, where $m(n, d)$

is as in Theorem 1.2. We do not have a proof of this even in the simplest non-trivial case of $n - d = 2$. We can only extend the proof of 2-connectivity from Theorem 1.1(i) to the oriented matroid setting, improving the result of [8] mentioned earlier.

Theorem 1.3. *If $\mathcal{L}$ is an oriented matroid of rank at least 3 and B is one of its topes then the graph of maximal chains in the tope poset of $\mathcal{L}$ based at B and elementary homotopies is 2-connected.*

Our next result proves that the lower bound of Theorem 1.2 is sharp in various special cases. In particular, this is proved for infinitely many values of n, given any fixed value of the codimension $n - d$.

Theorem 1.4. *If $n \geq d \geq 2$ are integers such that*

- *$n \leq d + 3$ or*
- *$n \geq d + 4$ and n is divisible by $n - d + 2$ or*
- *$d \leq 5$*

then there exists a d-zonotope Z with n generators in general position, a linear functional f and an f-monotone path on Z which admits only $m(n, d)$ flips.

The previous theorem has the following interesting consequence. While there are no flip deficient monotone paths on d-zonotopes with d or $d + 1$ generators, $\lceil d/2 \rceil$ flips can be achieved with $d + 2$ generators in any dimension (Theorem 5.1). Also, three flips can be achieved with $2d - 4$ generators in any dimension $d \geq 6$ (Corollary 5.4) and two flips are possible in the case of zonotopes of dimension 5 or less (Theorem 5.5). We conjecture in Section 5 that the lower bound $m(n, d)$ can be achieved for all n and d, in particular that for all $d \geq 3$ there exist monotone paths on zonotopes with $3d - 6$ generators which admit only two flips.

This paper is structured as follows. We begin with preliminaries on zonotopes and oriented matroids in Section 2. In Section 3 we characterize monotone paths on zonotopes and their flips in oriented matroid terms and establish some of their elementary properties. In particular, we prove that one can always add a generator and increase the length of a monotone path by one without altering the dimension or number of flips. Theorems 1.2, 1.4 and 1.3 are proved in Sections 4, 5 and 6, respectively. In Section 7 we point out an application of Theorem 1.1 to the graph of reduced decompositions of the maximal element in a finite Coxeter group.

2. Preliminaries

Throught the paper we use the notation $[m, n] := \{m, m + 1, \ldots, n\}$ for integers m, n with $m \leq n$ and let $[n] := [1, n]$.

2.1. **Sign vectors.** We denote by Λ_n the set $\{-, 0, +\}^{[n]}$ of sign vectors of length n and write $X = (X_1, X_2, \ldots, X_n)$ for $X \in \Lambda_n$. The set Λ_n is partially ordered by extending coordinatewise the partial order on $\{-, 0, +\}$ defined by the relations $0 < -$ and $0 < +$. Its unique minimal element is the zero vector, denoted by 0. The sign vector $-X$ is obtained by negating each coordinate of X. The *composition* $X \circ Y$ of two sign vectors is defined by $(X \circ Y)_i = X_i$ or Y_i if $X_i \neq 0$ or $X_i = 0$, respectively and their *separation set* is $S(X, Y) = \{i : X_i = -Y_i \neq 0\} = \{i : (X \circ Y)_i \neq (Y \circ X)_i\}$. Two sign vectors X and Y are called *orthogonal* if $S(X, Y)$ and $S(X, -Y)$ are either both empty or both nonempty.

2.2. **Oriented matroids.** A subset $\mathcal{L}$ of Λ_n is the set of *covectors* of an *oriented matroid* [5] if it satisfies the following axioms:

(L0) $0 \in \mathcal{L}$,
(L1) $X \in \mathcal{L}$ implies $-X \in \mathcal{L}$,
(L2) $X, Y \in \mathcal{L}$ implies $X \circ Y \in \mathcal{L}$,
(L3) if $X, Y \in \mathcal{L}$ and $e \in S(X,Y)$ then there exists a $C \in \mathcal{L}$ such that $C_e = 0$ and $C_f = (X \circ Y)_f = (Y \circ X)_f$ for $f \notin S(X,Y)$.

For example, let $A = (u_1, \ldots, u_n)$ be a configuration of n vectors in $\mathbb{R}^d$. Any linear functional $f : \mathbb{R}^d \to \mathbb{R}$ induces a sign vector X by $X_i = \mathrm{sign}(f(u_i))$. The set $\mathcal{L}(A)$ of all sign vectors induced on A by functionals forms an oriented matroid. The oriented matroids which can be obtained in this way are called *realizable* or *representable*. We will normally identify an oriented matroid with its set of covectors.

As a subposet of Λ_n, an oriented matroid $\mathcal{L}$ is a ranked poset whose rank (one less than the cardinality of any maximal chain in $\mathcal{L}$) we denote by r. The quantity $n-r$ is the *corank* of $\mathcal{L}$. The maximal elements of $\mathcal{L}$ are called *topes*. If a coordinate is zero in some tope then it is zero in every covector by axiom (L2). Thus, there is no loss of generality in assuming that topes have no zeros and, hence, that the topes are the covectors with no zeros. The *subtopes* of $\mathcal{L}$ are the elements of rank $r-1$. Any subtope is covered by exactly two topes T_1, T_2 and is said to *join* T_1 and T_2.

The separation set $S(T, T')$ of two topes is the set of coordinates where T and B have different sign. The *tope poset* $\mathcal{T}(\mathcal{L}, B)$ based at a given tope B is the set of all topes T of $\mathcal{L}$, partially ordered by inclusion of their separation sets $S(T, B)$ (see [5, Section 4.2] for more information). Thus $\mathcal{T}(\mathcal{L}, B)$ is a graded poset with B and $-B$ as its minimum and maximum elements, respectively. The rank of a tope T in this poset is the cardinality $d(T, B)$ of the seperation set $S(T, B)$, also called the *distance* between T and B.

2.3. **Duality.** If $\mathcal{L}$ is the set of covectors of an oriented matroid then the set of sign vectors orthogonal to all elements of $\mathcal{L}$ is again the set of covectors of an oriented matroid, called the *dual* of $\mathcal{L}$ and denoted by $\mathcal{L}^*$. The covectors of $\mathcal{L}^*$ are called *vectors* of $\mathcal{L}$. The rank of $\mathcal{L}^*$ equals the corank of $\mathcal{L}$.

Let $A = (u_1, \ldots, u_n)$ be a configuration of n vectors spanning $\mathbb{R}^d$. Up to a linear automorphism, there is a unique configuration $A^* = (u_1^*, \ldots, u_n^*)$ of n vectors spanning $\mathbb{R}^{n-d}$ such that $\sum_{i=1}^n u_i \otimes u_i^* = 0$ in $\mathbb{R}^d \otimes \mathbb{R}^{n-d}$. The two configurations are said to be *Gale transforms* of each other [16, Section 6.4]. Their oriented matroids are dual to one another. A vector $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ is the ordered sequence of values of a linear functional on A if and only if it is the sequence of coefficients of a linear dependence on A^*. Hence, the vectors of the oriented matroid realized by A are the sequences of signs of the coefficients in a linear dependence on A.

2.4. **Zonotopes.** Let $A = (u_1, \ldots, u_n)$ be a configuration of n vectors spanning $\mathbb{R}^d$. Let O be the origin in $\mathbb{R}^d$ and $[O, u_i] = \{\lambda u_i : 0 \leq \lambda \leq 1\}$ for each i. The zonotope $Z(A)$ generated by A is the polytope $Z(A) = \sum_{i=1}^n [O, u_i]$. This polytope depends only on the underlying multiset of elements of A. The map which sends a sign vector $X = (X_1, X_2, \ldots, X_n)$ to the sum $\sum_{X_i=+} u_i + \sum_{X_i=0} [O, u_i]$ induces an order reversing bijection between the poset of covectors $\mathcal{L}(A)$ and the poset of (nonempty) faces of $Z(A)$. Under this bijection, the topes of $\mathcal{L}(A)$ correspond to the vertices of $Z(A)$ and the subtopes correspond to the edges.

3. Flips for monotone paths on zonotopes

In this section we establish some elementary properties of monotone paths on zonotopes and their flips.

Let E be a configuration of n nonzero vectors in $\mathbb{R}^d$, with no two parallel, and $Z(E)$ be the zonotope generated by E. Let f be a linear functional which is not constant on any edge of $Z(E)$, in other words, which does not vanish on any element of E. Since changing a vector of E to its negative results only in a translation of $Z(E)$, we may assume without loss of generality that f is positive on E.

Lemma 3.1. *Under the previous assumptions on E and f:*

(i) *The sequence of vertices of $Z(E)$ defining an f-monotone path on $Z(E)$ is of the form O, v_1, v_1+v_2, $v_1+v_2+v_3, \ldots, v_1+\cdots+v_n$ for some permutation $(v_1,\ldots,v_n)$ of the elements of E.*

(ii) *A permutation $(v_1,\ldots,v_n)$ of E corresponds to an f-monotone path on $Z(E)$ if and only if for every $i \in [n]$ there exists a hyperplane containing v_i and having $\{v_1,\ldots,v_{i-1}\}$ and $\{v_{i+1},\ldots,v_n\}$, respectively, in its two complementary open halfspaces.*

Proof. Both parts follow from the characterization of vertices and edges of $Z(E)$ in terms of the covectors of $\mathcal{L}(E)$. $\square$

A sequence $A=(v_1,\ldots,v_n)$ of vectors satisfying the condition in Lemma 3.1(ii) for a given index i will be said to be *valid at i*. In oriented matroid terms, A is valid at i if the sign vector which is negative on $[i-1]$, zero on i and positive on $[i+1,n]$ is a covector of $\mathcal{L}(A)$. We call A a *valid sequence* if it is valid at every index i. The validity at the first and last index implies that the positive span of A is a pointed cone.

Remark 3.2. Another way of seeing how monotone paths are related to permutations is as follows. Let I^n denote the standard n-dimensional cube in $\mathbb{R}^n$ and let $\pi:\mathbb{R}^n \to \mathbb{R}^d$ be the projection which bijects the standard basis to the set of vectors E. Observe that $\pi(I^n)=Z(E)$. Moreover, $f\circ\pi$ is a linear functional in $\mathbb{R}^n$, generic for I^n. Every f-monotone path on $Z(E)$ lifts to a $(f\circ\pi)$-monotone path on I^n. As is well-known, (see [16, Example 9.8]) monotone paths in the n-cube correspond to permutations of the n coordinates.

This setting also helps to understand the monotone path polytope of $Z(E)$. The monotone path polytope $\Sigma(I^n, f\circ\pi)$ of the cube is the $(n-1)$-dimensional permutahedron (see again [16]). Lemma 2.3 in [3] says that the monotone path polytope of $Z(E)$ with respect to f equals $\pi(\Sigma(I^n, f\circ\pi))$.

Corollary 3.3. *If a sequence of n vectors is valid at $i-1$ and $i+1$, where indices are regarded modulo n, then it is also valid at i.*

Proof. For $i\in[n]$, let X^i be the sign vector which is negative on $[i-1]$, zero on i and positive on $[i+1,n]$. Let $X^0=-X^n$ and $X^{n+1}=-X^1$. With this notation, the elimination axiom (L3) applied to X^{i-1} and X^{i+1} produces exactly the sign vector X^i. The result follows. $\square$

Lemma 3.4. *Under the assumptions of Lemma 3.1 on E and f:*

(i) *Two valid permutations of E represent f-monotone paths on $Z(E)$ which differ by a flip if and only if one can be obtained from the other by swapping a subsequence of consecutive vectors which are coplanar.*

(ii) *The swapping of $v_i, v_{i+1}, \ldots, v_j$ on a valid permutation $(v_1, \ldots, v_n)$ of E corresponds to a flip of monotone paths if and only if $v_i, v_{i+1}, \ldots, v_j$ are coplanar and there exists a hyperplane which contains these vectors and has $\{v_1, \ldots, v_{i-1}\}$ and $\{v_{j+1}, \ldots, v_n\}$, respectively, in its two complementary open halfspaces.*

Proof. The forward statement of part (i) is clear from the definition of a flip. Its converse follows from the elimination axiom (L3) of oriented matroids applied to the covectors of a pair of distinct parallel edges of the two paths and orthogonality with the vectors provided by three-term linear relations, if any. Part (ii) follows from the characterization of faces of $Z(E)$ in terms of the covectors of $\mathcal{L}(E)$. □

We say that the valid sequence $A = (v_1, \ldots, v_n)$ has a flip at $(v_i, \ldots, v_j)$ if the condition in Lemma 3.4(ii) is satisfied. If no three consecutive vectors in the sequence are coplanar then the swapping referred to in the lemma is the transposition of two consecutive vectors v_i and v_{i+1}. The following lemma allows us to assume without loss of generality that this is always the case. We denote by $A \setminus v_i$ the sequence obtained from A by removing v_i.

Lemma 3.5. *Let $A = (v_1, v_2, \ldots, v_n)$ be a valid sequence of vectors in $\mathbb{R}^d$, with $d \geq 3$. If v_{i-1}, v_i and v_{i+1} are coplanar for some i, with indices regarded modulo n, then $A \setminus v_i$ has as many flips as A.*

Proof. For $i \in [n]$, coplanarity of (v_{i-1}, v_i, v_{i+1}) and validity of A at i imply that v_i is a positive combination of v_{i-1} and v_{i+1}, where $v_0 = -v_n$ and $v_{n+1} = -v_1$ if $i = 1$ or $i = n$, respectively. With this and Lemma 3.4 the proof is straightforward. □

In oriented matroid terms, A has a flip at the pair (v_i, v_{i+1}) if and only if the sign vector negative on $[i-1]$, zero on $\{i, i+1\}$ and positive on $[i+2, n]$ is a covector of $\mathcal{L}(A)$. The following lemma characterizes flips in terms of the vectors of $\mathcal{L}(A)$, i.e. the covectors of the dual oriented matroid $\mathcal{L}^*(A)$.

Lemma 3.6. *Let $A = (v_1, \ldots, v_n)$ be a valid sequence.*

(i) *If $X = (X_1, X_2, \ldots, X_n)$ is a nonzero vector of $\mathcal{L}(A)$ then there exist indices $i < j$ such that $X_i = +$ and $X_j = -$.*

(ii) *The sequence A has no flip at (v_i, v_{i+1}) if and only if there exists a vector of $\mathcal{L}(A)$ which is nonpositive on $[i-1]$, nonnegative on $[i+2, n]$ and restricts to $(+, -)$ on $\{i, i+1\}$.*

Proof. Part (i) follows from the fact that X is orthogonal to each of the covectors of $\mathcal{L}(A)$ provided by Lemma 3.4(ii). To prove (ii) observe that (v_i, v_{i+1}) is not a flip of A if and only if there exists a vector X of $\mathcal{L}(A)$ which is not orthogonal to $(-, \ldots, -, 0, 0, +, \ldots, +)$, the zeros located at coordinates i and $i+1$. Clearly, one of the vectors X and $-X$ has the first two properties claimed and is not identically zero on $[n] \setminus \{i, i+1\}$. That this sign vector restricts to $(+, -)$ on $\{i, i+1\}$ follows from (i). □

If A has no flip at (v_i, v_{i+1}) we call any vector of $\mathcal{L}(A)$ having the properties in part (ii) of the previous lemma a *witness* of the *nonflip* at this pair. An element v_i of A is *in general position* in A if v_i is not contained in any hyperplane spanned by vectors of A other than v_i. The sequence A is in general position if all its elements are in general position.

Proposition 3.7. *For any valid sequence $A = (v_1, \dots, v_n)$ of n vectors spanning $\mathbb{R}^d$, with $d \geq 3$, there exists a vector $v \in \mathbb{R}^d$ such that the sequence $(v_1, \dots, v_n, v)$ is valid, has the same flips as A and has the new vector v in general position.*

Proof. Let $v = v_n - \epsilon_1 v_1 - \cdots - \epsilon_{n-1} v_{n-1}$ for any positive, sufficiently small and sufficiently generic numbers $\epsilon_1, \dots, \epsilon_{n-1}$. Genericity implies that v is in general position in the sequence $B = (v_1, \dots, v_n, v)$.

Since v is very close to v_n, validity of A at any index $i \in [n-1]$ implies validity of B at the same index. For the same reason, any flip of A not involving v_n has to be a flip of B. Validity of the first sequence at n implies that there is a hyperplane containing v_n with all other v_i's on the same side. This hyperplane clearly has v on the other side, so the new sequence is valid at n. Corollary 3.3 implies that B is valid. The previous argument also implies that any flip of A involving v_n has to be a flip of B.

That every flip of B other than the transposition of v_n and v is also a flip of A is trivial by Lemma 3.4(ii). Hence, we only need to prove that the transposition of v_n and v is not a flip. This follows from the fact that v_n is a positive combination of v and the other v_i's: any functional vanishing on v_n and v must have both positive and negative values on the rest of the v_i's. $\square$

4. The lower bound

In this section we prove Theorem 1.2. We let $A = (v_1, v_2, \dots, v_n)$ be a valid sequence of n vectors in $\mathbb{R}^d$ and $\mathcal{L} = \mathcal{L}(A)$ be the corresponding oriented matroid on the ground set $[n]$. A set of indices $I \subseteq [n]$ is said to be dependent in $\mathcal{L}$ if the set of vectors $\{v_i : i \in I\}$ is linearly dependent.

Lemma 4.1. *Let $1 \leq k < l \leq n$ be two indices such that A has no flip at the pairs (v_i, v_{i+1}) with $k \leq i < l$ and suppose that no three elements of $(v_k, \dots, v_l)$, with at least two of them consecutive, are coplanar.*

(i) *The sequence $A \setminus v_k$ has no flip at the pairs (v_i, v_{i+1}) for $k < i < l$.*
(ii) *There exist three vectors of $\mathcal{L}$ which are nonpositive on $[k-1]$, nonnegative on $[l+1, n]$ and whose restrictions on $[k, l]$ are the sign vectors $(0, \dots, 0, +, -)$, $(+, -, 0, \dots, 0,)$ and $(+, 0, \dots, 0, -)$, respectively.*

Proof. We prove (i) and (ii) by induction on $l - k$. The result reduces to Lemma 3.6(ii) for $l - k = 1$, so let $l - k \geq 2$. Let $\mathcal{X}$ be the set of vectors of $\mathcal{L}$ which are nonpositive on $[k-1]$ and nonnegative on $[l+1, n]$. To prove (i) it suffices to show that for $k < i < l$, there exists a vector in $\mathcal{X}$ which is zero on k, nonpositive on $[k+1, i-1]$, nonnegative on $[i+2, n]$ and restricts to $(+, -)$ on $\{i, i+1\}$. By the inductive hypothesis for (ii), there exists a vector $X \in \mathcal{X}$ which restricts to $(+, 0, \dots, 0, -)$ on $[k, i]$ and is nonnegative on $[i+1, n]$. Let $Y = (Y_1, \dots, Y_n)$ be a witness of the nonflip of A at (v_i, v_{i+1}), so that $Y \in \mathcal{X}$. If $Y_k = 0$ then there is nothing to prove. Otherwise $Y_k = -$ and we can eliminate k between X and Y. Since $S(X, Y) \subseteq \{k, i, i+1\}$ is not dependent in $\mathcal{L}$, the vector produced by this elimination is nonzero. Lemma 3.6 (i) implies that this vector has the desired properties. To prove (ii) note that by (i), $A \setminus v_k$ satisfies the hypotheses of the lemma for $(v_{k+1}, \dots, v_l)$. By induction, there exist two vectors in $\mathcal{X}$ which restrict to $(0,\ldots,0,+,-)$ and $(0,+,0,\ldots,0,-)$ on $[k,l]$, respectively. By a symmetric argument, there exists a vector in $\mathcal{X}$ which restricts to $(+,-,0,\ldots,0)$ on $[k,l]$. Elimination of $k+1$ between the last two vectors produces a vector in $\mathcal{X}$ restricting to $(+,0,\ldots,0,-)$ and completes the induction. $\square$

Lemma 4.2. *Let $1 \leq k < l \leq n$ be two indices such that A has no flip at the pairs (v_i, v_{i+1}) with $k \leq i < l$ and suppose that $(v_k,\ldots,v_l)$ contains three coplanar elements, two of which are consecutive, but no three consecutive coplanar elements. There exists an index j with $k < j < l$ such that:*

(i) *no three consecutive elements of $(v_k,\ldots,v_l) \setminus v_j$ are coplanar,*
(ii) *$A \setminus v_j$ has no flip at (v_{j-1}, v_{j+1}) and*
(iii) *$A \setminus v_j$ has no flip at any pair (v_i, v_{i+1}) at which A has no flip.*

Proof. We choose a triple (v_{j_1}, v_j, v_{j_2}) of coplanar vectors, satisfying $k \leq j_1 < j < j_2 \leq l$ and $j = j_1 + 1$ or $j_2 - 1$, with $j_2 - j_1$ as small as possible. We assume with no loss of generality that $j = j_1 + 1$. Since $(v_j,\ldots,v_{j_2})$ contains no three coplanar elements, with at least two consecutive, we can apply Lemma 4.1 to A on this subsequence.

If $j_2 - j_1 \geq 4$ then (i) holds automatically. Otherwise $j_2 - j_1 = 3$, by our assumptions, and we can guarantee (i) by further choosing j_1 and j_2 to be as small as possible. Observe as in Lemma 3.5 that v_j is a positive combination of v_{j-1} and v_{j_2}. Let X be the vector of $\mathcal{L}(A)$ which is positive on $j-1$ and j_2, negative on j and zero elsewhere and let Y be a witness of the nonflip of A at (v_j, v_{j+1}). In view of Lemma 3.6(i), elimination of j between X and Y produces a vector of $\mathcal{L}(A)$ which witnesses the nonflip claimed in (ii). Finally, (iii) follows from Lemma 4.1(i) if $j < i < j'$ and is straightforward otherwise. $\square$

Lemma 4.3. *Let $1 \leq k < m < m+1 < l \leq n$ be indices such that A has a flip at (v_m, v_{m+1}) but no flip at other pairs (v_i, v_{i+1}) with $k \leq i < l$. If no three consecutive elements of $(v_k,\ldots,v_l)$ are coplanar then the corank of $\mathcal{L}$ is at least $l-k-1$.*

Proof. By Lemma 4.2 we may assume that none of the subsequences $(v_k,\ldots,v_m)$ or $(v_{m+1},\ldots,v_l)$ contains three coplanar elements, with at least two of them consecutive. Hence we can apply Lemma 4.1 on these subsequences. Let $\mathcal{X}$ be the set of vectors of $\mathcal{L}$ which are nonpositive on $[k-1]$ and nonnegative on $[l+1,n]$. We first claim that there exist two vectors in $\mathcal{X}$ which are zero on $[k,l] \setminus \{k,m,m+1,l\}$ and have the following additional properties:

(i) The first vector is nonnegative on $\{m+1,l\}$ and restricts to $(+,-)$ on $\{k,m\}$.
(ii) The second vector is nonpositive on $\{k,m\}$ and restricts to $(+,-)$ on $\{m+1,l\}$.

We establish the existence of the vector claimed in (i) by induction on $l-k$. If $l = m+2$ then this follows from Lemma 4.1 (ii), so let $l \geq m+3$. By induction, there exist two vectors of $\mathcal{X}$ which are zero on $[k,l] \setminus \{k,m,m+1,l-1,l\}$ and have the following additional properties:

(i′) The first vector is nonnegative on $\{m+1,l-1,l\}$ and restricts to $(+,-)$ on $\{k,m\}$. We may assume that it is positive on $l-1$ since otherwise it has the properties claimed in (i).
(ii′) The second vector is nonpositive on $\{k,m\}$, nonnegative on l and restricts to $(+,-)$ on $\{m+1,l-1\}$.

In view of Lemma 3.6(i), elimination of $l-1$ between the two vectors produces a vector of $\mathcal{L}$ with the desired properties. The case of the vector claimed in (ii) is analogous.

Given the claim, we complete the proof as follows. One can further choose the vector in (i) to be zero on l by eliminating l between the vectors in (i) and (ii), if needed. By successively composing the vectors (i) and (ii) for different subintervals of $[k,l]$ we can find a chain of nonzero vectors of $\mathcal{L}$ of cardinality $l-k-1$. This implies that the corank of $\mathcal{L}$ is at least $l-k-1$.

□

Proof of Theorem 1.2. Let $A=(v_1,v_2,\dots,v_n)$ be the valid permutation of the set of generators of Z representing a given f-monotone path on Z and $\mathcal{L}=\mathcal{L}(A)$ be the corresponding oriented matroid of rank d. The case $d=2$ is trivial so we assume that $d\geq 3$. Lemma 3.5 and the fact that $m(n,d)\leq m(n-1,d)$ allow us to assume further that A does not contain three coplanar vectors v_{i-1},v_i,v_{i+1}, with indices regarded modulo n.

The case $n=d$ is trivial. If $n=d+1$ then $\mathcal{L}$ has only two nonzero vectors, one being the negative of the other. Two such vectors cannot be witnesses of distinct nonflips unless these are at consecutive pairs (v_{i-1},v_i) and (v_i,v_{i+1}) and $i-1,i,i+1$ are the only nonzero coordinates of the vectors. Since this would imply that v_{i-1}, v_i and v_{i+1} are coplanar, contrary to our assumption, it follows that A has at least $d-1$ flips. Finally, suppose that $n\geq d+2$. We say that A has a flip at the pair (v_n,v_1) if there exists a hyperplane which contains v_1 and v_n and has the rest of the v_i in the same open halfspace. With this extension the notion of flip becomes cyclic, in the sense that for all i, the sequence $(v_{i+1},\dots,v_n,-v_1,\dots,-v_i)$ is valid and has flips which biject to those of A under shifting by i. Lemma 4.3 implies that any $n-d+2$ consecutive pairs (v_i,v_{i+1}) include at least two flips. This implies a lower bound of $\lceil 2n/(n-d+2)\rceil = m(n,d)+1$ on the number of flips of A, in the cyclic sense, and of $m(n,d)$ if the pair (v_n,v_1) is not considered as a possible flip. □

5. Monotone paths with few flips

In this section we use the technique of Gale transforms to construct valid sequences with the number of flips claimed in Theorem 1.4.

We will tacitly use the fact that if $A=(v_1,\dots,v_n)$ is a sequence of vectors spanning $\mathbb{R}^d$, $X=(X_1,\dots,X_n)$ is a vector of $\mathcal{L}(A)$ such that $\{v_i : X_i\neq 0\}$ spans $\mathbb{R}^d$ and $X<Y$ for a sign vector Y then Y is a vector of $\mathcal{L}(A)$.

Theorem 5.1. *For any $d\geq 2$ there exists a valid sequence of $d+2$ vectors in general position in $\mathbb{R}^d$ which has only $\lceil d/2\rceil$ flips.*

Proof. We may assume that $d\geq 4$, since the cases $d=2$ and $d=3$ are trivial. We will construct explicitly a Gale transform of a sequence with the desired properties.

Suppose first that d is even. Let $v_2^*:=(1,-2)$, $v_4^*:=(1,0)$, $v_{d-1}^*:=(0,1)$ and $v_{d+1}^*:=(-2,1)$. Let the other $d-2$ vectors be any vectors in general position which, together with v_4^* and v_{d-1}^*, form the sequence

$$v_4^*,v_1^*,v_6^*,v_3^*,v_8^*,v_5^*,\dots,v_d^*,v_{d-3}^*,v_{d+2}^*,v_{d-1}^*$$

when taken in anti-clockwise order. This sequence alternates even and odd indices and both the even and the odd appear monotonically. The configuration is shown in

Figure 1 for $d = 6$. We claim that the Gale transform $(v_1, \ldots, v_{d+2})$ of $(v_1^*, \ldots, v_{d+2}^*)$ is valid and has only $d/2$ flips. Indeed, the fact that v_2^*, v_{d+1}^* and any other v_i^* are positively dependent implies that the sequence is valid at 1 and $d+2$. The fact that v_{i-1}^* is a positive combination of v_{i+2}^* and v_{i+3}^* for even $i < d$ and v_{d+2}^* is a positive combination of v_{d-2}^* and v_{d-1}^* implies validity at all other indices and that the $d/2$ pairs (v_i, v_{i+1}) for even $i \leq d$ are flips. On the other hand there is no flip at (v_1, v_2) or (v_{d+1}, v_{d+2}), since the positive spans of $\{v_3^*, \ldots, v_{d+2}^*\}$ and $\{v_1^*, \ldots, v_d^*\}$ are pointed cones, and no flip at (v_{i-1}, v_i) for even $i \in [4, d]$, since the positive spans of $\{v_1^*, \ldots, v_{i-2}^*\}$ and $\{v_{i+1}^*, \ldots, v_{d+2}^*\}$ intersect only at the zero vector. This completes the proof in the even case.

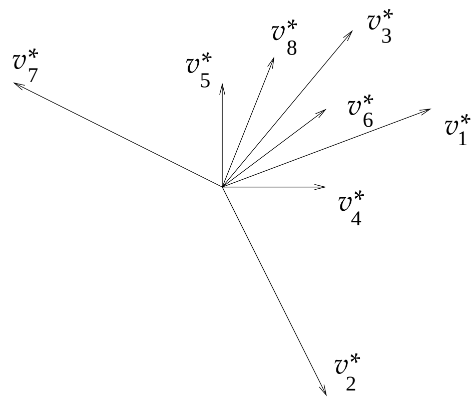


FIGURE 1. Few flips in corank 2.

For the case that d is odd, let $v_1^*, \ldots, v_{d+1}^*$ be as before and let v_0^* be any generic positive combination of v_2^* and v_4^*. The sequence $(v_0, \ldots, v_{d+1})$ dual to $(v_0^*, \ldots, v_{d+1}^*)$ is valid and the only new flip is at the pair (v_0, v_1). $\square$

The construction in Theorem 5.1 generalizes the following example of P. Edelman and V. Reiner [11], refered to (but not explicitly described) in Remark 3.4 of [1]. Let (v_1, v_3, v_4, v_6) be a linear basis of $\mathbb{R}^4$, $v_2 = v_1 + 3v_3 + v_4 + 2v_6$ and $v_5 = 2v_1 + v_3 + 3v_4 + v_6$. The sequence $(v_1, v_2, v_3, v_4, v_5, v_6)$ is valid and has only two flips, namely at (v_2, v_3) and (v_4, v_5).

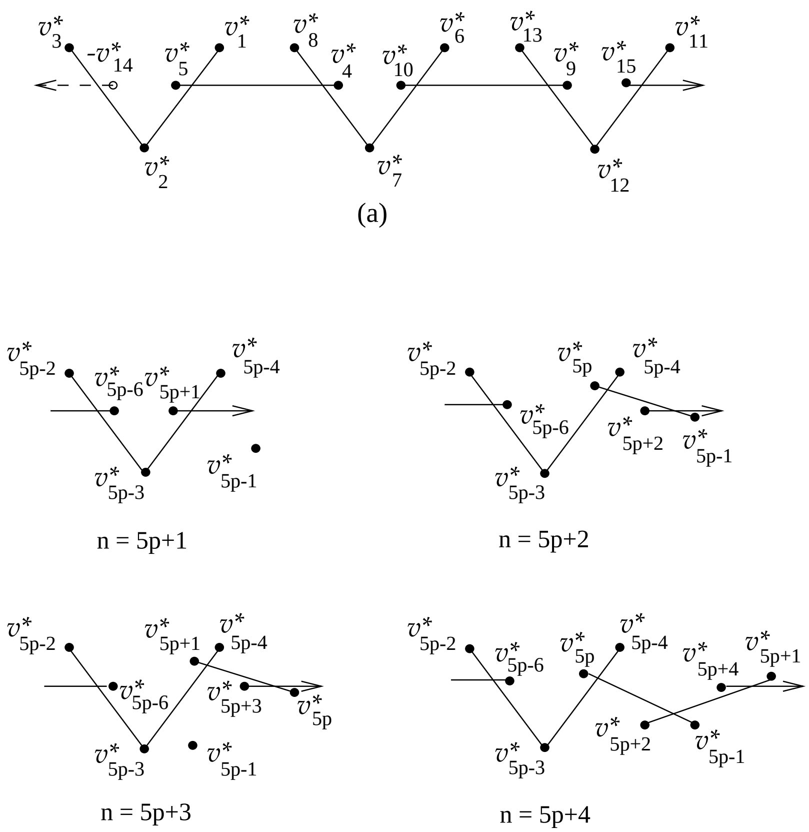
In the following constructions we will use the fact that the validity of a vector sequence A, as well as witnesses of nonflips having no zero coordinate, are preserved under any sufficiently small perturbation of the elements of A.

Theorem 5.2. *For any $d \geq 2$ there exists a valid sequence of $n = d + 3$ vectors in general position in $\mathbb{R}^d$ which has only $\lceil 2n/5 \rceil - 1$ flips.*

Proof. Since the claimed number of flips is equal to $\lceil d/2 \rceil$ for $n \leq 9$, we may assume that $n \geq 10$. Suppose first that n is divisible by 5. Let $v_1^* = (2, 1, 1)$, $v_2^* = (0, -2, 1)$, $v_3^* = (-2, 1, 1)$, $v_4^* = (5, 0, 1)$, $v_5^* = (1, 0, 1)$, $v_{n-1}^* = -(-1, 0, 1)$ and $v_i^* = v_{i-5}^* + (6, 0, 0)$ for all $i \in \{6, \ldots, n-2, n\}$. Figure 2(a) shows the vector configuration in the affine plane $x_3 = 1$ in $\mathbb{R}^3$ for $n = 15$. The white dot represents the vector $-v_{14}^*$. We leave it to the reader to verify that the Gale transform $(v_1, \ldots, v_n)$ of $(v_1^*, \ldots, v_n^*)$ is a valid sequence and has exactly $\frac{2n}{5} - 1$ flips, namely at the pairs

(v_i, v_{i+1}) for $i = n-2$ and $i = 5j-2, 5j$, where $1 \le j \le \frac{n}{5} - 1$. To help with this task, two points i and $i+1$ in Figure 2 are joined by a segment if and only if the pair (v_i, v_{i+1}) is not a flip (the segment joining v_{15}^* and $-v_{14}^*$ is meant to go through infinity). As an example, a witness for the nonflip at (v_7, v_8) is produced by any line in Figure 2 (a) passing below v_2^*, v_4^* and v_6^* and above v_{10}^* and v_{13}^*. If n is not divisible by 5, the last few vectors can be modified as in Figure 2(b) in order to get the desired number of flips.

Since lines witnessing the nonflips can be chosen not passing through any point v_i^*, the dual sequence can be perturbed to general position without affecting its validity or number of flips. □



(b)

FIGURE 2. Few flips in corank 3.

Theorem 5.3. *If $n \ge d+2 \ge 4$ and $n = p\,(n-d+2)$, with $p \in \mathbb{Z}$, then there exists a valid sequence of n vectors in general position in $\mathbb{R}^d$ which has only $2p-1$ flips.*

Proof. We may assume that $d \ge 3$, in other words that $p \ge 2$. The following construction generalizes the main ideas in the proofs of Theorems 5.1 and 5.2. Let $k = n - d$ and

$$(u_{-1}, u_1, \ldots, u_k, u_{k+2}, w)$$

be a configuration of vectors in $\mathbb{R}^k$ whose oriented matroid contains the $k+2$ vectors (1) and $k+1$ covectors (2) listed below.

$$
(1)\quad
\begin{array}{c}
(-,\ +,\ldots,+,+,+,\ 0,\quad 0)\\
(-,\ 0,+,\ldots,+,+,\ +,\quad 0)\\
(-,\ -,0,+,\ldots,+,\ +,\quad 0)\\
\vdots\\
(-,\ -,\ldots,-,-,0,\ +,\quad 0)\\
(0,\ -,-,\ldots,-,-,\ +,\quad 0)
\end{array}
\qquad\qquad
(2)\quad
\begin{array}{c}
(-,\ +,-,+,+,\ldots,\ +,\quad +)\\
(-,\ -,+,-,+,\ldots,\ +,\quad +)\\
\vdots\\
(-,\ \ldots,-,-,+,-,\ +,\quad +)\\
(-,\ -,-,\ldots,-,-,\ -,\quad +)\\
(+,\ +,+,\ldots,+,+,\ +,\quad 0)
\end{array}
$$

For instance, take a Gale transform of the rank 3 vector configuration whose affine picture appears in Figure 3. We assume further that there exists a functional $g : \mathbb{R}^k \to \mathbb{R}$ with $g(u_i) = 1$ for all i and $g(w) = 0$. This is possible since we can choose g as any functional which produces the last covector listed in (2) and then scale each vector u_i appropriately.

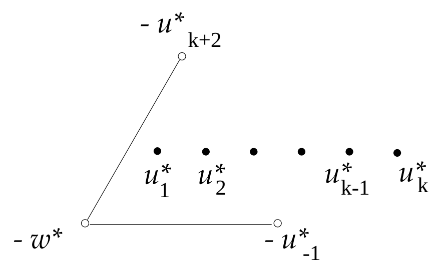


FIGURE 3. A vector configuration of rank 3.

Let $u_{n-1} = -u_{-1}$ and $u_i = u_{i-k-2} + t$ for all $i \in [k+1, n] \setminus \{k+2, n-1\}$, where $t = \lambda w$ for some positive $\lambda \in \mathbb{R}$. Let $A^* = (u_1, \ldots, u_n)$ and $A = (v_1, \ldots, v_n)$ be a Gale transform of A^*. We claim that A is valid for any value of λ and that it has exactly $2p - 1$ flips if λ is sufficiently large, namely at the pairs (v_{q-2}, v_{q-1}) and (v_q, v_{q+1}) for indices q divisible by $k + 2$.

Indeed, let $q = a(k + 2)$ with $a \in [0, p - 1]$. By our choice of g, the sum of the coefficients in any linear dependence on $(u_{-1}, u_1, \ldots, u_k, u_{k+2})$ is zero and hence such a dependence is preserved under translation of all vectors by t. As a result, the sign vectors (1) are vectors of the sequence $(u_{q-1}, u_{q+1}, \ldots, u_{q+k}, u_{q+k+2})$, with $u_{-1} = -u_{n-1}$ if $q = 0$. Since any $k + 1$ elements of this sequence span $\mathbb{R}^k$, this implies that A is valid at the indices $q, \ldots, q + k + 1$. For the second claim let $f_1, \ldots, f_k$ be functionals on $\mathbb{R}^k$ which induce the first k covectors listed in (2) and choose λ so that $f_i(t) > |f_i(u_j)|$ for all i and $j = -1, 1, \ldots, k, k + 2$. A simple computation shows that, under this assumption, the functional $f_i - a f_i(t)\, g$ induces a witness for a nonflip at (v_{q+i}, v_{q+i+1}) if $1 \leq i < k$ and at (v_{q+k+1}, v_{q+k+2}) if $i = k$. The other $n - 1 - kp = 2p - 1$ pairs are flips by Theorem 1.2.

Since the witnesses of the nonflips have no zero coordinates, they are preserved under any sufficiently small perturbation of A into general position. $\square$

Corollary 5.4. *For any $d \geq 6$ and any $n \geq 2d - 4$ there exists a valid sequence of n vectors in general position in $\mathbb{R}^d$ which has only three flips.*

Proof. For the case $n = 2d - 4$ take $p = 2$ in the previous theorem. The general case follows from Proposition 3.7. □

Theorem 5.5. *For any $n \geq 9$ there exists a valid sequence of n vectors in general position in $\mathbb{R}^5$ which has only two flips.*

Proof. In view of Proposition 3.7 it suffices to treat the case $n = 9$. Let $A^* = (v_1^*, \dots, v_9^*)$ be the sequence of columns of the following matrix, which we denote again by A^*:

$$A^* := \begin{bmatrix} 6 & 0 & -6 & 14 & 8 & 2 & 2 & 5 & 10 \\ 1 & -5 & 1 & -1 & 5 & -1 & 1 & -2 & 1 \\ 1 & -5 & 1 & 1 & -5 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 0 & 1 \end{bmatrix}.$$

Note that $v_1^*, v_2^*, \dots, v_6^*, -v_7^*$ and v_9^* lie in the affine hyperplane $x_4 = 1$ in $\mathbb{R}^4$ while v_8^* is parallel to it. Part (a) of Figure 4 shows a two-dimensional projection of this hyperplane along the direction of the third coordinate x_3. The white dot represents $-v_7^*$ and v_8^* is drawn as lying at infinity. The linking of the triangles $\{v_1^*, v_2^*, v_3^*\}$, $\{v_4^*, v_5^*, v_6^*\}$ and $\{v_7^*, v_8^*, v_9^*\}$ shown should help the reader to understand the construction.

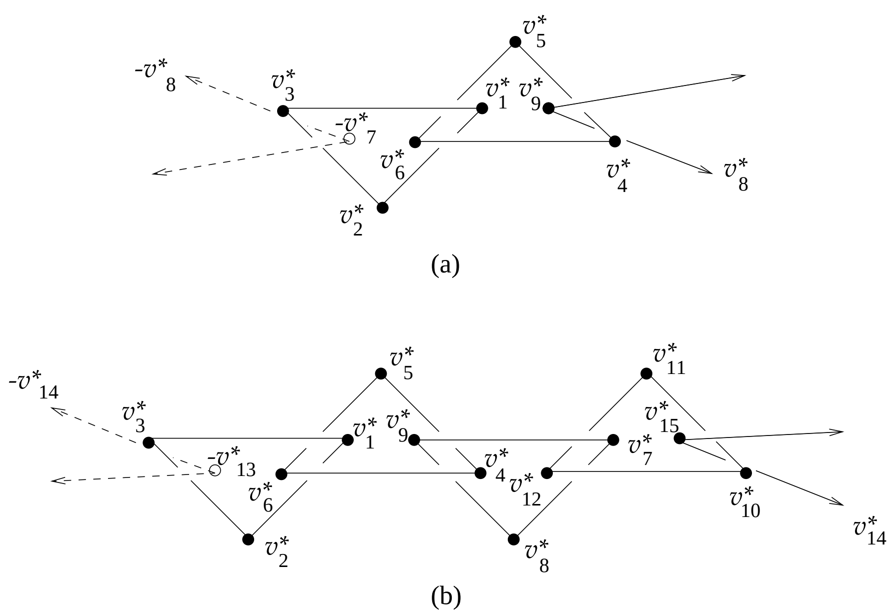


FIGURE 4. Two vector configurations of rank 4.

Let $A = (v_1, \dots, v_9)$ be a Gale transform of A^*. We claim that A is valid and has only two flips, namely at the pairs (v_3, v_4) and (v_6, v_7). To see the latter we consider the matrix

$$W := \begin{bmatrix} 1 & 2+e & 1+e & 3-e \\ 1 & -2-e & -1-e & -3+e \\ 1 & -2-e & 1+e & -5-e \\ 1 & 2+e & -1-e & -11+e \\ 2 & 8 & 6 & -27 \\ -2 & -8 & 6 & -11 \end{bmatrix}$$

where ϵ is any positive constant smaller than $1/2$. The six rows of W represent linear functionals over $\mathbb{R}^4$ which witness, respectively, nonflips at the pairs $(1,2)$, $(2,3)$, $(4,5)$, $(5,6)$, $(7,8)$ and $(8,9)$. This is shown by the following matrix multiplication:

$$W \cdot A^* = \begin{bmatrix} 12+\epsilon & -12-11\epsilon & \epsilon & 16-\epsilon & 16-\epsilon & 4-\epsilon & 1+2\epsilon & 1-2\epsilon & 15 \\ -\epsilon & 12+11\epsilon & -12-\epsilon & 12+\epsilon & \epsilon & \epsilon & 3-2\epsilon & 9+2\epsilon & 5 \\ -\epsilon & -\epsilon & -12-\epsilon & 12+\epsilon & -12-11\epsilon & \epsilon & 5 & 9+2\epsilon & 3-2e \\ -4+\epsilon & -16+\epsilon & -16+\epsilon & -\epsilon & 12+11\epsilon & -12-\epsilon & 15 & 1-2\epsilon & 1+2e \\ -1 & -97 & -25 & -1 & -1 & -25 & 39 & -6 & 1 \\ -25 & -1 & -1 & -25 & -97 & -1 & -1 & 6 & -39 \end{bmatrix}$$

To see that the sequence is valid consider the following matrix.

$$V := \begin{bmatrix} 0 & 1 & 4 & 0 & 0 & 1 & 6 & 2 & 0 \\ -85 & -17 & 0 & 0 & 17 & 85 & 12 & 12 & 12 \\ -1 & 0 & 0 & -4 & -1 & 0 & 0 & 2 & 6 \end{bmatrix}$$

The rows of V are orthogonal to those of A^*, hence they represent linear dependences among the vectors v_i^*. Since $\det(v_2^*, v_6^*, v_7^*, v_8^*) = \det(v_1^*, v_5^*, v_8^*, v_9^*)$=24, in each dependence the vectors in A^* with nonzero coefficients span $\mathbb{R}^4$. Hence, the three dependences can be perturbed keeping the signs of the nonzero coefficients and producing arbitrary signs in the zero ones, to get dependences which show validity of A at 1 and 9 (first row of V), 3 and 4 (second row) and 6 and 7 (third row). Validity at the other three indices follows from Corollary 3.3.

As in the previous constructions, nonflips are preserved under any sufficiently small perturbation of A into general position. □

Remark 5.6. The construction in the proof of Theorem 5.5 can be generalized to show that for any $p \in \mathbb{N}$ and for $d = 6p - 1$, there exists a valid sequence of $n = d + 4$ vectors in general position in $\mathbb{R}^d$ which has only $\frac{n}{3} - 1$ flips. This again matches the lower bound of Theorem 1.2. More precisely, if $(v_1^*, \ldots, v_6^*)$ is as in that proof, $v_{n-2}^* = -(-2, -1, 0, 1)$, $v_{n-1}^* = (5, -2, 0, 0)$, $v_n^* = (16p - 6, 1, 0, 1)$ and $v_i^* = v_{i-6}^* + (16, 0, 0, 0)$ for all i with $7 \leq i \leq n - 3$ then the Gale transform $(v_1, \ldots, v_n)$ of $(v_1^*, \ldots, v_n^*)$ is valid and has flips only at the pairs (v_q, v_{q+1}) for indices q divisible by 3. Part (b) of Figure 4 shows A^* in the case $p = 2$, with the same conventions adopted for part (a) in the proof of Theorem 5.5.

Proof of Theorem 1.4. Combine Theorems 5.1, 5.2, 5.3 and 5.5. □

In a special case, part (ii) of the following conjecture claims that two flips can be achieved in the case of zonotopes with $3d-6$ generators in any dimension $d \geq 3$.

Conjecture 5.7. *For integers $n \geq d \geq 2$, let $h(n, d)$ denote the minimum number of flips among all monotone paths on d-zonotopes with n generators.*

(i) $h(n,d)$ *is a nondecreasing function of* n *in each fixed corank* $n-d$.
(ii) $h(n,d) = m(n,d)$ *for all* n *and* d *and the minimum can be achieved in general position.*

Clearly, part (ii) of the conjecture implies part (i). Conversely, Theorem 5.3 and part (i) imply that

$$h(n,d) \;\leq\; 2\left\lceil \frac{n}{n-d+2} \right\rceil - 1 \;\leq\; m(n,d)+1$$

for $n \geq d+2$, hence that the lower bound $m(n,d)$ can be off by at most one. Theorem 5.3 implies that part (i) of the conjecture is true in a weak asymptotic sense.

6. Oriented matroids and chains of topes

In this section we adapt the ideas of [1, Section 4] to the setting of oriented matroids and prove Theorem 1.3.

Let $\mathcal{L}$ be an oriented matroid of rank r on the ground set $[n]$, $\mathcal{T}$ be the set of topes of $\mathcal{L}$ and $B \in \mathcal{T}$. We assume without loss of generality that $\mathcal{L}$ has no loops or parallel elements. We first define carefully the graph which appears in the statement of Theorem 1.3. Two maximal chains $B = T_0 < T_1 < \cdots < T_n = -B$ and $B = T'_0 < T'_1 < \cdots < T'_n = -B$ in the tope poset $\mathcal{T}(\mathcal{L}, B)$ are said to differ by an *elementary homotopy* or *deformation* [5, Section 4.4], or *flip* if there exist indices $0 \leq i < j \leq n$ such that $T_k = T'_k$ for all k with either $k \leq i$ or $k \geq j$ and

$$\{T_i, T_{i+1}, \ldots, T_j, T'_{j-1}, \ldots, T'_{i+1}\} = \mathrm{star}(X)$$

for some $X \in \mathcal{L}$ of rank $r-2$, where $\mathrm{star}(X) = \{T \in \mathcal{T} : T \geq X\}$. We denote by $\mathcal{G}(\mathcal{L}, B)$ the graph with vertices the maximal chains of $\mathcal{T}(\mathcal{L}, B)$ and edges defined by the flips. If $\mathcal{L}$ is the realizable oriented matroid associated to the zonotope Z and f is a linear functional on Z taking its minimum at the vertex of Z which corresponds to the tope B then $\mathcal{G}(\mathcal{L}, B)$ is naturally isomorphic to the graph $\mathcal{G}(Z, f)$ of f-monotone paths on Z and polygon moves.

For any closed interval $I = [T, T']$ in $\mathcal{T}(\mathcal{L}, B)$ we can define similarly the graph $\mathcal{G}(\mathcal{L}, I)$ with vertices the maximal chains of I and edges the flips among them. To be more precise, if C and C' are any maximal chains in the intervals $[B, T]$ and $[T', -B]$, respectively, then two maximal chains in I are connected by an edge in $\mathcal{G}(\mathcal{L}, I)$ if their extensions by C and C' are connected by an edge in $\mathcal{G}(\mathcal{L}, B)$. In the special case that $I = \mathcal{T}(\mathcal{L}, B)$, the following result of Cordovil and Moreira [8] states that the graph $\mathcal{G}(\mathcal{L}, B)$ is connected.

Proposition 6.1. ([8] [5, Proposition 4.4.7]) *For any closed interval* I *of* $\mathcal{T}(\mathcal{L}, B)$, *the graph* $\mathcal{G}(\mathcal{L}, I)$ *is connected.*

Let us recall the construction of pullbacks from [1, Section 4]. We will think of graphs as finite one-dimensional simplicial complexes, so that a simplicial map of graphs sends vertices to vertices and edges to either edges or vertices. Furthermore, such a map is surjective if it is surjective both on vertices and edges. Given simplicial maps of graphs $\alpha : G_1 \to H$ and $\beta : G_2 \to H$, we define a graph G as follows. The vertices of G are the ordered pairs (a, b) of vertices of G_1 and G_2, respectively, such that $\alpha(a) = \beta(b)$. Two vertices (a, b) and (a', b') are connected by an edge in G if either

(i) $a = a'$ and $\{b, b'\}$ is an edge of G_2,
(ii) $b = b'$ and $\{a, a'\}$ is an edge of G_1 or
(iii) $\{a, a'\}$ and $\{b, b'\}$ are edges of G_1 and G_2, respectively, which both map homeomorphically onto the same edge of H.

Note that G is the cartesian product of G_1 and G_2 if H has a single vertex. The diagram

$$
\begin{array}{ccc}
G & \longrightarrow & G_2 \\
\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\
G_1 & \xrightarrow{\alpha} & H,
\end{array}
\tag{1}
$$

where the unlabeled arrows denote projections to the first and second coordinate, is called a *pullback diagram.*

The following lemma will be needed, as in [1, Section 4].

Lemma 6.2. ([1, Proposition 4.8]) *Let (1) be a pullback diagram. If*

(i) *G_1 and G_2 are 2-connected,*
(ii) *α and β are surjective and*
(iii) *the fibers $\alpha^{-1}(v), \beta^{-1}(v)$ are connected for every vertex v of H*

then G is 2-connected.

Remember that in Section 2.2 we defined the distance $d(B,T)$ between B and another tope T as the number of coordinates at which the signs of B and T differ. Given $\mathcal{L}$ and B as before, we define graphs H_i, for $0 \le i \le n$, and $H_{i-1,i}$, for $1 \le i \le n$, as follows. The vertices of H_i are the topes T of $\mathcal{L}$ with $d(B,T) = i$ and those of $H_{i-1,i}$ are the subtopes of $\mathcal{L}$ which join a vertex of H_{i-1} to a vertex of H_i. In either case, two vertices X_1 and X_2 are connected by an edge of the graph if there exists a covector X of rank $r-2$ with $X < X_1$ and $X < X_2$ in $\mathcal{L}$. Note that there is a natural surjective simplicial map $\beta_{i-1} : H_{i-1,i} \to H_{i-1}$.

We postpone the proof of the following lemma until the end of this section.

Lemma 6.3. *The graph $H_{i-1,i}$ is $(r-1)$-connected for all $1 \le i \le n$.*

Proof of Theorem 1.3. For $1 \le i \le n$ we define a graph $\mathcal{G}_i$ of partial chains in $\mathcal{T}(\mathcal{L}, B)$ as follows. The vertices of $\mathcal{G}_i$ are the chains $T_0 < T_1 < \cdots < T_i$ of $\mathcal{T}(\mathcal{L}, B)$ with $d(B, T_j) = j$ for all j, where $T_0 = B$ is implied. Two such vertices are connected by an edge in $\mathcal{G}_i$ if they can be extended to maximal chains of $\mathcal{T}(\mathcal{L}, B)$ which are connected by an edge in $\mathcal{G}(\mathcal{L}, B)$. Note that $\mathcal{G}_i = \mathcal{G}(\mathcal{L}, B)$ for $i = n$, so it suffices to prove that $\mathcal{G}_i$ is 2-connected for all i by induction on i.

At the basis of the induction, $\mathcal{G}_1$ is 2-connected by Lemma 6.3, since it is isomorphic to $H_{0,1}$ and $r \ge 3$. Let $1 \le i < n$ and note that projection to the last coordinate defines a surjective simplicial map $\alpha_i : \mathcal{G}_i \to H_i$. The maps $\alpha = \alpha_i$ and $\beta = \beta_i$ give rise to the pullback diagram

$$
\begin{array}{ccc}
\mathcal{G}_{i+1} & \longrightarrow & H_{i,i+1} \\
\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\
\mathcal{G}_i & \xrightarrow{\alpha} & H_i.
\end{array}
\tag{2}
$$

The graph $\mathcal{G}_i$ is 2-connected by induction and $H_{i,i+1}$ is 2-connected by Lemma 6.3 and the assumption $r \geq 3$. The fibers $\alpha^{-1}(v)$ are connected by Proposition 6.1 while connectivity of the fibers $\beta^{-1}(v)$ is the content of [5, Lemma 4.4.4]. Lemma 6.2 then implies that $\mathcal{G}_{i+1}$ is 2-connected. This completes the inductive step and the proof of the theorem. □

The next corollary follows also from the proof of Theorem 1.2 in Section 4.

Corollary 6.4. *If $\mathcal{L}$ is an oriented matroid of rank $r \geq 3$ and B is one if its topes then any maximal chain in the tope poset $\mathcal{T}(\mathcal{L}, B)$ admits at least two flips.*

We now come back to the proof of Lemma 6.3. For the background on regular cell complexes needed we refer to [5, Section 4.7]. We will use in particular the fact that the augmented face poset of a regular cell decomposition of a PL d-sphere is a combinatorial d-manifold, in the sense of [7] (i.e. a finite lattice L of height $d+2$ such that each nonempty open interval in L is either an antichain of two elements or connected with at least two comparable elements). This follows easily from the fact that the class of (regular cell decompositions of) PL spheres is closed under taking links [5, Theorem 4.7.21 (iv)].

Proof of Lemma 6.3. Let $\mathcal{L}_{i-1}$ and $\mathcal{L}_{i-1,i}$ be the order ideals in $\mathcal{L}$ generated by the set R_{i-1} of topes at distance at most $i-1$ from B and the vertex set of $H_{i-1,i}$, respectively. Since R_{i-1} is an order ideal in the tope poset $\mathcal{T}(\mathcal{L}, B)$, it follows from [5, Theorem 4.3.3] and [5, Proposition 4.7.26 (ii)] that $\mathcal{L}_{i-1} - \hat{0}$ is the face poset of a shellable, regular cell decomposition of a PL $(r-1)$-ball. The boundary of this ball is a regular cell decomposition of a PL $(r-2)$-sphere whose face poset is isomorphic to $\mathcal{L}_{i-1,i} - \hat{0}$. It follows that $\mathcal{L}_{i-1,i} \cup \hat{1}$ is a combinatorial $(r-2)$-manifold. A theorem of Barnette [2] (see also [7]) then implies that the graph formed by the top two levels of $\mathcal{L}_{i-1,i}$ is $(r-1)$-connected. This graph coincides with $H_{i-1,i}$ by definition. □

7. Simple zonotopes and Coxeter arrangements

In this section we point out an application of the result of [1] on simple polytopes to finite Coxeter groups.

A d-dimensional polytope P is *simple* if every vertex of P is incident to exactly d edges. If P is a zonotope, this is equivalent to the statement that the polar arrangement of linear hyperplanes is *simplicial*. An important class of simplicial arrangements of hyperplanes is the class of Coxeter arrangements [5, Section 2.3] [13, Chapter 6]. Let W be a finite *Coxeter group*, i.e. a finite group presented by a finite set of generators S and the relations

(i) $s^2 = 1$, $s \in S$ and
(ii) $(ss')^{m(s,s')} = 1$, $s \neq s'$, $s, s' \in S$

for some integers $m(s, s') \geq 2$. The minimum size of such a set S is called the rank of W. Relations (i) and (ii) are known as the *Coxeter relations*. The class of finite Coxeter groups coincides with that of finite reflection groups. The associated reflection arrangement, or *Coxeter arrangement* $\mathcal{A}_W$, is known to be simplicial and hence its polar zonotope is simple. Its dimension is the rank of W. The regions of $\mathcal{A}_W$ are naturally in bijection with the elements of W. Moreover, if B is the region corresponding to the identity element then the maximal chains in the poset of regions of $\mathcal{A}_W$ are in bijection with the reduced decompositions of the highest element w_0 of W, i.e. expressions of minimal length of the form $w_0 = s_1 s_2 \cdots s_\ell$, with $s_i \in S$ for all i. The flips correspond to moves which replace m successive entries $s, s', s, s', \ldots$ with $s', s, s', s, \ldots$, where $s \neq s'$ and $m = m(s, s')$, known as *Coxeter moves.*

The following corollary of Theorem 1.1(iii) strengthens the well known fact (cf. [6, 12]) that any two reduced decompositions of w_0 can be obtained from each other by a sequence of Coxeter moves.

Corollary 7.1. *If W is a finite Coxeter group of rank r and w_0 is its heighest element then the graph of reduced decompositions of w_0 and Coxeter moves is $(r-1)$-connected.*

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Christos A. Athanasiadis, Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm, Sweden
E-mail address: athana@math.kth.se

Francisco Santos, Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria, Santander, E-39005, Spain
E-mail address: santos@matesco.unican.es