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A POINT SET WHOSE SPACE OF TRIANGULATIONS IS DISCONNECTED

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Introduction

In this paper we construct explicitly a triangulation of a 6-dimensional point configuration of 324 points which admits no *geometric bistellar operations* (or *flips*, for short). This triangulation is an isolated element in the *graph of triangulations* of the point configuration. It has been a central open question in polytope combinatorics in the last decade whether point configurations exist for which this graph is not connected (see, e.g. [37, Question 1.2] and [48, Challenge 3]). We also construct a 234-dimensional polytope with 552 vertices whose graph of triangulations has an isolated element.

Our construction is likely to have an impact in algebraic geometry too, via the connections between lattice polytopes and toric varieties [21, 23, 31, 43]. For example, in [2, Section 2] and [24, Section 4] the different authors study algebraic schemes based on the poset of subdivisions of an integer point configuration. The connectivity of these schemes and of the graph of triangulations are equivalent. See Section 4.3, in particular Corollary 4.9.

The graph of triangulations is also related to the Baues poset, which appears in *oriented matroid theory*, *zonotopal tilings* and *hyperplane arrangements*, so our result has implications in these areas.

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Let $\mathcal{A}$ be a finite point set in the real affine space $\mathbb{R}^k$. A *polyhedral subdivision* of $\mathcal{A}$ is a geometric polyhedral complex with vertices in $\mathcal{A}$ which covers the convex hull of $\mathcal{A}$. If all the cells are simplices then it is a *triangulation*. More combinatorial definitions are convenient if $\mathcal{A}$ is not in *convex position*, i.e. if some element of $\mathcal{A}$ is not a vertex of the convex hull. See Definitions 4.1 and 1.1 for details, and also [6], [21, Chapter 7], [36], [47, Chapter 9], or the monograph in preparation [14].

There are at least the following three ways to give a structure to the collection of all triangulations of a point configuration $\mathcal{A}$:

(A) *Flips.* Geometric bistellar operations, or flips, are the minimal changes which can be made in a triangulation of $\mathcal{A}$ to produce a new one. See Definition 1.3. A particular case is the familiar diagonal edge flip in two-dimensional triangulations, of frequent use in computational geometry and geometric combinatorics.

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The *graph of triangulations of* $\mathcal{A}$ has the triangulations of $\mathcal{A}$ as vertices and the flips between them as edges. Graphs of triangulations in dimension two are known to be connected since the early days of computational geometry [25]. For the vertex set of a convex polygon, the graph is a classical object in combinatorics, first studied by Stasheff and Tamari [46, 42] and related to associativity structures and to binary trees [26, 41]. It is disturbing that in dimension three, and even assuming convex position, we do not know whether the graph is always connected or not.

The graph of triangulations of $\mathcal{A}$ contains as an induced subgraph the 1-skeleton of the *secondary polytope* of $\mathcal{A}$ introduced by Gel'fand et al. [20]. This is a polytope of dimension $|\mathcal{A}| - \dim(\mathcal{A}) - 1$ whose vertices are in bijection with the *regular* (or *coherent*) triangulations of $\mathcal{A}$; see [6, 21, 27, 47] and also Section 4.2. Triangulations with less than $|\mathcal{A}| - \dim(\mathcal{A}) - 1$ geometric bistellar operations are called *flip-deficient*. Flip-deficiency cannot occur either in dimension two or in convex position in dimension three [15]. In non-convex position in dimension three there are triangulations with more than n^2 vertices and less than $4n$ flips for arbitrarily large n [38, Section 2]. In dimension four there are triangulations with arbitrarily many vertices and bounded number of flips [38, Sections 3 and 4]. The precise statement of our main result is:

Theorem 1 *There is a triangulation without geometric bistellar neighbors of an integer point configuration* $\mathcal{A}_1 \times \mathcal{A}_2$ *of dimension* 6 *with* 324 *points.* $\mathcal{A}_1 \times \mathcal{A}_2$ *is the product of two point configurations* $\mathcal{A}_1 \subset \mathbb{R}^4$ *and* $\mathcal{A}_2 \subset \mathbb{R}^2$ *with* 81 *and four points respectively. The triangulation has* $9 \times 64 \times 3 \times \binom{6}{2}$ *maximal simplices.*

Jörg Rambau has checked, with his computer program `TOPCOM` [34], that our six dimensional triangulation with the integer coordinates described in Section 3.4 is in fact a triangulation and has no flips. The current release of `TOPCOM` includes files which generate the triangulation and carry out this checking.

The point configuration of Theorem 1 is not in convex position. Only 96 of its 324 points are *extremal*, i.e. vertices in the convex hull. But the "reoriented Lawrence construction" described in [39, Section 4.4] and [15, Proposition 3.3] substitutes any element of a point configuration by two extremal points, increasing the dimension by one and preserving the graph of triangulations and poset of subdivisions. Applying it 228 times we obtain:

Corollary 2 *There is a triangulation without flips of a lattice polytope of dimension* 234 *with* 552 *vertices.*

Connectivity of the graph of triangulations is an important question in computational geometry, where flips are used to enumerate triangulations or to search for optimal ones [12, 17]. But it is also relevant theoretically: if the graph is connected, properties which hold for a particular triangulation and are preserved under flips must hold for any other one. Our negative result contrasts the following two positive results in more algebraic-geometric and topological settings:

Morelli's factorization theorem of toric birational maps [1, 30] has as an implication that any two triangulations of a rational point configuration are connected by geometric bistellar operations if we allow the use of arbitrarily many additional rational vertices in the intermediate steps. (Geometric bistellar flips in triangulations with rational vertices correspond to pairs blow-up/blow-down in toric varieties).

Pachner [32] has shown that any two PL-homeomorphic combinatorial manifolds can be connected by a sequence of the (non-geometric) bistellar operations used in combinatorial topology [10, 19].

(B) *The Baues poset.* The polyhedral subdivisions of $\mathcal{A}$ form a partially ordered set (*poset*) with the refinement relation. Its minimal elements are the triangulations and its unique maximal element is the trivial subdivision. It is usually referred to as the *Baues poset* of $\mathcal{A}$ and its study is the *generalized Baues problem*. We denote this poset by $\Omega(\mathcal{A})$ and call *strict Baues poset* the one obtained by removing from it the trivial subdivision.

To be precise, Baues posets were introduced implicitly in [8] and explicitly in [7] in a more general situation where one has an affine projection π from the vertex set of a polytope P to $\mathcal{A}$. The *Baues poset of* π is the poset of subdivisions of $\mathcal{A}$ which are *induced* by π, meaning that only projections of faces of P are allowed as cells in the subdivisions. (A more careful definition is needed in degenerate cases). When P is a simplex, hence of dimension $|\mathcal{A}|-1$, all subdivisions of $\mathcal{A}$ are π-induced and this is the case of interest to us. When P is a cube its projection is a *zonotope* and the π-induced subdivisions are its *zonotopal tilings* [47]. See [37] for a very complete account of the different contexts in which Baues posets appear.

The *Baues complex* of $\mathcal{A}$ is the *order complex* [9] of the strict Baues poset. That is to say, the abstract simplicial complex whose simplices are the chains in the poset. Billera et al. [7] and Rambau and Ziegler [36] proved, respectively, that if $\dim(\mathcal{A}) = 1$ or $\dim(P) - \dim(\mathcal{A}) = 2$, then the Baues complex of any projection $\pi : \mathrm{vert}(P) \to \mathcal{A}$ is homotopy equivalent to a $(\dim(P) - \dim(\mathcal{A}) - 1)$-sphere. The first case solved a conjecture of Baues [5]. The conjecture that this holds for arbitrary P and $\mathcal{A}$ was since called the *generalized Baues conjecture*, or GBC. It was inspired by the fact that the fiber polytope of the projection [8] (a generalization of the secondary polytope) has dimension $\dim(P) - \dim(\mathcal{A})$ and its face lattice is naturally embedded in the Baues poset. The GBC was disproved in [36], but the particularly interesting cases of P being a simplex or a cube remain open. They have been solved (positively) only if $\dim(\mathcal{A}) = 2$, if $\dim(P) - \dim(\mathcal{A}) = 3$ or if π is the natural projection onto a cyclic polytope or cyclic zonotope [4, 16, 35, 45]. The cube case with $\dim(P) = n$ and $\dim(\mathcal{A}) = d$ is a special case of the simplex case with $\dim(P) = n-1$ and $\dim(\mathcal{A}) = d-1$. See below.

Triangulations of $\mathcal{A}$ and bistellar flips between them are precisely the minimal and next-to-minimal elements in the Baues poset (Lemma 4.2). In particular, connectivity of the graph of triangulations implies connectivity of the Baues complex. The converse is only (almost) true if $\mathcal{A}$ is in general position (Corollary 4.3). In the case of our triangulation without flips we can only say that it is an isolated element in the poset of proper refinements of any minimal element among the subdivisions of $\mathcal{A}$ refined by it. But the strict Baues poset is probably connected.

(C) *The coherent poset of subdivisions.* In Section 4.2 we introduce a modified Baues poset $\Omega_c(\mathcal{A})$. It again contains all the polyhedral subdivisions of $\mathcal{A}$, but a subdivision Π' refining another one Π is considered smaller only if the refinement is coherent in a certain sense, first studied by Alexeev [2, Definition 2.12.10].

The poset $\Omega_c(\mathcal{A})$ is nicer than the Baues poset in the following respects:

- It is the right poset in order to study the algebraic schemes introduced in [2, Section 2] and [24, Section 4]. In particular, the order complex of $\Omega_c(\mathcal{A})$ is connected if and only if the schemes are connected (Corollary 4.9).

- The topology of the whole poset is much more related to that of its lower ideals in the case of $\Omega_c(\mathcal{A})$ than in $\Omega(\mathcal{A})$ (compare Corollaries 4.3 and 4.7). In particular, $\Omega_c(\mathcal{A})$ is connected if and only if the graph of triangulations is connected, and this is true regardless of whether the trivial subdivision is considered an element of the poset or not.

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The structure of the paper is as follows. Section 1 contains preliminaries on triangulations and geometric bistellar operations. The rest of the paper is divided into Sections 2, 3 and 4 which can be read independently. The first two combine to produce Theorem 1.

In Section 2.1 we study the relation between the flips in triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ and the flips in any triangulation which refines the product $\mathcal{T}_1 \times \mathcal{T}_2$. In Section 2.2 we introduce a particular way of refining such a product based on the *staircase triangulation* of the product of two simplices. In Section 2.3 we prove that this refinement has no flips when $\mathcal{T}_1$ is any triangulation satisfying certain hypotheses and $\mathcal{T}_2$ is a very simple triangulation of dimension two (Corollary 2.7).

In Section 3.1 we describe a triangulation of a four-dimensional point configuration with 81 points. Then, we compute its flips (Section 3.2) and introduce a *locally acyclic orientation* of its 1-skeleton (Section 3.3). The construction is based on the one which appeared in Section 4 of [38], a connection detailed in Remark 3.4. The triangulation satisfies the requirements needed in Corollary 2.7. Although in its original form the point configuration is not rational, in Section 3.4 we modify it so that it becomes integer, loosing part of its symmetry but none of its special properties. This gives Theorem 1.

Sections 4.1 and 4.2 deal with the relation between the graph of triangulations and the posets $\Omega(\mathcal{A})$ and $\Omega_c(\mathcal{A})$. The latter is explicitly introduced here for the first time, although it is implicit in [2, Section 2.12]. Section 4.3 shows the relation between $\Omega_c(\mathcal{A})$ and the aforementioned algebraic schemes.

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It would be desirable to improve our construction of a triangulation without flips in any of the following ways, in case this is possible:

Smaller dimension: Connectivity of the graph of triangulations and the Baues poset is unknown starting in dimension three. Theorem 2.6 cannot provide a triangulation without flips in dimension less than five: every locally acyclic orientation of the graph of a planar triangulation has reversible edges [38, Remark 9] and a 1-dimensional triangulation always contains sinks, sources and "sandwich edges". In dimension three, we know triangulations which admit locally acyclic orientations without reversible edges, sinks or sources (Remark 3.4). But we have not obtained one whose flippable circuits are "sandwich circuits", as required in Corollary 2.7.

It is also interesting to know whether triangulations without flips exist in convex position and reasonably small dimension. Flip-deficiency occurs in convex position starting in dimension four [15].

Small size or corank: By corank we mean the dimension $|\mathcal{A}| - d - 1$ of the secondary polytope. The fact that the GBC is false already for the projection of a 5-dimensional polytope with 10 vertices onto dimension 2 [36] indicates that the simplex case might also be false in relatively small examples.

General position: A point configuration of dimension d is said to be in *general position* if every $d+1$ points are affinely independent.

Flip-deficient triangulations in general position exist starting in dimension three [15]. A triangulation without flips of a point configuration $\mathcal{A}$ in general position would disprove the GBC for projections from simplices (Corollary 4.3). The techniques in this paper cannot be adapted to general position in any obvious way, since they are based on taking products. Actually, we have reasons to believe that any perturbation of our triangulation into general position creates flips.

Primitiveness: We say that $\mathcal{A}$ is primitive if $\mathcal{A} = \mathrm{conv}(\mathcal{A}) \cap \mathbb{Z}^k$. This is of interest in the algebraic-geometric context, since it is equivalent to normality of the corresponding affine toric variety. See Corollary 4.9, part (iv).

Unimodular configurations and $\mathcal{A}$-graded ideals: Let $\mathcal{A} \subset (\mathbb{Z}_{\geq 0})^k$ be a non-negative, integer point configuration. Sturmfels [43, Chapter 10] has shown that the radical of every *monomial $\mathcal{A}$-graded ideal* (*mono-$\mathcal{A}$-GI* for short) equals the Stanley ideal of some triangulation of $\mathcal{A}$. The converse is not always true: the Stanley ideal of a triangulation may not be the radical of any mono-$\mathcal{A}$-GI. We do not know whether our triangulation without flips has this property.

Specially interesting in this context is the case when $\mathcal{A}$ is *unimodular*, meaning that all its affine bases have the same determinant, up to a sign. Then the Stanley ideal of any triangulation of $\mathcal{A}$ is a mono-$\mathcal{A}$-GI and every mono-$\mathcal{A}$-GI is radical [43, Lemma 10.14]. Our point configuration is not unimodular.

The $\mathcal{A}$-Graded ideals are the closed points of the toric Hilbert scheme of $\mathcal{A}$, defined by Peeva and Stillman [33] (see also [44, Section 5]). Using a notion of flip between mono-$\mathcal{A}$-GI's, Maclagan and Thomas [28] have shown that a triangulation without flips whose Stanley ideal equals the radical of some mono-$\mathcal{A}$-GI would imply that the toric Hilbert scheme is not connected.

Lifting triangulations and Lawrence polytopes; relation to zonotopal tilings and oriented matroids: Let $\mathcal{A}$ be a point configuration, let $\mathcal{A}^*$ denote a Gale transform of $\mathcal{A}$ and let $\mathcal{M}(\mathcal{A}^*)$ be the *oriented matroid* of affine dependences in $\mathcal{A}^*$ (roughly speaking, its set of circuits). Every generic one-element extension of $\mathcal{M}(\mathcal{A}^*)$ induces a triangulation of $\mathcal{A}$. The triangulations which can be obtained in this way are called *lifting triangulations* (see [11, Section 9.6] or [39, Section 4]). It can be easily proved that our triangulation is not lifting.

Specially interesting in this context is the case when the Gale transform $\mathcal{A}^*$ is centrally symmetric, i.e. when $\mathcal{A}$ is the vertex set of a *Lawrence polytope* [11, 47]. Then all its triangulations are lifting (see [39, Section 4.3] or [22, Section 4]). Moreover, the poset of subdivisions of a Lawrence polytope is isomorphic to the poset of zonotopal tilings of a certain zonotope, and vice versa.

A lifting triangulation without flips would imply that the graph of cubical flips between zonotopal tilings of a certain zonotope is disconnected, thus answering question 1.3 in [37]. If, moreover, the triangulation is in general position, it would disprove the GBC for projections from cubes, which by the above mentioned results is equivalent to the GBC for the projections from simplices to Lawrence polytopes.

Via the Bohne-Dress Theorem [11, 47], these two cases of the GBC are also equivalent to the *extension space conjecture* of oriented matroid theory [11, pp. 295–296], stating that the poset of one-element extensions of a realizable oriented matroid of rank r is homotopy equivalent to an $(r-1)$-sphere.

Finally, the extension space conjecture is the case $k = d-1$ of the following far-reaching conjecture by MacPherson, Mnëv and Ziegler [37, Conjecture 11]: that the poset of all strong images of rank k of any realizable oriented matroid $\mathcal{M}$ of rank d (the *OM-Grassmannian of rank* k of $\mathcal{M}$) is homotopy equivalent to the real Grassmannian $G^k(\mathbb{R}^d)$. This conjecture is relevant in matroid bundle theory [3] and the *combinatorial differential geometry* introduced by MacPherson [29].

1. Triangulations and flips

Let $\mathcal{A}$ be a finite subset of the real affine space $\mathbb{R}^k$ and let d denote the dimension of the affine subspace $\operatorname{aff}(\mathcal{A})$ spanned by $\mathcal{A}$. This is what we call a *point configuration of dimension d.*

Definition 1.1. A *triangulation* of $\mathcal{A}$ is any collection $\mathcal{T}$ of affinely independent subsets of $\mathcal{A}$ with the following properties: (i) if S is in $\mathcal{T}$, then every subset of S is in $\mathcal{T}$, i.e. $\mathcal{T}$ is an abstract simplicial complex; (ii) if S and S' are in $\mathcal{T}$, then $\operatorname{conv}(S) \cap \operatorname{conv}(S')$ is a face of both $\operatorname{conv}(S)$ and $\operatorname{conv}(S')$, i.e. $\mathcal{T}$ induces a geometric simplicial complex in $\mathbb{R}^k$; (iii) $\cup_{S\in\mathcal{T}}\operatorname{conv}(S) = \operatorname{conv}(\mathcal{A})$, i.e. $\mathcal{T}$ covers the convex hull of $\mathcal{A}$.

We will call *simplices* of $\mathcal{T}$ all its elements, *maximal simplices* of $\mathcal{T}$ the elements of dimension d and *facets* of $\mathcal{T}$ the facets of the maximal simplices, i.e. the simplices of dimension $d-1$. This deviates from standard use in simplicial complexes, where the facets are the maximal simplices. With our conventions, every facet of $\operatorname{conv}(\mathcal{A})$ is triangulated by facets of $\mathcal{T}$. A facet is *interior* if its convex hull intersects the interior of $\operatorname{conv}(\mathcal{A})$ and is a *boundary facet* otherwise.

It has some combinatorial advantages to consider as elements of $\mathcal{T}$ only the maximal simplices, as is done in [6, 13, 21]. Here (but not in Section 4) we prefer to use the convention that lower dimensional ones are also elements, to work more easily with the links and stars of simplices. If $S \in \mathcal{T}$, then $\operatorname{star}_{\mathcal{T}}(S) = \{S' \in \mathcal{T} : S \cup S' \in \mathcal{T}\}$ and $\operatorname{link}_{\mathcal{T}}(S) = \{S' \in \mathcal{T} : S \cup S' \in \mathcal{T}, S \cap S' = \emptyset\}$.

Lemma 1.2 ([13])**.** *Let $\mathcal{A}$ be a point configuration and let $\mathcal{T}$ be a pure abstract simplicial complex of dimension* $\dim(\mathcal{A})$ *with vertices contained in $\mathcal{A}$ and containing only affinely independent subsets. $\mathcal{T}$ is a triangulation of $\mathcal{A}$ if and only if:*

(i) *The link of every interior facet S of $\mathcal{T}$ has exactly two elements, which lie on opposite sides of the hyperplane of* $\operatorname{aff}(\mathcal{A})$ *spanned by S.*
(ii) *There exists a point in* $\operatorname{conv}(\mathcal{A})$ *which lies in the convex hull of exactly one maximal simplex of $\mathcal{T}$.* □

Geometric bistellar operations arose in the following terms in the work of Gel'fand, Kapranov and Zelevinsky [20] [21, Chapter 7], who called them *modifications.*

Following the terminology of matroid theory, we call a minimal affinely dependent subset of $\mathcal{A}$ a *circuit* (see [11] or [47] for details). The *Radon partition* of a circuit Z is the unique partition $Z = Z_+ \cup Z_-$ such that the convex hulls of the two parts intersect. Equivalently, Z_+ and Z_- contain respectively the elements with positive and negative coefficient in the unique (up to a scalar multiple) affine dependence equation on Z. The pair (Z_+, Z_-) is called an *oriented circuit*; of course, if (Z_+, Z_-) is an oriented circuit, then so is (Z_-, Z_+), and the two of them are the only orientations of $Z = Z_+ \cup Z_-$. Since our interest will always be in oriented circuits we will use the word circuits for the pairs (Z_+, Z_-) and will call the underlying unoriented circuit the *support* of (Z_+, Z_-).

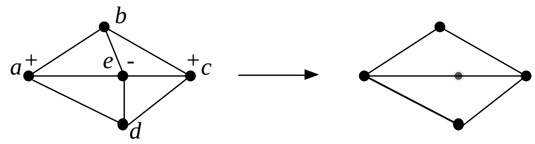

FIGURE 1. A flip supported on the circuit $(\{a,c\},\{e\})$. It has two flippable facets, $\{b,e\}$ and $\{d,e\}$. The reverse flip, supported on $(\{e\},\{a,c\})$, has no flippable facets.

The support Z of a circuit (Z_+, Z_-) admits exactly two triangulations $\mathcal{T}_+(Z)$ and $\mathcal{T}_-(Z)$, which have Z_+ and Z_- as their unique minimal non-faces. I.e:

$$\mathcal{T}_+(Z) := \{S \subseteq Z : Z_+ \not\subseteq S\}, \qquad \mathcal{T}_-(Z) := \{S \subseteq Z : Z_- \not\subseteq S\}.$$

Definition 1.3. Let $\mathcal{T}$ be a triangulation of $\mathcal{A}$ and let $(Z_+, Z_-) \subseteq \mathcal{A}$ be a circuit of $\mathcal{A}$. Suppose that the following conditions are satisfied:

(i) The triangulation $\mathcal{T}_+(Z)$ is a subcomplex of $\mathcal{T}$.
(ii) All the maximal simplices of $\mathcal{T}_+(Z)$ have the same link L in $\mathcal{T}$. In particular, $\mathcal{T}_+(Z) * L$ is a subcomplex of $\mathcal{T}$. Here $A * B := \{S \cup T : S \in A, T \in B\}$.

Then, we can obtain a new triangulation $\mathcal{T}'$ of $\mathcal{A}$ by replacing the subcomplex $\mathcal{T}_+(Z) * L$ of $\mathcal{T}$ by the complex $\mathcal{T}_-(Z) * L$. This operation of changing the triangulation is called a *geometric bistellar operation*, *geometric bistellar flip* (or a *flip*, for short) supported on the circuit (Z_+, Z_-). We call (Z_+, Z_-) a *flippable circuit* and we say that $\mathcal{T}$ and $\mathcal{T}'$ are *geometric bistellar neighbors*.

Our definition of flip supported on a circuit explicitly assumes that the circuit is oriented so that the star of the *negative part* is "flipped out" and the star of the *positive part* is "flipped in". This convention is not made by other authors (see [21, page 231]) and will be relevant in our exposition. See [40] for a generalization of the concept of flip in the framework of Baues posets and fiber polytopes.

It is sometimes more convenient to focus on *flippable facets*, which we now introduce, instead of *flippable circuits*. Let S be the common facet of two maximal simplices $S \cup \{a_1\}$ and $S \cup \{a_2\}$ of a triangulation $\mathcal{T}$. The $d+2$ elements of $S \cup \{a_1, a_2\}$ affinely span $\mathrm{aff}(\mathcal{A})$. Hence, there is a unique circuit (Z_+, Z_-) with support contained in them. This circuit can be oriented so that $a_1, a_2 \in Z_+$. We say that (Z_+, Z_-) is the circuit *supported on* S. We say that S is a *flippable facet of* $\mathcal{T}$ if there is a flip supported on (Z_+, Z_-). See an example in Figure 1.

In the following statement, if $(Z_+ \cup Z_-)$ is full-dimensional, then the link L of $\mathcal{T}_+(Z)$ has one maximal simplex: the empty set.

Lemma 1.4. *Let $\mathcal{T}$ be a triangulation of a point configuration $\mathcal{A}$ of dimension d. Let (Z_+, Z_-) be a flippable circuit, with $|Z_+| = k$. Let l be the number of maximal simplices in the common link L of $\mathcal{T}_+(Z)$ in $\mathcal{T}$. Then, (Z_+, Z_-) is supported on $l\binom{k}{2}$ facets of $\mathcal{T}$. In particular:*

(i) *If $\mathcal{T}$ uses all the elements of $\mathcal{A}$ as vertices, then every flippable circuit is supported on at least one flippable facet.*
(ii) *If $\mathrm{conv}(Z_+ \cup Z_-)$ has dimension d' and intersects the interior of $\mathrm{conv}(\mathcal{A})$, then (Z_+, Z_-) is supported on at least $(d+1-d')\binom{k}{2}$ flippable facets.*

Proof. The flippable facets of $\mathcal{T}$ on which the circuit (Z_+, Z_-) is supported are those of the form $S * T$ where S is an interior facet of $\mathcal{T}_+(Z)$ and T is a maximal simplex in L. The triangulation $\mathcal{T}_+(Z)$ has $\binom{k}{2}$ interior facets since it has k maximal simplices and every pair of them are adjacent.

If every element of $\mathcal{A}$ is a vertex in $\mathcal{T}$, then Z_+ has at least two elements. If $\mathrm{conv}(Z_+ \cup Z_-)$ intersects the interior of $\mathrm{conv}(\mathcal{A})$, then L is a triangulation of a $(d-d')$-sphere and, hence, it has at least $d+1-d'$ maximal simplices. □

2. Refining the product of two triangulations

Throughout this section, let $\mathcal{T}_1$ and $\mathcal{T}_2$ be triangulations using all the elements of respective point configurations $\mathcal{A}_1$ of dimension d_1 and $\mathcal{A}_2$ of dimension d_2. Their product $\mathcal{T}_1 \times \mathcal{T}_2$ is the polyhedral subdivision of $\mathcal{A}_1 \times \mathcal{A}_2$ into products $S_1 \times S_2$ of simplices $S_1 \in \mathcal{T}_1$ and $S_2 \in \mathcal{T}_2$. Clearly, $\mathcal{T}_1 \times \mathcal{T}_2$ uses all the elements of $\mathcal{A}_1 \times \mathcal{A}_2$.

2.1. Flippable circuits in refinements of $\mathcal{T}_1 \times \mathcal{T}_2$.

Here we let $\mathcal{T}$ be any triangulation of $\mathcal{A}_1 \times \mathcal{A}_2$ refining $\mathcal{T}_1 \times \mathcal{T}_2$. We are going to study the relation between the flips in $\mathcal{T}$ and the flips in $\mathcal{T}_1$ and $\mathcal{T}_2$. We consider the two natural projections $\pi_1 : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_1$ and $\pi_2 : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_2$ and will often use the following properties.

Lemma 2.1. (i) $\mathcal{T}_1 = \{\pi_1(S) : S \in \mathcal{T}\}$ *and* $\mathcal{T}_2 = \{\pi_2(S) : S \in \mathcal{T}\}$.

(ii) *For every simplex* $S \in \mathcal{T}$ *and for* $i \in \{1,2\}$*:*

$$\mathrm{star}_{\mathcal{T}_i}(\pi_i(S)) = \pi_i(\mathrm{star}_{\mathcal{T}}(S)) \quad \text{and} \quad \mathrm{link}_{\mathcal{T}_i}(\pi_i(S)) \subseteq \pi_i(\mathrm{link}_{\mathcal{T}}(S)). \qquad \square$$

By an *affine equality* on a point configuration $\mathcal{A}$ we mean a valid expression of the form $\sum_{a \in Z_+} \lambda_a a = \sum_{b \in Z_-} \mu_b b$, in which Z_+ and Z_- are disjoint subsets of $\mathcal{A}$, all the coefficients λ_a and μ_b are strictly positive reals, and $\sum_a \lambda_a = \sum_b \mu_b$. For example, there is a unique affine equality (up to a scalar multiple) associated to any circuit (Z_+, Z_-) of $\mathcal{A}$.

Any affine equality C on $\mathcal{A}_1 \times \mathcal{A}_2$ is projected to affine equalities on $\mathcal{A}_i$, that we denote $\pi_i(C)$ $(i \in \{1,2\})$, in the following natural way: substitute every point $(a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ which appears in the equality by its projection a_i and combine the coefficients of points which at the end appear more than once. One or both of the projected equalities might result in a trivial equality $0 = 0$. This happens, for example, for the equality $(a, a') + (b, b') = (a, b') + (b, a')$ for any given points $a, b \in \mathcal{A}_1$ and $a', b' \in \mathcal{A}_2$.

We are interested in the affine equalities on $\mathcal{A}_1 \times \mathcal{A}_2$ associated to the flippable circuits of $\mathcal{T}$. It is more convenient to look at flippable facets of $\mathcal{T}$ instead, which can be done by part (i) of Lemma 1.4. Interior facets of $\mathcal{T}$ fall into the following three categories:

(A) *Facets which are interior to a maximal cell* $S_1 \times S_2$ *of* $\mathcal{T}_1 \times \mathcal{T}_2$. The points involved in the equality are contained in $S_1 \times S_2$, whose two projections are affinely independent. Hence, the projected equalities are trivial.

(B_1) *Facets common to two maximal cells* $S_1 \times S_2$ *and* $S'_1 \times S_2$ *of* $\mathcal{T}_1 \times \mathcal{T}_2$ *with the same second factor.* Now the points involved in the equality are contained in $(S_1 \cup S'_1) \times S_2$. The second projection has to be a trivial equality as in the previous case. The first projection could in principle be trivial or can be the equality associated to the circuit contained in $S_1 \cup S'_1$. Lemmas 2.2 and 2.3 below show that the second is always the case and that the circuit in question is flippable in $\mathcal{T}_1$, if the original facet is flippable in $\mathcal{T}$. See Figure 2.

(B_2) *Facets common to two maximal cells $S_1 \times S_2$ and $S_1 \times S_2'$ of $\mathcal{T}_1 \times \mathcal{T}_2$ with the same first factor.* This case is analogous to (B_1).

Lemma 2.2. *Let S be an interior facet common to two maximal cells $S_1 \times S_2$ and $S_1' \times S_2$ of $\mathcal{T}_1 \times \mathcal{T}_2$ with the same second factor. Let (Z_+, Z_-) be the circuit supported on that facet of $\mathcal{T}$ and let C denote the affine equality associated to the circuit. Then, $\pi_1(C)$ is the affine equality on a circuit (Z_+', Z_-') of $\mathcal{A}_1$, with $Z_+' \subseteq \pi_1(Z_+)$ and $Z_-' \subseteq \pi_1(Z_-)$.*

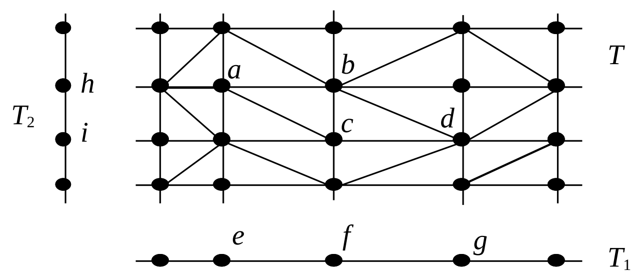

FIGURE 2. The circuit $(\{a, d\}, \{b, c\})$ of $\mathcal{T}$ projects to the circuit $(\{e, g\}, \{f\})$ of $\mathcal{T}_1$. Here $S_1 = \{e, f\}$, $S_1' = \{f, g\}$ and $S_2 = \{h, i\}$.

Proof. Let Z_+' and Z_-' be the sets of points on one and the other side of the projected equality $\pi_1(C)$. The conditions $Z_+' \subseteq \pi_1(Z_+)$ and $Z_-' \subseteq \pi_1(Z_-)$ are obvious. Let (a, b) and (a', b') be the two points in $\text{link}_{\mathcal{T}}(S)$. Clearly $a \neq a'$ and they are not in $\pi_1(S)$. Hence, a and a' appear in $\pi_1(C)$ and $\pi_1(C)$ is not trivial. Since $Z_+' \cup Z_-' \subseteq \pi_1(Z_+ \cup Z_-)$ is contained in the union of two adjacent simplices of $\mathcal{T}_1$, (Z_+', Z_-') must be a circuit. ☐

Lemma 2.3. *In the conditions of Lemma 2.2, if (Z_+, Z_-) supports a flip of $\mathcal{T}$, then:*

- (i) *For any $a \in \pi_1(Z_+)$ there is a unique $b \in \mathcal{A}_2$ such that $(a, b) \in Z_+$ and there is no $b \in \mathcal{A}_2$ such that $(a, b) \in Z_-$.*
- (ii) *$Z_+' = \pi_1(Z_+)$ and $Z_-' = \pi_1(Z_-)$.*
- (iii) *(Z_+', Z_-') supports a flip of $\mathcal{T}_1$.*
- (iv) *There is no pair of elements $a \in Z_+'$ and $b \in \mathcal{A}_2$ such that (a, b) is in the common link in $\mathcal{T}$ of the positive triangulation of (Z_+, Z_-).*
- (v) *$\pi_2(Z_+) = \pi_2(Z_-)$.*

Proof. We denote $Z = Z_+ \cup Z_-$ and $Z' = Z_+' \cup Z_-'$.

Let $a \in \pi_1(Z_+)$. Let $b \in \mathcal{A}_2$ with $(a, b) \in Z_+$. Since (Z_+, Z_-) is flippable, $Z \setminus (a, b)$ is a simplex of $\mathcal{T}$ and hence projects to a simplex of $\mathcal{T}_1$. On the other hand, Z projects to a dependent set. This implies that no other point (a, b'') is contained in Z because otherwise $\pi_1(Z) = \pi_1(Z \setminus (a, b))$. This finishes part (i).

The containments $Z_+' \subseteq \pi_1(Z_+)$ and $Z_-' \subseteq \pi_1(Z_-)$ appear in the statement of Lemma 2.2. Their converses hold because any element in $\pi_1(Z_+) \setminus Z_+'$ or in $\pi_1(Z_-) \setminus Z_-'$ must appear as the first coordinate of points both in Z_+ and Z_-, contradicting part (i). This proves part (ii).

Let us now see part (iii). By parts (i) and (ii), for any $a \in Z'_+$ there is a $b \in \mathcal{A}_2$ with $(a,b) \in Z_+$ and with $Z' \setminus a = \pi_1(Z \setminus (a,b))$. Since (Z_+, Z_-) is flippable in $\mathcal{T}$, $Z' \setminus a$ is in $\mathcal{T}_1$. Hence, $\mathcal{T}_1$ contains as a subcomplex the positive triangulation of Z'.

Let a and a' be two different elements in Z'_+. We have to prove that $\operatorname{link}_{\mathcal{T}_1}(Z' \setminus a) = \operatorname{link}_{\mathcal{T}_1}(Z' \setminus a')$, for which we check only one of the two containments. Let $b, b' \in \mathcal{A}_2$ with $(a,b), (a',b') \in Z_+$, as before. We have $\operatorname{link}_{\mathcal{T}_1}(Z' \setminus a) = \operatorname{link}_{\mathcal{T}_1}(\pi_1(Z \setminus (a,b))) \subseteq \pi_1(\operatorname{link}_{\mathcal{T}}(Z \setminus (a,b))) = \pi_1(\operatorname{link}_{\mathcal{T}}(Z \setminus (a',b')))$, where the first equality holds by part (ii) of the statement, the containment in the middle by part (ii) of Lemma 2.1 and the last equality because (Z_+, Z_-) is flippable. With similar arguments $\pi_1(\operatorname{link}_{\mathcal{T}}(Z \setminus (a',b'))) \subseteq \operatorname{star}_{\mathcal{T}_1}(\pi_1(Z \setminus (a',b'))) = \operatorname{star}_{\mathcal{T}_1}(Z' \setminus a')$. In conclusion, $\operatorname{link}_{\mathcal{T}_1}(Z' \setminus a) \subseteq \operatorname{star}_{\mathcal{T}_1}(Z' \setminus a')$. Since no element of $Z' \setminus a'$ can appear in $\operatorname{link}_{\mathcal{T}_1}(Z' \setminus a)$, we have $\operatorname{link}_{\mathcal{T}_1}(Z' \setminus a) \subseteq \operatorname{link}_{\mathcal{T}_1}(Z' \setminus a')$.

For part (iv), if (a,b') is a point in the common link of the positive triangulation of (Z_+, Z_-) with $a \in Z'_+ = \pi_1(Z_+)$, then let $(a,b) \in Z_+$ as before. We have that $Z \cup (a,b') \setminus (a,b)$ is a simplex in $\mathcal{T}$. But $Z' = \pi_1(Z) = \pi_1(Z \cup (a,b') \setminus (a,b))$. This is impossible since Z' is dependent in $\mathcal{A}_1$.

Part (v) is trivial: the second projection of the affine equality on the circuit (Z_+, Z_-) is the trivial equality. This implies that every element in $\pi_2(Z)$ is the second coordinate of points in both Z_+ and Z_-. □

2.2. **The staircase refinement of $\mathcal{T}_1 \times \mathcal{T}_2$.** We first recall what the staircase triangulation of the product of two simplices is. Let S_1 and S_2 be two simplices of dimensions d_1 and d_2, and suppose that their vertices are given in a specific order. The total order on the vertices of S_1 and S_2 induces the following partial order on the vertices of the product $S_1 \times S_2$: $(a,b) \le (a',b')$ if and only if $a \le a'$ and $b \le b'$. Every chain in this partial order is an affinely independent subset of $S_1 \times S_2$. The collection of all such chains is a triangulation of $S_1 \times S_2$, described for example in [21, Section 7.3.D] and [27, p. 282] and used also in algebraic topology.

All the maximal simplices contain the vertices (a_{min}, b_{min}) and (a_{max}, b_{max}), where a_{min} and a_{max} (respectively b_{min} and b_{max}) are the minimum and maximum vertices of S_1 (respectively of S_2). More visually, if we arrange the vertices of $S_1 \times S_2$ in a $(d_1+1) \times (d_2+1)$ rectangular grid, then the maximal simplices of our triangulation are the monotone staircases from (a_{min}, b_{min}) to (a_{max}, b_{max}). For this reason we call this triangulation the *staircase triangulation* of $S_1 \times S_2$ and we denote it $\operatorname{stair}(S_1 \times S_2)$. We will encounter an example of a staircase triangulation in Remark 3.4.

Any pair of adjacent maximal simplices in $\operatorname{stair}(S_1 \times S_2)$ differ in the replacement of a point (a,b') by a point (a',b), for consecutive pairs of vertices $a < a'$ and $b < b'$ in the total orderings of S_1 and S_2. In particular, every flip of $\operatorname{stair}(S_1 \times S_2)$ is supported on a circuit $(\{(a,b'),(a',b)\}, \{(a,b),(a',b')\})$, for consecutive pairs of vertices $a < a'$ and $b < b'$. The converse is also true although we do not need it: $\operatorname{stair}(S_1 \times S_2)$ has exactly $d_1 \times d_2$ flips, supported on the circuits of that form.

For the sequel, it is convenient to reformulate the construction of the staircase triangulation as follows: We will express the ordering of the vertices of S_1 and S_2 by giving an acyclic orientation to their respective 1-skeletons. Then, the circuits which produce flips are those of the form $(\{(a,b'),(a',b)\}, \{(a,b),(a',b')\})$ where $\{a,a'\}$ and $\{b,b'\}$ are edges of S_1 and S_2 whose reversal produces no directed cycle. We call such edges *reversible edges*.

Suppose now that we have two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of two point configurations $\mathcal{A}_1$ and $\mathcal{A}_2$ as in Section 2.1, and that we are given respective *locally acyclic* orientations of the 1-skeletons of $\mathcal{T}_1$ and $\mathcal{T}_2$. By this we mean orientations which are acyclic on every simplex. Then, refining each product $S_1 \times S_2 \in \mathcal{T}_1 \times \mathcal{T}_2$ in its staircase way (according to the given locally acyclic orientations) produces a triangulation of $\mathcal{A}_1 \times \mathcal{A}_2$. That this is indeed a triangulation follows from the fact that in the staircase triangulation of the product of two simplices $S_1 \times S_2$, each face $F_1 \times F_2$ is triangulated according to the staircase triangulation corresponding to the restrictions of the orderings of S_1 to F_1 and of S_2 to F_2.

We call this triangulation the *staircase refinement* of $\mathcal{T}_1 \times \mathcal{T}_2$ and denote it $\mathrm{stair}(\mathcal{T}_1 \times \mathcal{T}_2)$. In our notation, $\mathcal{T}_1$ and $\mathcal{T}_2$ represent not only the triangulations but also specific locally acyclic orientations of their 1-skeletons, since the staircase refinement depends on them. In [18, p. 67], a simplicial complex together with a locally acyclic orientation is called an *ordered simplicial complex* and the (combinatorial) staircase refinement of two ordered simplicial complexes is called the *Cartesian product* of them. It is used as the standard way to give the structure of a simplicial complex to the product of (the underlying spaces of) two simplicial complexes.

The analysis of flippable circuits carried out in Section 2.1 gives the following:

Lemma 2.4. *If the locally acyclic orientation of at least one of $\mathcal{T}_1$ or $\mathcal{T}_2$ does not have any reversible edge, then every flippable circuit of* $\mathrm{stair}(\mathcal{T}_1 \times \mathcal{T}_2)$ *projects, in the sense of Lemmas 2.2 and 2.3, either to a flippable circuit of $\mathcal{T}_1$ or to a flippable circuit of $\mathcal{T}_2$.*

Proof. Recall the three types (A), (B_1) and (B_2) of possible interior facets in a refinement of $\mathcal{T}_1 \times \mathcal{T}_2$ that we mentioned before Lemma 2.2. Those of types (B_1) or (B_2) project to flippable facets of $\mathcal{T}_1$ or $\mathcal{T}_2$ by Lemmas 2.2 and 2.3.

Hence, we only need to deal with facets of type (A), i.e. facets interior to a cell $S_1 \times S_2$ of $\mathcal{T}_1 \times \mathcal{T}_2$. A flip in one of these facets will be in particular a flip in the staircase triangulation of $S_1 \times S_2$. Such a flip is supported in a circuit of the form $(\{(a,b'),(a',b)\},\{(a,b),(a',b')\})$ where $\{a,a'\}$ is a reversible edge in the orientation of S_1 and $\{b,b'\}$ a reversible edge in the orientation of S_2. For this circuit to be flippable it is clearly necessary (and sufficient, although we do not need it) that both edges $\{a,a'\}$ and $\{b,b'\}$ be reversible *in every simplex* of $\mathcal{T}_1$ and $\mathcal{T}_2$ in which they appear. $\square$

2.3. **Towards a triangulation without flips.** Let $\mathcal{T}$ be a triangulation of a point configuration $\mathcal{A}$, with its 1-skeleton oriented locally acyclically. A *source* (resp. a *sink*) of $\mathcal{T}$ is a vertex which is the source (resp. the sink) of every simplex incident to it. An edge of $\mathcal{T}$ will be called a *sandwich edge* if its source and sink are the source and sink of every simplex containing the edge.

Let (Z_+, Z_-) be a flippable circuit of $\mathcal{T}$ and suppose that Z_+ has at least two elements. This is automatic if all the points of $\mathcal{A}$ are vertices in $\mathcal{T}$. Then, every pair $\{a_+, a_-\}$ with $a_+ \in Z_+$ and $a_- \in Z_-$ is an edge in $\mathcal{T}$.

Definition 2.5. In the above conditions, we say that an element $a_+ \in Z_+$ is a *source* (respectively, a *sink*) in (Z_+, Z_-) if it is the source (respectively, the sink) in all the edges $\{a_+, a_-\}$ with $a_- \in Z_-$. We say that

- (Z_+, Z_-) is a *source circuit* (respectively, *a sink circuit*) if every $a_+ \in Z_+$ is a source (respectively, a sink) in the circuit.

- (Z_+, Z_-) is a *sandwich circuit* if every $a_+ \in Z_+$ is either a source or a sink in the circuit and there is at least one of each.

Theorem 2.6. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be triangulations with their 1-skeletons oriented in locally acyclic ways. Let (Z_+, Z_-) be a circuit which supports a flip in* $\mathrm{stair}(\mathcal{T}_1 \times \mathcal{T}_2)$ *and suppose that it projects onto a flippable circuit (Z'_+, Z'_-) of $\mathcal{T}_1$, in the sense of Lemmas 2.2 and 2.3.*

(i) *If (Z'_+, Z'_-) is a sink circuit, then $\pi_2(Z_+)$ is a sink vertex of $\mathcal{T}_2$.*
(ii) *If (Z'_+, Z'_-) is a source circuit, then $\pi_2(Z_+)$ is a source vertex of $\mathcal{T}_2$.*
(iii) *If (Z'_+, Z'_-) is a sandwich circuit, then $\pi_2(Z_+)$ is a sandwich edge of $\mathcal{T}_2$.*

Proof. Let S_2 be any simplex in $\mathcal{T}_2$ containing $\pi_2(Z_+)$. For any $a \in Z'_+$ there is a unique point $b \in \pi_2(Z_+)$ with $(a,b) \in Z_+$, by part (i) of Lemma 2.3. We will prove that if a is a source in (Z'_+, Z'_-) then b is the source of S_2. This and the analog statement when a is a sink imply parts (i), (ii) and (iii).

Let b_0 be the source of S_2. Let $a' \in Z'_+ \setminus \{a\}$, so that $S_1 := Z'_+ \cup Z'_- \setminus \{a'\}$, is a simplex in $\mathcal{T}_1$. Let b' be such that $(a', b') \in Z_+$. Since the staircase triangulation of $S_1 \times S_2$ is a subcomplex of $\mathrm{stair}(\mathcal{T}_1 \times \mathcal{T}_2)$ and since (Z_+, Z_-) is flippable, there is a maximal staircase S in $S_1 \times S_2$ containing $Z_+ \cup Z_- \setminus \{(a', b')\}$.

The sources of (Z'_+, Z'_-) contained in S_1 form an initial segment of S_1, and in each of their columns there is only one element of S by parts (i) and (iv) of Lemma 2.3. Hence, for any source a'' of (Z'_+, Z'_-) contained in S_1 (in particular for a), (a'', b_0) is in Z_+. ☐

Corollary 2.7. *Let $\mathcal{T}_1$ be a triangulation using all the points of a point configuration $\mathcal{A}_1$ and whose 1-skeleton has been oriented in a locally acyclic way, without reversible edges or global sinks, and with the property that all its flippable circuits are sandwich circuits.*

Let $\mathcal{A}_2 = \{o, p, q, r\} \subset \mathbb{Z}^2$ with $o = (0,0)$, $p = (1,0)$, $q = (0,1)$ and $r = (-1,-1)$. Let $\mathcal{T}_2$ be the triangulation of $\mathcal{A}$ whose maximal simplices are $\{o,p,q\}$, $\{o,p,r\}$, and $\{o,q,r\}$ with its 1-skeleton oriented by $o \to p$, $o \to q$, $o \to r$ and $p \to q \to r \to p$.

Then, the staircase refinement $\mathrm{stair}(\mathcal{T}_1 \times \mathcal{T}_2)$ *of $\mathcal{T}_1 \times \mathcal{T}_2$ does not have any flips.*

Proof. By Lemma 2.4, and since $\mathcal{T}_1$ has no reversible edges, every flippable circuit of $\mathrm{stair}(\mathcal{T}_1 \times \mathcal{T}_2)$ projects either to a flippable circuit of $\mathcal{T}_1$, which is a sandwich circuit, or to the unique flippable circuit of $\mathcal{T}_2$, which is a sink circuit. Both cases are forbidden by Theorem 2.6, since $\mathcal{T}_1$ has no sinks and $\mathcal{T}_2$ has no sandwich edges. ☐

3. A construction in dimension four

In this section we describe a triangulation in dimension four which satisfies the hypotheses of the $\mathcal{T}_1$ of Corollary 2.7.

3.1. **Description of the triangulation.** Throughout Section 3, let $\mathcal{A}$ be the following point configuration in $\mathbb{R}^4$, with 81 elements. Indices are regarded modulo 8:

(i) The origin $O = (0,0,0,0)$.
(ii) The 8 points $v_j := \left(0, 0, \cos\left(\frac{\pi}{4}j\right), \sin\left(\frac{\pi}{4}j\right)\right)$, for $j = 0, \dots, 7$.
(iii) The 8 points $h_i := \left(\cos\left(\frac{\pi}{4}i\right), \sin\left(\frac{\pi}{4}i\right), 0, 0\right)$, for $i = 0, \dots, 7$.
(iv) The 32 points $t_{i,j} := h_i + v_j = \left(\cos\left(\frac{\pi}{4}i\right), \sin\left(\frac{\pi}{4}i\right), \cos\left(\frac{\pi}{4}j\right), \sin\left(\frac{\pi}{4}j\right)\right)$, for all values of $i, j \in \{0, \dots, 7\}$ with $i + j$ odd.

(v) The 32 points $s_{i+\frac{1}{2},j+\frac{1}{2}} = (t_{i,j+1} + t_{i+1,j})/2$, for all values of $i, j \in \{0, \dots, 7\}$ with $i + j$ even.

The convex hull of $\mathcal{A}$ is a 4-polytope whose vertices are the 32 points $t_{i,j}$ and with the following 48 facets:

- Eight square anti-prisms $P_{k+\frac{1}{2}}$ $(k = 0, \dots, 7)$ whose eight vertices are the points $t_{i,j}$ with $j \in \{k, k+1\}$,
- Another eight square anti-prisms $P'_{k+\frac{1}{2}}$ $(k = 0, \dots, 7)$ whose eight vertices are the points $t_{i,j}$ with $i \in \{k, k+1\}$, and
- 32 tetrahedra $T_{i,j}$ $(i, j \in \{0, \dots, 7\}, i+j$ even) having as vertices $t_{i-1,j}$, $t_{i+1,j}$, $t_{i,j-1}$ and $t_{i,j+1}$.

The two series of 8 anti-prisms form two solid tori in the three-dimensional topological sphere $\partial(\text{conv}(\mathcal{A}))$ which are glued to one another along edges, leaving some spaces for the 32 tetrahedra $T_{i,j}$. The point $s_{i+\frac{1}{2},j+\frac{1}{2}}$ lies in an edge incident to the tetrahedra $T_{i,j}$ and $T_{i+1,j+1}$ and to the anti-prisms $P_{j+\frac{1}{2}}$ and $P'_{i+\frac{1}{2}}$. The point v_j (respectively h_i) lies in the square between the two anti-prisms $P_{j-\frac{1}{2}}$ and $P_{j+\frac{1}{2}}$ (respectively $P'_{i-\frac{1}{2}}$ and $P'_{i+\frac{1}{2}}$).

The affine symmetry group of $\mathcal{A}$ has 128 elements: any of the 16 anti-prisms can be sent to any other one in eight ways. We will be interested in the following subgroup G with 64 elements. Let g_h and g_v be the rotations of angle $\pi/4$ on the first two and on the last two coordinates respectively, and let g_t be the exchange of the first two and last two coordinates. Then, G is generated by $g_1 := {g_h}^2$, $g_2 := {g_v}^2$, $g_3 := g_h \circ g_v = g_v \circ g_h$, and $g_4 := g_t$. Either of the first two is redundant. All these isometries fix the origin O. Table 1 shows the permutations induced on the other points of $\mathcal{A}$.

$g_1:$	$(h_i \mapsto h_{i+2};$	$v_j \mapsto v_j;$	$t_{i,j} \mapsto t_{i+2,j};$	$s_{i+\frac{1}{2},j+\frac{1}{2}} \mapsto s_{i+\frac{5}{2},j+\frac{1}{2}}\ )$
$g_2:$	$(h_i \mapsto h_i;$	$v_j \mapsto v_{j+2};$	$t_{i,j} \mapsto t_{i,j+2};$	$s_{i+\frac{1}{2},j+\frac{1}{2}} \mapsto s_{i+\frac{1}{2},j+\frac{5}{2}}\ )$
$g_3:$	$(h_i \mapsto h_{i+1};$	$v_j \mapsto v_{j+1};$	$t_{i,j} \mapsto t_{i+1,j+1};$	$s_{i+\frac{1}{2},j+\frac{1}{2}} \mapsto s_{i+\frac{3}{2},j+\frac{3}{2}}\ )$
$g_4:$	$(h_i \mapsto v_i;$	$v_j \mapsto h_j;$	$t_{i,j} \mapsto t_{j,i};$	$s_{i+\frac{1}{2},j+\frac{1}{2}} \mapsto s_{j+\frac{1}{2},i+\frac{1}{2}}\ )$

TABLE 1. The affine symmetries of the triangulation $\mathcal{T}$.

We now define a triangulation $\mathcal{T}$ of $\mathcal{A}$ which has G as its symmetry group. For this, we give in Table 2 a representative of each of the nine orbits of maximal simplices in $\mathcal{T}$. We name the orbits with the first nine letters of the capital Greek alphabet. The indices i and j in Table 2 range over all the possibilities modulo 8 with $i+j$ even. This produces 32 simplices in each orbit. The other 32 are obtained by applying the transformation g_4 and we name them as follows: $\mathrm{A}'_{j,i} := g_4(\mathrm{A}_{i,j})$, $\mathrm{B}'_{j,i} := g_4(\mathrm{B}_{i,j})$, $\Gamma'_{j,i} := g_4(\Gamma_{i,j})$, $\Delta'_{j,i} := g_4(\Delta_{i,j})$, $\mathrm{E}'_{j,i} := g_4(\mathrm{E}_{i,j})$, $\mathrm{Z}'_{j,i} := g_4(\mathrm{Z}_{i,j})$, $\mathrm{H}'_{j,i} := g_4(\mathrm{H}_{i,j})$, $\Theta'_{j,i} := g_4(\Theta_{i,j})$, and $\mathrm{I}'_{j,i} := g_4(\mathrm{I}_{i,j})$. That these 9×64 simplices of dimension four are indeed the maximal simplices of a triangulation of $\mathcal{A}$ will be proved afterwards. Remark 3.4 at the end of Section 3.3 may help to clarify this construction and its genesis.

There are 7 orbits of boundary facets in $\mathcal{T}$, with representatives given in Table 3. The first five orbits triangulate the 16 square anti-prisms. The last two orbits triangulate the 32 tetrahedra $T_{i,j}$. For example, the tetrahedron $T_{0,0}$ is triangulated

$\mathrm{A}_{i,j} := \{O$	v_{j+1}	$t_{i,j+1}$	$s_{i-\frac{1}{2},j-\frac{1}{2}}$	$s_{i+\frac{1}{2},j+\frac{1}{2}}\}$
$\mathrm{B}_{i,j} := \{O$	v_{j+1}	$t_{i+1,j}$	$s_{i-\frac{1}{2},j-\frac{1}{2}}$	$s_{i+\frac{1}{2},j+\frac{1}{2}}\}$
$\Gamma_{i,j} := \{O$	v_{j+1}	$t_{i+1,j}$	$t_{i+2,j+1}$	$s_{i+\frac{1}{2},j+\frac{1}{2}}\}$
$\Delta_{i,j} := \{O$	v_{j+1}	$t_{i-1,j}$	$t_{i,j+1}$	$s_{i-\frac{1}{2},j-\frac{1}{2}}\}$
$\mathrm{E}_{i,j} := \{O$	v_{j+1}	v_j	$t_{i+1,j}$	$s_{i-\frac{1}{2},j-\frac{1}{2}}\}$
$\mathrm{Z}_{i,j} := \{O$	v_{j+1}	v_j	$t_{i-1,j}$	$s_{i-\frac{1}{2},j-\frac{1}{2}}\}$
$\mathrm{H}_{i,j} := \{v_{j+1}$	v_j	$t_{i-1,j}$	$t_{i+1,j}$	$s_{i-\frac{1}{2},j-\frac{1}{2}}\}$
$\Theta_{i,j} := \{v_{j+1}$	$t_{i-1,j}$	$t_{i+1,j}$	$s_{i-\frac{1}{2},j-\frac{1}{2}}$	$s_{i+\frac{1}{2},j+\frac{1}{2}}\}$
$\mathrm{I}_{i,j} := \{v_{j+1}$	$t_{i,j+1}$	$t_{i-1,j}$	$s_{i-\frac{1}{2},j-\frac{1}{2}}$	$s_{i+\frac{1}{2},j+\frac{1}{2}}\}$

TABLE 2. Representatives for the nine orbits of maximal simplices in $\mathcal{T}$.

with $\Theta_{0,0} \setminus v_1$, $\Theta'_{0,0} \setminus h_1$, $\mathrm{I}_{0,0} \setminus v_1$ and $\mathrm{I}'_{0,0} \setminus h_1$. Each of these four 3-simplices joins the two points $s_{-\frac{1}{2},-\frac{1}{2}}$ and $s_{i+\frac{1}{2},j+\frac{1}{2}}$, which lie in two opposite edges of $T_{0,0}$, to one of the remaining four edges of $T_{0,0}$.

$\Gamma_{i,j} \setminus O$	$=$	$\{v_{j+1},\ t_{i+1,j},\ t_{i+2,j+1},\ s_{i+\frac{1}{2},j+\frac{1}{2}}\}$	$\subset$	$P_{j+\frac{1}{2}}$
$\mathrm{H}_{i,j} \setminus s_{i-\frac{1}{2},j-\frac{1}{2}}$	$=$	$\{v_{j+1},\ v_j,\ t_{i-1,j},\ t_{i+1,j}\}$	$\subset$	$P_{j+\frac{1}{2}}$
$\mathrm{H}_{i+1,j+1} \setminus v_{j+2}$	$=$	$\{v_{j+1},\ t_{i,j+1},\ t_{i+2,j+1},\ s_{i+\frac{1}{2},j+\frac{1}{2}}\}$	$\subset$	$P_{j+\frac{1}{2}}$
$\Theta_{i,j} \setminus s_{i-\frac{1}{2},j-\frac{1}{2}}$	$=$	$\{v_{j+1},\ t_{i-1,j},\ t_{i+1,j},\ s_{i+\frac{1}{2},j+\frac{1}{2}}\}$	$\subset$	$P_{j+\frac{1}{2}}$
$\mathrm{I}_{i,j} \setminus s_{i-\frac{1}{2},j-\frac{1}{2}}$	$=$	$\{v_{j+1},\ t_{i-1,j},\ t_{i,j+1},\ s_{i+\frac{1}{2},j+\frac{1}{2}}\}$	$\subset$	$P_{j+\frac{1}{2}}$
$\Theta_{i,j} \setminus v_{j+1}$	$=$	$\{t_{i-1,j},\ t_{i+1,j},\ s_{i-\frac{1}{2},j-\frac{1}{2}},\ s_{i+\frac{1}{2},j+\frac{1}{2}}\}$	$\subset$	$T_{i,j}$
$\mathrm{I}_{i,j} \setminus v_{j+1}$	$=$	$\{t_{i,j+1},\ t_{i-1,j},\ s_{i-\frac{1}{2},j-\frac{1}{2}},\ s_{i+\frac{1}{2},j+\frac{1}{2}}\}$	$\subset$	$T_{i,j}$

TABLE 3. Representatives for the seven orbits of boundary facets in $\mathcal{T}$.

There are 19 orbits of interior facets in the triangulation $\mathcal{T}$, with representatives given in Table 4. For each one we write its two representations as the complement of a vertex in a maximal simplex of $\mathcal{T}$ followed by its four vertices. As before, the indices i and j range over all the possibilities modulo 8 and with $i+j$ even. This gives 32 elements of each orbit and the other 32 are obtained applying the transformation g_4 to them.

3.2. **$\mathcal{T}$ is a triangulation, with two orbits of flips.** In order to prove that $\mathcal{T}$ is a triangulation and compute its flips, in Table 5 we display a list of affine equalities valid on $\mathcal{A}$. Each equality involves only the points of $\mathcal{A}$ in the star of one representative of one of the 19 orbits of interior facets in $\mathcal{T}$, although different orbits may produce the same equality. In other words, the points on the left and right part of each equality are the positive and negative parts of the circuit supported on the corresponding interior facet of $\mathcal{T}$. The group G acts freely over the circuits supported on facets except for those of types γ (same circuit as τ), ι and λ, in which the stabilizer has exactly two elements, as indicated in Table 5.

The reader can check the equalities with a symbolic computation program or can verify them by hand in the following way: first check that in every equation the sum of coefficients is equal on both sides; second, forget the appearances of

$\alpha_{i,j} := \mathrm{A}_{i,j} \setminus O =$	$\mathrm{I}_{i,j} \setminus t_{i-1,j} =$	$\{v_{j+1},\ t_{i,j+1},\ s_{i+\frac12,j+\frac12},\ s_{i-\frac12,j-\frac12}\}$
$\beta_{i,j} := \mathrm{A}_{i,j} \setminus v_{j+1} =$	$\mathrm{B}'_{j,i} \setminus h_{i+1} =$	$\{O,\ t_{i,j+1},\ s_{i-\frac12,j-\frac12},\ s_{i+\frac12,j+\frac12}\}$
$\gamma_{i,j} := \mathrm{A}_{i,j} \setminus t_{i,j+1} =$	$\mathrm{B}_{i,j} \setminus t_{i+1,j} =$	$\{O,\ v_{j+1},\ s_{i-\frac12,j-\frac12},\ s_{i+\frac12,j+\frac12}\}$
$\delta_{i,j} := \mathrm{A}_{i,j} \setminus s_{i-\frac12,j-\frac12} =$	$\mathrm{Z}_{i+1,j+1} \setminus v_{j+2} =$	$\{O,\ v_{j+1},\ t_{i,j+1},\ s_{i+\frac12,j+\frac12}\}$
$\epsilon_{i,j} := \mathrm{A}_{i,j} \setminus s_{i+\frac12,j+\frac12} =$	$\Delta_{i,j} \setminus t_{i-1,j} =$	$\{O,\ v_{j+1},\ t_{i,j+1},\ s_{i-\frac12,j-\frac12}\}$
$\zeta_{i,j} := \mathrm{B}_{i,j} \setminus O =$	$\Theta_{i,j} \setminus t_{i-1,j} =$	$\{v_{j+1},\ t_{i+1,j},\ s_{i-\frac12,j-\frac12},\ s_{i+\frac12,j+\frac12}\}$
$\eta_{i,j} := \mathrm{B}_{i,j} \setminus s_{i-\frac12,j-\frac12} =$	$\Gamma_{i,j} \setminus t_{i+2,j+1} =$	$\{O,\ v_{j+1},\ t_{i+1,j},\ s_{i+\frac12,j+\frac12}\}$
$\theta_{i,j} := \mathrm{B}_{i,j} \setminus s_{i+\frac12,j+\frac12} =$	$\mathrm{E}_{i,j} \setminus v_j =$	$\{O,\ v_{j+1},\ t_{i+1,j},\ s_{i-\frac12,j-\frac12}\}$
$\iota_{i,j} := \Gamma_{i-1,j-1} \setminus v_j =$	$\Delta'_{j,i} \setminus h_{i+1} =$	$\{O,\ t_{i,j-1},\ t_{i+1,j},\ s_{i-\frac12,j-\frac12}\}$
$\kappa_{i,j} := \Gamma_{i-1,j-1} \setminus t_{i,j-1} =$	$\mathrm{E}_{i,j} \setminus v_{j+1} =$	$\{O,\ v_j,\ t_{i+1,j},\ s_{i-\frac12,j-\frac12}\}$
$\lambda_{i,j} := \Gamma_{i-1,j-1} \setminus s_{i-\frac12,j-\frac12} =$	$\Delta_{i+1,j-1} \setminus s_{i+\frac12,j-\frac32} =$	$\{O,\ v_j,\ t_{i,j-1},\ t_{i+1,j}\}$
$\mu_{i,j} := \Delta_{i,j} \setminus O =$	$\mathrm{I}_{i,j} \setminus s_{i+\frac12,j+\frac12} =$	$\{v_{j+1},\ t_{i-1,j},\ t_{i,j+1},\ s_{i-\frac12,j-\frac12}\}$
$\nu_{i,j} := \Delta_{i,j} \setminus t_{i,j+1} =$	$\mathrm{Z}_{i,j} \setminus v_j =$	$\{O,\ v_{j+1},\ t_{i-1,j},\ s_{i-\frac12,j-\frac12}\}$
$\xi_{i,j} := \mathrm{E}_{i,j} \setminus O =$	$\mathrm{H}_{i,j} \setminus t_{i-1,j} =$	$\{v_{j+1},\ v_j,\ t_{i+1,j},\ s_{i-\frac12,j-\frac12}\}$
$o_{i,j} := \mathrm{E}_{i,j} \setminus t_{i+1,j} =$	$\mathrm{Z}_{i,j} \setminus t_{i-1,j} =$	$\{O,\ v_{j+1},\ v_j,\ s_{i-\frac12,j-\frac12}\}$
$\pi_{i,j} := \mathrm{E}_{i,j} \setminus s_{i-\frac12,j-\frac12} =$	$\mathrm{Z}_{i+2,j} \setminus s_{i+\frac32,j-\frac12} =$	$\{O,\ v_{j+1},\ v_j,\ t_{i+1,j}\}$
$\rho_{i,j} := \mathrm{Z}_{i,j} \setminus O =$	$\mathrm{H}_{i,j} \setminus t_{i+1,j} =$	$\{v_{j+1},\ v_j,\ t_{i-1,j},\ s_{i-\frac12,j-\frac12}\}$
$\sigma_{i,j} := \mathrm{H}_{i,j} \setminus v_j =$	$\Theta_{i,j} \setminus s_{i+\frac12,j+\frac12} =$	$\{v_{j+1},\ t_{i-1,j},\ t_{i+1,j},\ s_{i-\frac12,j-\frac12}\}$
$\tau_{i,j} := \Theta_{i,j} \setminus t_{i+1,j} =$	$\mathrm{I}_{i,j} \setminus t_{i,j+1} =$	$\{v_{j+1},\ t_{i-1,j},\ s_{i+\frac12,j+\frac12},\ s_{i-\frac12,j-\frac12}\}$

TABLE 4. Representatives for the nineteen orbits of interior facets in $\mathcal{T}$.

the point O and perform the following substitutions: $t_{k,l} = h_k + v_l$, $s_{k+\frac12,l+\frac12} = (h_k + h_{k+1} + v_l + v_{l+1})/2$, $h_{k+1} + h_{k-1} = \sqrt{2}h_k$, and $v_{l+1} + v_{l-1} = \sqrt{2}v_l$. After this is done all terms should cancel out in all the equations.

Lemma 3.1. *$\mathcal{T}$ is a triangulation of $\mathcal{A}$.*

Proof. We first check that $A_{0,0} = \{O, v_1, t_{0,1}, s_{-\frac12,-\frac12}, s_{\frac12,\frac12}\}$ is affinely independent:

$$\begin{vmatrix} 0 & 0 & 1 & \frac12 + \frac{\sqrt2}{4} & \frac12 + \frac{\sqrt2}{4} \\ 0 & 0 & 0 & -\frac{\sqrt2}{4} & \frac{\sqrt2}{4} \\ 0 & \frac{\sqrt2}{2} & \frac{\sqrt2}{2} & \frac12 + \frac{\sqrt2}{4} & \frac12 + \frac{\sqrt2}{4} \\ 0 & \frac{\sqrt2}{2} & \frac{\sqrt2}{2} & -\frac{\sqrt2}{4} & \frac{\sqrt2}{4} \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = - \begin{vmatrix} 0 & -\frac{\sqrt2}{4} & \frac{\sqrt2}{4} \\ \frac{\sqrt2}{2} & \frac12 + \frac{\sqrt2}{4} & \frac12 + \frac{\sqrt2}{4} \\ \frac{\sqrt2}{2} & -\frac{\sqrt2}{4} & \frac{\sqrt2}{4} \end{vmatrix} = \frac{2+\sqrt2}{8}$$

Since $A_{i,j}$ is an affine basis, for any point $a \in \mathcal{A} \setminus \mathrm{A}_{i,j}$ and $b \in A_{i,j}$, if a and b have non-zero coefficient in the unique affine equality supported on $A_{i,j} \cup \{a\}$, then $A_{i,j} \cup \{a\} \setminus \{b\}$ is again a basis. Hence, the affine equalities $C(\alpha_{i,j})$, $C(\gamma_{i,j})$, $C(\delta_{i,j})$, and $C(\epsilon_{i,j})$ imply, respectively, that $\mathrm{I}_{i,j}$, $\mathrm{B}_{i,j}$, $\mathrm{Z}_{i+1,j+1}$, and $\Delta_{i,j}$ are also bases. Using the same argument and the other equalities we conclude that $\Gamma_{i,j}$, $\mathrm{E}_{i,j}$, $\mathrm{H}_{i,j}$, and $\Theta_{i,j}$ are bases too. Thus, $\mathcal{T}$ consists only of independent subsets of $\mathcal{A}$ and we can apply Lemma 1.2 to it.

The lists of boundary and interior facets of $\mathcal{T}$ given in Tables 3 and 4 are complete and non-redundant in the sense that exactly one representative of each of the five facets of each of the nine orbits of 4-simplices appears exactly once in them. Moreover, in each of $C(\alpha_{i,j}), \ldots, C(\tau_{i,j})$ the two vertices joined to the interior facet

$C(\alpha_{0,0}), C(\epsilon_{0,0}), C(\mu_{0,0})$:	$(\sqrt{2}-1)O + \frac{1+\sqrt{2}}{2}t_{-1,0} + s_{\frac{1}{2},\frac{1}{2}} = \frac{\sqrt{2}}{2}v_1 + \frac{1}{2}t_{0,1} + \sqrt{2}s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\beta_{0,0})$:	$(\frac{\sqrt{2}}{2}+1)v_1 + (\frac{\sqrt{2}}{2}+1)h_1 + s_{-\frac{1}{2},-\frac{1}{2}} = 2O + (1+\sqrt{2})s_{\frac{1}{2},\frac{1}{2}}$
$C(\gamma_{0,0}) = C(\gamma'_{0,0}), C(\tau_{0,0}) = C(\tau'_{0,0})$:	$t_{1,0} + t_{0,1} = 2s_{\frac{1}{2},\frac{1}{2}}$
$C(\delta_{0,0})$:	$s_{-\frac{1}{2},-\frac{1}{2}} + (1+\frac{\sqrt{2}}{2})v_2 + s_{\frac{1}{2},\frac{1}{2}} = (2-\frac{\sqrt{2}}{2})O + \frac{\sqrt{2}}{2}v_1 + (1+\frac{\sqrt{2}}{2})t_{0,1}$
$C(\zeta_{0,0}), C(\xi_{0,0}), C(o_{0,0}), C(\rho_{0,0})$:	$(2-\sqrt{2})O + \frac{\sqrt{2}}{2}t_{1,0} + (1+\frac{\sqrt{2}}{2})t_{-1,0} = v_1 + 2s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\eta_{0,0})$:	$2s_{-\frac{1}{2},-\frac{1}{2}} + (1+\sqrt{2})t_{2,1} = (2-\sqrt{2})O + \sqrt{2}v_1 + (1+\sqrt{2})t_{1,0}$
$C(\theta_{0,0})$:	$\sqrt{2}s_{\frac{1}{2},\frac{1}{2}} + v_0 + (3-2\sqrt{2})O = v_1 + t_{1,0} + (2-\sqrt{2})s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\iota_{0,0}) = C(\iota'_{1,-1})$:	$v_0 + h_1 = O + t_{1,0}$
$C(\kappa_{0,0})$:	$(1+\sqrt{2})t_{0,-1} + \sqrt{2}v_1 = (2\sqrt{2}-2)O + t_{1,0} + 2s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\lambda_{0,0}) = C(\lambda'_{1,-1})$:	$\sqrt{2}s_{-\frac{1}{2},-\frac{1}{2}} + \sqrt{2}s_{\frac{1}{2},-\frac{3}{2}} = (\sqrt{2}-1)O + (\sqrt{2}+1)t_{0,-1}$
$C(\nu_{0,0})$:	$t_{0,1} + \sqrt{2}v_0 + (2-\sqrt{2})O + t_{-1,0} = 2v_1 + 2s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\pi_{0,0})$:	$s_{-\frac{1}{2},-\frac{1}{2}} + (1+\sqrt{2})s_{\frac{3}{2},-\frac{1}{2}} + (1+\frac{\sqrt{2}}{2})v_1 = O + (1+\frac{\sqrt{2}}{2})v_0 + (1+\sqrt{2})t_{1,0}$
$C(\sigma_{0,0})$:	$\sqrt{2}v_0 + 2s_{\frac{1}{2},\frac{1}{2}} = v_1 + \frac{\sqrt{2}}{2}t_{-1,0} + \frac{1+\sqrt{2}}{2}t_{1,0}$

TABLE 5. Affine equalities for the circuits supported on interior facets of $\mathcal{T}$.

in question appear both on the left-hand side of the equality. This implies that these two points are in opposite sides of the interior facet. With this and Lemma 1.2 we only have to check that some point of $\mathrm{conv}(\mathcal{A})$ lies in the convex hull of exactly one simplex of $\mathcal{T}$. For a generic point in the interior of $\mathrm{conv}(\mathcal{A})$ but sufficiently close to a facet of $\mathrm{conv}(\mathcal{A})$, this follows from the fact that $\mathcal{T}$ induces a triangulation on that facet. $\square$

Lemma 3.2. *$\mathcal{T}$ has two orbits of flips under the action of its symmetry group G, with representatives supported on the circuits*

$$(\{O, t_{i-1,j}, s_{i+\frac{1}{2},j+\frac{1}{2}}\}, \{v_{j+1}, t_{i,j+1}, s_{i-\frac{1}{2},j-\frac{1}{2}}\}) \text{ and}$$

$$(\{v_j, s_{i+\frac{1}{2},j+\frac{1}{2}}\}, \{v_{j+1}, t_{i-1,j}, t_{i+1,j}\}).$$

Proof. That $(\{O, t_{i-1,j}, s_{i+\frac{1}{2},j+\frac{1}{2}}\}, \{v_{j+1}, t_{i,j+1}, s_{i-\frac{1}{2},j-\frac{1}{2}}\})$ and $(\{v_j, s_{i+\frac{1}{2},j+\frac{1}{2}}\}, \{v_{j+1}, t_{i-1,j}, t_{i+1,j}\})$ are circuits follows respectively from the equalities $C(\alpha_{0,0})$ and $C(\sigma_{0,0})$ of Table 5.

The first circuit spans $\mathrm{conv}(\mathcal{A})$ and the maximal simplices in its positive triangulation are $\mathrm{A}_{i,j}$, $\Delta_{i,j}$, and $\mathrm{I}_{i,j}$. The second circuit spans the anti-prism facet $P_{j+\frac{1}{2}}$ of $\mathrm{conv}(\mathcal{A})$. The maximal simplices in its positive triangulation are $\mathrm{H}_{i,j} \setminus s_{i-\frac{1}{2},j-\frac{1}{2}}$ and $\Theta_{i,j} \setminus s_{i-\frac{1}{2},j-\frac{1}{2}}$. In other words, $\mathcal{T}$ contains its positive triangulation and all the maximal simplices in it have the same link. Hence, both circuits are flippable.

We now check that the other 15 orbits of interior facets are not flippable, using Lemma 1.4 for this. In the first place, if a flippable circuit has more than two positive elements, then it must be supported on at least $\binom{3}{2}$ facets of $\mathcal{T}$. This implies that the circuits supported on the facets of types β, δ, θ, ν, and π are not flippable, because they are supported only on one facet of $\mathcal{T}$ each and their positive parts have at least three elements.

The circuits supported on facets of types ζ (same circuit as ξ, o, and ρ), η, ι, κ, and λ all contain the element O and hence we can apply part (ii) of Lemma 1.4 to them. According to it we would need at least $2\cdot 3$, $2\cdot 1$, $3\cdot 1$, $2\cdot 1$, and $3\cdot 1$ flippable facets for them respectively, but they are supported in only 4, 1, 2, 1, and 2 facets.

Only the facets of types γ and τ remain to be checked. $\gamma_{0,0}$ and $\tau_{0,0}$ are both supported on the circuit $(\{t_{1,0}, t_{0,1}\}, \{s_{\frac{1}{2},\frac{1}{2}}\})$.This circuit is not flippable since the link of $\{t_{1,0}, s_{\frac{1}{2},\frac{1}{2}}\}$ in $\mathcal{T}$ contains $\{O, v_1, v_2\}$ (simplex $\mathrm{Z}_{1,1}$), the link of $\{t_{0,1}, s_{\frac{1}{2},\frac{1}{2}}\}$ contains $\{O, v_1, t_{2,1}\}$ (simplex $\Gamma_{0,0}$) and the points v_2 and $t_{2,1}$ lie both on the same side of the hyperplane spanned by $\{O, v_1, t_{1,0}, t_{0,1}, s_{\frac{1}{2},\frac{1}{2}}\}$ (equation $C(\kappa_{1,1})$). $\square$

3.3. **A locally acyclic orientation of the edges of $\mathcal{T}$.** We now introduce an orientation of the 1-skeleton of the triangulation $\mathcal{T}$. There are 14 orbits of edges. In Table 6 below we show a representative for each orbit, and a particular maximal simplex containing it. We orient the representatives from the vertex which appears first to the second and let each orbit be oriented by the action of the symmetry group G.

$\{O, v_j\} \in \mathrm{A}_{i,j+1}$	$\{v_j, v_{j+1}\} \in \mathrm{E}_{i,j}$
$\{O, t_{i,j+1}\} \in \mathrm{A}_{i,j}$	$\{O, s_{i+\frac{1}{2},j+\frac{1}{2}}\} \in \mathrm{A}_{i,j}$
$\{v_j, t_{i+1,j}\} \in \Gamma_{i+1,j+1}$	$\{t_{i,j-1}, v_j\} \in \Delta_{i+1,j-1}$
$\{v_j, s_{i-\frac{1}{2},j-\frac{1}{2}}\} \in \mathrm{A}_{i-1,j-1}$	$\{s_{i+\frac{1}{2},j-\frac{3}{2}}, v_j\} \in \mathrm{E}_{i+1,j-1}$
$\{t_{i-1,j}, t_{i+1,j}\} \in \mathrm{H}_{i,j}$	$\{t_{i,j-1}, t_{i+1,j}\} \in \Gamma_{i-1,j-1}$
$\{s_{i-\frac{1}{2},j-\frac{1}{2}}, t_{i,j+1}\} \in \mathrm{A}_{i,j}$	$\{s_{i-\frac{1}{2},j-\frac{1}{2}}, s_{i+\frac{1}{2},j+\frac{1}{2}}\} \in \mathrm{A}_{i,j}$
$\{t_{i-1,j}, s_{i+\frac{1}{2},j+\frac{1}{2}}\} \in \Theta_{i,j}$	$\{t_{i,j-1}, s_{i-\frac{1}{2},j-\frac{1}{2}}\} \in \mathrm{B}_{i-1,j-1}$

TABLE 6. The fourteen orbits of edges in $\mathcal{T}$.

That this orientation *is locally acyclic* can be seen in Table 7, where the total order induced on the vertices of each maximal representative simplex is shown. It *has no reversible edges* since the reversal of any edge produces a cycle in the simplex included after it in Table 6. The only vertices which are sinks of some maximal simplex in Table 7 are v_{j+1}, $t_{i,j+1}$ and $s_{i+\frac{1}{2},j+\frac{1}{2}}$, but the first two are not sinks in $\mathrm{A}_{i,j}$ and the last one is not a sink in $\Gamma_{i,j}$. Hence, the orientation *has no sinks.*

Finally, let us see that the two types of flippable circuits of $\mathcal{T}$ (see Lemma 3.2) are *sandwich circuits*, in the terms of Definition 2.5. This holds since the orientation of the 1-skeleton of $\mathcal{T}$ is compatible only with the following total order on the supports of the circuits:

$$\overset{+}{O}\to\overset{+}{t}_{i-1,j}\to\overset{-}{s}_{i-\frac{1}{2},j-\frac{1}{2}}\to\overset{-}{v}_{j+1}\to\overset{-}{t}_{i,j+1}\to\overset{+}{s}_{i+\frac{1}{2},j+\frac{1}{2}},$$

$$\overset{+}{v}_j\to\overset{-}{t}_{i-1,j}\to\overset{-}{t}_{i+1,j}\to\overset{-}{v}_{j+1}\to\overset{+}{s}_{i+\frac{1}{2},j+\frac{1}{2}}\ .$$

The following statement sums up the key facts on $\mathcal{T}$ proved throughout this section:

Theorem 3.3. *$\mathcal{T}$ is a triangulation of the point configuration $\mathcal{A}\subset\mathbb{R}^4$. With the locally acyclic orientation we have given to its edges it has no global sinks, it has no reversible edges and every flippable circuit is a sandwich circuit.*

$\mathrm{A}_{i,j}$:	$\{O$	$\to$	$s_{i-\frac12,j-\frac12}$	$\to$	v_{j+1}	$\to$	$t_{i,j+1}$	$\to$	$s_{i+\frac12,j+\frac12}\}$
$\mathrm{B}_{i,j}$:	$\{O$	$\to$	$s_{i-\frac12,j-\frac12}$	$\to$	$t_{i+1,j}$	$\to$	v_{j+1}	$\to$	$s_{i+\frac12,j+\frac12}\}$
$\Gamma_{i,j}$:	$\{O$	$\to$	$t_{i+1,j}$	$\to$	v_{j+1}	$\to$	$s_{i+\frac12,j+\frac12}$	$\to$	$t_{i+2,j+1}\}$
$\Delta_{i,j}$:	$\{O$	$\to$	$t_{i-1,j}$	$\to$	$s_{i-\frac12,j-\frac12}$	$\to$	v_{j+1}	$\to$	$t_{i,j+1}\}$
$\mathrm{E}_{i,j}$:	$\{O$	$\to$	v_j	$\to$	$s_{i-\frac12,j-\frac12}$	$\to$	$t_{i+1,j}$	$\to$	$v_{j+1}\}$
$\mathrm{Z}_{i,j}$:	$\{O$	$\to$	v_j	$\to$	$t_{i-1,j}$	$\to$	$s_{i-\frac12,j-\frac12}$	$\to$	$v_{j+1}\}$
$\mathrm{H}_{i,j}$:	$\{v_j$	$\to$	$t_{i-1,j}$	$\to$	$s_{i-\frac12,j-\frac12}$	$\to$	$t_{i+1,j}$	$\to$	$v_{j+1}\}$
$\Theta_{i,j}$:	$\{t_{i-1,j}$	$\to$	$s_{i-\frac12,j-\frac12}$	$\to$	$t_{i+1,j}$	$\to$	v_{j+1}	$\to$	$s_{i+\frac12,j+\frac12}\}$
$\mathrm{I}_{i,j}$:	$\{t_{i-1,j}$	$\to$	$s_{i-\frac12,j-\frac12}$	$\to$	v_{j+1}	$\to$	$t_{i,j+1}$	$\to$	$s_{i+\frac12,j+\frac12}\}$

TABLE 7. The orientation of the 1-skeleton of $\mathcal{T}$ is locally acyclic.

Hence, the staircase refinement stair$(\mathcal{T}\times\mathcal{T}_2)$*, does not have any flips, where* $\mathcal{T}_2$ *is the 2-dimensional triangulation of Corollary 2.7.* ☐

Remark 3.4. The triangulation $\mathcal{T}$ described in the last pages is based on the construction in Section 4 of [38]. We show here this connection because understanding that (simpler) triangulation will help to understand this one.

Let $n \ge 3$ be an integer. Let $\mathcal{A}_n \subset \mathbb{R}^4$ consist of the following $(n+1)^2$ points: $O := (0,0,0,0)$, $v_j := \left(0,0,\cos\left(\frac{\pi}{4}j\right),\sin\left(\frac{\pi}{4}j\right)\right)$, $h_i := \left(\cos\left(\frac{\pi}{4}i\right),\sin\left(\frac{\pi}{4}i\right),0,0\right)$, and $t_{i,j} := h_i + v_j - O = \left(\cos\left(\frac{\pi}{4}i\right),\sin\left(\frac{\pi}{4}i\right),\cos\left(\frac{\pi}{4}j\right),\sin\left(\frac{\pi}{4}j\right)\right)$. The indices i and j range over all the possibilities modulo n. Let $\mathcal{T}_n$ be the triangulation having the following $6n^2$ maximal simplices. For each $i,j = 0,\dots,n-1$:

$$\begin{array}{ll}
\Phi_{i,j} := \{O, v_j, t_{i,j+1}, t_{i,j}, t_{i+1,j+1}\}, & \Phi'_{i,j} := \{O, h_i, t_{i+1,j}, t_{i,j}, t_{i+1,j+1}\},\\
\Psi_{i,j} := \{O, v_j, t_{i+1,j}, t_{i,j}, t_{i+1,j+1}\}, & \Psi'_{i,j} := \{O, h_i, t_{i,j+1}, t_{i,j}, t_{i+1,j+1}\},\\
\Omega_{i,j} := \{O, v_{j+1}, v_j, t_{i+1,j+1}, t_{i,j+1}\}, & \Omega'_{i,j} := \{O, h_{i+1}, h_i, t_{i+1,j+1}, t_{i+1,j}\}.
\end{array}$$

The configuration $\mathcal{A}_n$ is the product with itself of the two-dimensional point configuration $\mathcal{B}_n$ consisting of the vertices and the center of a regular n-gon. The triangulation $\mathcal{T}_n$ described above equals the staircase triangulation stair$(\mathcal{S}_n\times\mathcal{S}_n)$, where $\mathcal{S}_n$ is the radial triangulation of $\mathcal{B}_n$, with the following locally acyclic orientation: orient all interior edges from the center to the boundary and orient the boundary in the direction of positive angles. Indeed, the six 4-simplices in the list above are the monotone staircases in the following grid:

v_{j+1}	$t_{i,j+1}$	$t_{i+1,j+1}$
v_j	$t_{i,j}$	$t_{i+1,j}$
O	h_i	h_{i+1}

A different description of $\mathcal{T}_n$ is as follows: let $P = \mathrm{conv}(\mathcal{A})$, which equals the product of two n-gons. Each of the $2n$ facets of P is a prism over a regular n-gon (the facets form two cycles with n prisms each; each cycle is a solid torus and the two solid tori are glued along their boundaries). If we divide each of the $2n$ prisms into n-triangular prisms by its medial axis, we have the boundary of P decomposed into $2n^2$ equal triangular prisms. In $\mathcal{T}_n$, the cone of each such prism to the centroid O of P is triangulated with three 4-simplices, one from each of the three orbits Φ, Ψ and Ω.

To go from the triangulation $\mathcal{T}_8$ to our triangulation $\mathcal{T}$ proceed as follows: In the point configuration $\mathcal{A}_8$, perform a positive rotation of order 8 to the octagons containing the points h_i and v_j (but leaving the $t_{i,j}$'s fixed) and substitute each point $t_{i,j}$ with $i+j$ even by the point $s_{i-\frac{1}{2},j-\frac{1}{2}}$. This gives the point configuration $\mathcal{A}$. The triangulation $\mathcal{T}_8$ becomes a geometric simplicial complex with non-convex boundary but whose simplices still intersect properly. Each of the three orbits of 4-simplices in $\mathcal{T}_8$ divides into two orbits of simplices present in $\mathcal{T}$: the orbits A and Γ for Φ, B and Δ for Ψ, and E and Z for Ω. The other three orbits H, Θ and I of $\mathcal{T}$ fill in the convex hull of $\mathcal{A}$.

The above modification on $\mathcal{T}_n$ is in principle possible for any even n. But with $n = 4$ bad intersections occur between simplices and with $n = 6$ some of the flippable circuits are not sandwich circuits.

When considered in $\mathcal{T}_8$ (or in any other $\mathcal{T}_n$), the orientation of the 1-skeleton described in Section 3.3 is still locally acyclic and has no reversible edges or sinks. With $n = 3$ this is exactly the triangulation and orientation which appeared in [38, Section 4]. It is interesting that the triangulation induced by $\mathcal{T}_n$ in the Schlegel diagram of $\operatorname{conv}(\mathcal{A}_n)$ with respect to any of its facets has dimension three and its 1-skeleton oriented in a locally acyclic way, without reversible edges, sinks or sources (but with non-sandwich circuits). This might be a step towards a triangulation without flips in dimension five.

3.4. **Integer coordinates.** Consider the following sixteen points in $\mathbb{R}^4$:

$$\begin{array}{llll}
h_0 := (4,-4,0,0), & h_1 := (6,0,0,0), & h_2 := (4,4,0,0), & h_3 := (0,6,0,0),\\
h_4 := (-4,4,0,0), & h_5 := (-6,0,0,0), & h_6 := (-4,-4,0,0), & h_7 := (0,-6,0,0),\\
v_0 := (0,0,6,0), & v_1 := (0,0,4,4), & v_2 := (0,0,0,6), & v_3 := (0,0,-4,4),\\
v_4 := (0,0,-6,0), & v_5 := (0,0,-4,-4), & v_6 := (0,0,0,-6), & v_7 := (0,0,4,-4).
\end{array}$$

Let $\mathcal{A}_{\text{int}}$ be the point configuration consisting of them, together with the origin $O = (0,0,0,0)$, the 32 points $t_{i,j} := h_i + v_j$ for all values of $i,j \in \{0,\dots,7\}$ with $i+j$ odd, and the 32 points $s_{i+\frac{1}{2},j+\frac{1}{2}} = (t_{i,j+1} + t_{i+1,j})/2$ for all values of $i,j \in \{0,\dots,7\}$ with $i+j$ even. Indices are regarded modulo 8. The essential difference between $\mathcal{A}_{\text{int}}$ and the point configuration $\mathcal{A}$ of the previous sections is that the regular octagons in the first two and last two coordinate planes have been modified to have integer coordinates (using the approximations $4/3 \simeq \sqrt{2} \simeq 3/2$) and hence they are not regular anymore. But, for example, $\operatorname{conv}(\mathcal{A})$ and $\operatorname{conv}(\mathcal{A}_{\text{int}})$ are combinatorially equivalent polytopes. We construct a triangulation $\mathcal{T}_{\text{int}}$ of $\mathcal{A}_{\text{int}}$ following word by word the description in Section 3.1, except that the symmetries displayed in Table 1 are only combinatorial. With the notation of Section 3.1, the affine symmetry group of $\mathcal{T}_{\text{int}}$ is a subgroup of G of index 2, generated by ${g_h}^2$, ${g_v}^2$, and $g_t \circ g_h \circ g_v$.

The only other thing to be modified with respect to what we said in Sections 3.1, 3.2 and 3.3 are the affine equalities displayed in Table 5. Each orbit of interior facets of $\mathcal{T}$, say α, breaks into two orbits in $\mathcal{T}_{\text{int}}$ having as representatives $\alpha_{0,0}$ and $\alpha'_{0,0}$. These two orbits may produce different equalities, and all of them are displayed in Table 8. They can be verified as follows: first check that in every equation the sum of coefficients is equal on both sides; second, forget the appearances of the point O and perform the following substitutions:

$$\begin{array}{llll}
t_{k,l} = h_k + v_l, & s_{k+\frac{1}{2},l+\frac{1}{2}} = (h_k + h_{k+1} + v_l + v_{l+1})/2 & & \forall k,l \in \{0,\dots,7\};\\
2h_{k+1} + 2h_{k-1} = 3h_k & \text{and} \quad 3v_{k+1} + 3v_{k-1} = 4v_k & & \text{if } k \text{ is even};\\
3h_{k+1} + 3h_{k-1} = 4h_k & \text{and} \quad 2v_{k+1} + 2v_{k-1} = 3v_k & & \text{if } k \text{ is odd}.
\end{array}$$

$C(\alpha_{0,0}), C(\epsilon_{0,0}), C(\mu_{0,0})$:	$2O + 5t_{-1,0} + 4s_{\frac{1}{2},\frac{1}{2}} = 3v_1 + 2t_{0,1} + 6s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\alpha'_{0,0}), C(\epsilon'_{0,0}), C(\mu'_{0,0})$:	$2O + 7t_{0,-1} + 6s_{\frac{1}{2},\frac{1}{2}} = 4h_1 + 3t_{1,0} + 8s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\beta_{0,0})$:	$21v_1 + 20h_1 + 12s_{-\frac{1}{2},-\frac{1}{2}} = t_{1,0} + 24O + 28s_{\frac{1}{2},\frac{1}{2}}$
$C(\beta'_{0,0})$:	$20h_1 + 21v_1 + 12s_{-\frac{1}{2},-\frac{1}{2}} + t_{0,1} = 24O + 30s_{\frac{1}{2},\frac{1}{2}}$
$C(\gamma_{0,0}) = C(\gamma'_{0,0}), C(\tau_{0,0}) = C(\tau'_{0,0})$:	$t_{1,0} + t_{0,1} = 2s_{\frac{1}{2},\frac{1}{2}}$
$C(\delta_{0,0})$:	$12s_{-\frac{1}{2},-\frac{1}{2}} + 20v_2 + 12s_{\frac{1}{2},\frac{1}{2}} = 14O + 9v_1 + 21t_{0,1}$
$C(\delta'_{0,0})$:	$12s_{-\frac{1}{2},-\frac{1}{2}} + 21h_2 + 12s_{\frac{1}{2},\frac{1}{2}} = 17O + 8h_1 + 20t_{1,0}$
$C(\zeta_{0,0}), C(\xi_{0,0}), C(o_{0,0}), C(\rho_{0,0})$:	$2O + 5t_{1,0} + 2t_{-1,0} = 3v_1 + 6s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\zeta'_{0,0}), C(\xi'_{0,0}), C(o'_{0,0}), C(\rho'_{0,0})$:	$2O + 7t_{0,1} + 3t_{0,-1} = 4h_1 + 8s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\eta_{0,0})$:	$12s_{-\frac{1}{2},-\frac{1}{2}} + 15t_{2,1} = 4O + 9v_1 + 14t_{1,0}$
$C(\eta'_{0,0})$:	$12s_{-\frac{1}{2},-\frac{1}{2}} + 14t_{1,2} = 3O + 8h_1 + 15t_{0,1}$
$C(\theta_{0,0})$:	$30s_{\frac{1}{2},\frac{1}{2}} + 20v_0 + 4O = 21v_1 + 21t_{1,0} + 12s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\theta'_{0,0})$:	$28s_{\frac{1}{2},\frac{1}{2}} + 21h_0 + 3O = 20h_1 + 20t_{0,1} + 12s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\iota_{0,0}) = C(\iota'_{1,-1})$:	$v_0 + h_1 = O + t_{1,0}$
$C(\iota'_{0,0}) = C(\iota_{-1,1})$:	$h_0 + v_1 = O + t_{0,1}$
$C(\kappa_{0,0})$:	$5t_{0,-1} + 3v_1 = 2O + 2t_{1,0} + 4s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\kappa'_{0,0})$:	$7t_{-1,0} + 4h_1 = 2O + 3t_{0,1} + 6s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\lambda_{0,0}) = C(\lambda'_{1,-1})$:	$4s_{-\frac{1}{2},-\frac{1}{2}} + 4s_{\frac{1}{2},-\frac{3}{2}} = O + 7t_{0,-1}$
$C(\lambda'_{0,0}) = C(\lambda_{-1,1})$:	$3s_{-\frac{1}{2},-\frac{1}{2}} + 3s_{\frac{-3}{2},\frac{1}{2}} = O + 5t_{-1,0}$
$C(\nu_{0,0})$:	$3t_{0,1} + 4v_0 + 2O + 3t_{-1,0} = 6v_1 + 6s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\nu'_{0,0})$:	$2t_{1,0} + 3h_0 + O + 2t_{0,-1} = 4h_1 + 4s_{-\frac{1}{2},-\frac{1}{2}}$
$C(\pi_{0,0})$:	$12s_{-\frac{1}{2},-\frac{1}{2}} + 30s_{\frac{3}{2},-\frac{1}{2}} + 21v_1 = 14O + 20v_0 + 29t_{1,0}$
$C(\pi'_{0,0})$:	$12s_{-\frac{1}{2},-\frac{1}{2}} + 28s_{-\frac{1}{2},\frac{3}{2}} + 20h_1 = 10O + 21h_0 + 29t_{0,1}$
$C(\sigma_{0,0})$:	$4v_0 + 6s_{\frac{1}{2},\frac{1}{2}} = 3v_1 + 2t_{-1,0} + 5t_{1,0}$
$C(\sigma'_{0,0})$:	$6h_0 + 8s_{\frac{1}{2},\frac{1}{2}} = 4h_1 + 3t_{0,-1} + 7t_{0,1}$

TABLE 8. Affine equalities for the circuits supported on interior facets of $\mathcal{T}_{\text{int}}$.

Comparing Tables 5 and 8 we see that the only circuits that change are those supported on facets of type β. But, since $C(\beta_{0,0})$ and $C(\beta'_{0,0})$ in Table 8 have both at least three elements in their positive parts and are supported in only one facet of $\mathcal{T}_{\text{int}}$ each, Lemma 1.4 implies that they are still not flippable. Hence, in Theorem 3.3 we can take $\mathcal{T}_{\text{int}}$ and $\mathcal{A}_{\text{int}}$ instead of $\mathcal{T}$ and $\mathcal{A}$.

4. The graph of triangulations versus the poset of subdivisions

4.1. **The Baues poset.** Let $\mathcal{A} \subset \mathbb{R}^k$ be a point configuration of dimension d. A *cell* of $\mathcal{A}$ is any spanning subset B of $\mathcal{A}$. A *face* of a cell B is the intersection $F \cap B$ of B with any face F of the convex hull of B.

Definition 4.1. A *(polyhedral) subdivision* of $\mathcal{A}$ is any family Π of cells of $\mathcal{A}$ such that $\cup_{B\in\Pi}\mathrm{conv}(B) = \mathrm{conv}(\mathcal{A})$ and for any $B_1, B_2 \in \Pi$ one has $\mathrm{conv}(B_1) \cap \mathrm{conv}(B_2) = \mathrm{conv}(B_1 \cap B_2)$ and $B_1 \cap B_2$ is a (possibly empty) face of B_1 and B_2.

The subdivisions of $\mathcal{A}$ are partially ordered by the following *refinement* relation:

$$\Pi \leq \Pi' \qquad \Longleftrightarrow \qquad \forall B \in \Pi \quad \exists B' \in \Pi' : B \subseteq B'.$$

The partially ordered set (poset) so obtained is the *Baues poset* of $\mathcal{A}$, which we denote $\Omega(\mathcal{A})$. We call *strict Baues poset* the Baues poset without the trivial subdivision $\{\mathcal{A}\}$. A *lower ideal* in a poset X is any subposet Y with the property that $x \in Y$ and $y < x$ imply $y \in Y$. For any $x \in X$, the *principal lower ideal* $X_{\leq x}$ (resp. *strict principal lower ideal* $X_{<x}$) of x is the subposet of elements below x, including x (resp. excluding x).

The following lemma is proved as Corollary 4.5 and Proposition 5.3 in [40].

Lemma 4.2. *Let Π be a polyhedral subdivision of $\mathcal{A}$. The following conditions are equivalent:*

(i) *Π has only two proper refinements.*
(ii) *All the proper refinements of Π are triangulations.*
(iii) *Π has only two proper refinements and they are triangulations which differ by a geometric bistellar flip.*

Moreover, any pair of triangulations which differ by a geometric bistellar flip are the two refinements of a subdivision satisfying these conditions. □

From now on we call geometric bistellar flip (or flip, for short) a subdivision in the conditions of Lemma 4.2. For any $X \subseteq \Omega(\mathcal{A})$ we denote $G(X)$ the Baues poset restricted to the triangulations and flips in X. If X is a lower ideal then $G(X)$ is a lower ideal as well, and is naturally a subgraph of the graph of triangulations of $\mathcal{A}$. The following is a version of Lemma 3.1 in [37] or Proposition 4.8 in [40].

Corollary 4.3. *Let X be any lower ideal in $\Omega(\mathcal{A})$.*

(i) *If $G(X)$ is connected, then X is connected.*
(ii) *If X is connected and $X_{<\Pi}$ is connected for every $\Pi \in X \setminus G(X)$, then $G(X)$ is connected.*
(iii) *If the graph of triangulations $G(\Omega(\mathcal{A}))$ is disconnected, then there is a $\Pi \in \Omega(\mathcal{A}) \setminus G(\Omega(\mathcal{A}))$ whose strict lower ideal is disconnected. If, moreover, $\mathcal{A}$ is in general position, then there is a subconfiguration of $\mathcal{A}$ whose strict Baues poset is disconnected.*

Proof. Parts (i) and (ii) hold for any finite poset X and any lower ideal $G(X)$ of it containing all the minimal elements. Part (ii) implies the first half of (iii). The "moreover" holds since in general position $\Omega(\mathcal{A})_{\leq\Pi}$ is the product of the Baues posets of the cells of Π. □

4.2. **The coherent poset of subdivisions.** A *height function* on $\mathcal{A}$ is any map $h : \mathcal{A} \to \mathbb{R}$. A height function is *affine* if it is the restriction of an affine map $\mathbb{R}^k \to \mathbb{R}$. The *coherent* [21, 37, 47] or *regular* [6, 26] subdivision of $\mathcal{A}$ induced by a height function h is

$$\{(h - f)^{-1}(0) : f \text{ is an affine height function with } h \geq f\}.$$

In other words, it is the subdivision of $\mathcal{A}$ obtained by projection of the lower facets of the configuration $\{(a, h(a)) \in \mathbb{R}^{k+1} : a \in \mathcal{A}\}$ onto the first k coordinates.

Definition 4.4 (Alexeev)**.** Let Π and Π' be two subdivisions of $\mathcal{A}$. A *system of height functions* on Π is a family $\{h_B : B \in \Pi\}$, where h_B is a height function on the cell B and $h_B - h_{B'}$ is an affine height function on $B \cap B'$ for every $B, B' \in \Pi$.

We say that Π' is a *coherent refinement* of Π if Π' is a refinement of Π and there is a system of height functions on Π such that Π' restricted to each $B \in \Pi$ is the coherent subdivision of B given by h_B.

Coherent refinements are called *regular subdecompositions* in [2, Section 2.12]. We say that Π' is a *strongly coherent refinement* of Π if Π' is a coherent refinement of Π for a system of height functions with $h_B = h_{B'}$ on $B \cap B'$ for any two cells B and B' of Π. Strongly coherent refinements are studied in [40]. All the results in this section are true (and easier to prove, since the relation is transitive) for strongly coherent refinements.

Example 4.5. Let $\mathcal{A} = \{(4,0,0),(0,4,0),(0,0,4),(2,1+\epsilon,1-\epsilon),(1-\epsilon,2,1+\epsilon),(1+\epsilon,1-\epsilon,2)\}$, where ϵ is a sufficiently small real number, possibly zero. This is the smallest example of a configuration with non-coherent subdivisions [6, 21, 37, 47]. If $\epsilon = 0$, then $\mathcal{A}$ consists of the vertices of two homothetic triangles one inside another. If $\epsilon \neq 0$, then the interior triangle is slightly rotated, but the Baues poset is independent of this rotation since the oriented matroid is preserved.

Let the points in $\mathcal{A}$ be labelled $1,\dots,6$ in the order we have written them. The subdivision $\Pi = \{456, 1245, 2356, 1346\}$ is coherent (with height function $(1,1,1,0,0,0)$) if and only if $\epsilon = 0$. Regardless of the value of ϵ all the refinements of Π are coherent refinements and they are in poset isomorphism with the faces of a 3-cube. If $\epsilon \neq 0$, then they are all also strongly coherent refinements. But for $\epsilon = 0$ only the coherent subdivisions refining Π are strongly coherent refinements of Π, and they are in poset isomorphism with the faces of a hexagon.

Lemma 4.6. *Let Π be a polyhedral subdivision of $\mathcal{A}$.*

(i) *There is a polytope $\Sigma_c(\Pi,\mathcal{A})$ whose poset of non-empty faces is isomorphic to the (refinement) poset of all coherent refinements of Π.*
(ii) *Vertices of $\Sigma_c(\Pi,\mathcal{A})$ correspond bijectively to triangulations.*
(iii) *The edges of $\Sigma_c(\Pi,\mathcal{A})$ are geometric bistellar flips.*
(iv) *Let Π' be a coherent refinement of Π. Then every coherent refinement of Π which refines Π' is also a coherent refinement of Π'.*

Proof. (i) This is Lemma 2.12.11 in [2], where $\Sigma_c(\Pi,\mathcal{A})$ is constructed in a way similar to the construction of fiber polytopes in [8]. For our purposes here it would be sufficient to prove that the poset of coherent refinements of Π is anti-isomorphic to the face lattice of a certain complete polyhedral fan, as follows: The systems of height functions on Π form a real vector space, which is divided into convex polyhedral cones by the property of defining the same coherent refinement. These cones form a polyhedral fan.

(ii) A coherent refinement is a triangulation if and only if it can be obtained with a sufficiently generic system of height functions.

(iii) Let v_0 be a system of height functions on Π which produces Π'. Any system of height functions sufficiently close to v_0 produces a coherent refinement of Π which refines Π'. We will prove that if Π' is not a flip then more than two refinements can be produced in this way.

If a certain cell B of Π' has more than $d+2$ vertices, then every coherent subdivision of B appears in some coherent refinement of Π, and there are more than two of them. Hence, we can assume that each non-simplicial cell of Π' contains the support of a unique (up to sign reversal) circuit. If the same circuit is contained in all of them, then Π' is a flip on that circuit. If there are two different supports of circuits Z_1 and Z_2 in cells of Π', then let $a \in Z_1 \setminus Z_2$ and let $b \in Z_2 \setminus Z_1$. Modifying v_0 by the addition of two sufficiently small global constants to the heights of a and b we can triangulate the two circuits independently, producing at least four coherent refinements of Π which refine Π'.

(iv) Every system v of height functions on Π can be regarded as a system of height functions on Π' as well. Moreover, if v produces Π'' as a coherent refinement of Π, then it produces Π'' as a coherent refinement of Π' as well, since for any $B \in \Pi''$ the linear constraints on v needed for B to be in the coherent refinement of Π' are weaker (more local) than those for the coherent refinement of Π. □

Let $\Omega_{sc}(\mathcal{A})$ and $\Omega_c(\mathcal{A})$ be the posets of subdivisions of $\mathcal{A}$ partially ordered by strongly coherent and (the transitive closure of) coherent refinement, respectively.

Corollary 4.7. *Let X be a lower ideal in $\Omega_c(\mathcal{A})$. $G(X)$ is connected if and only if X is connected. In particular, $\Omega_c(\mathcal{A})$ is connected if and only if the graph of triangulations of $\mathcal{A}$ is connected.*

Proof. If $G(X)$ is connected, then X is connected because any subdivision in X can be coherently refined to a triangulation in X and $G(X)$ is a subposet of X.

Suppose now that X is connected and let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulations in X. There is a sequence of subdivisions $\{\mathcal{T} = \Pi_0, \Pi_1, \dots, \Pi_{k-2}, \Pi_{k-1}, \Pi_k = \mathcal{T}'\}$ in X such that for every $i \in \{1, \dots, k\}$ either Π_{i-1} is a coherent refinement of Π_i or vice versa. We proceed by induction on k. If $k = 2$, then $\mathcal{T}$ and $\mathcal{T}'$ are coherent refinements of a subdivision in X and parts (i), (ii) and (iii) of Lemma 4.6 imply that they are connected in $G(X)$. For general k, let $\mathcal{T}''$ be a triangulation which coherently refines both Π_{k-2} and Π_{k-1}. It exists by part (iv) of Lemma 4.6. Using $\mathcal{T}''$, our sequence can be broken into two shorter ones: $\{\mathcal{T}, \Pi_1, \dots, \Pi_{k-2}, \mathcal{T}''\}$ and $\{\mathcal{T}'', \Pi_{k-1}, \mathcal{T}'\}$. □

Following Alexeev, we call the polytopes $\Sigma_c(\Pi, \mathcal{A})$ of Lemma 4.6 *generalized secondary polytopes*. Similar polytopes $\Sigma_{sc}(\Pi, \mathcal{A})$ are constructed in [40] for the strong-coherent case. The main difference between the two is that $\Sigma_{sc}(\Pi', \mathcal{A})$ equals the face of $\Sigma_{sc}(\Pi, \mathcal{A})$ corresponding to Π', while $\Sigma_c(\Pi', \mathcal{A})$ only *projects* onto the face of $\Sigma_c(\Pi, \mathcal{A})$ corresponding to Π', whenever Π' is a strongly coherent (resp. coherent) refinement of Π. The secondary polytope of $\mathcal{A}$ equals both $\Sigma_c(\{\mathcal{A}\}, \mathcal{A})$ and $\Sigma_{sc}(\{\mathcal{A}\}, \mathcal{A})$, and has dimension $|\mathcal{A}| - d - 1$. $\Sigma_c(\Pi, \mathcal{A})$ can have dimension higher than that (examples exist with $|\mathcal{A}| = 10$ and $d = 2$), but $\Sigma_{sc}(\Pi, \mathcal{A})$ cannot.

The existence and properties of generalized secondary polytopes for $\Omega_c(\mathcal{A})$ and $\Omega_{sc}(\mathcal{A})$ provides the following interpretation of the order complexes of these posets: they are homeomorphic to the union of all the generalized secondary polytopes, each polytope glued to faces of the generalized secondary polytopes of the subdivisions of which it is a coherent (resp. strongly coherent) refinement. In the strong-coherent case the gluing is face to face in the polytopes, and easy to understand. In the coherent case, part (iv) of Lemma 4.6 implies that if Π' coherently refines Π then every chain of coherent refinements of Π bounded above by Π' is also a chain of coherent refinements of Π'. This induces a simplicial embedding of the barycentric subdivision of the face of $\Sigma_c(\Pi, \mathcal{A})$ corresponding to Π' into the barycentric subdivision of $\Sigma_c(\Pi', \mathcal{A})$.

Example 4.5 (continued) If $\epsilon \neq 0$, then $\Sigma_c(\Pi, \mathcal{A})$ and $\Sigma_{sc}(\Pi, \mathcal{A})$ are combinatorial 3-cubes, with three facets in common with the secondary polytope of $\mathcal{A}$. If $\epsilon = 0$, then $\Sigma_{sc}(\Pi, \mathcal{A})$ is a hexagonal facet of the secondary polytope while $\Sigma_c(\Pi, \mathcal{A})$ is still a 3-cube and has this hexagon glued through its interior.

If $\epsilon \neq 0$, then $\Omega_c(\mathcal{A}) = \Omega_{sc}(\mathcal{A})$ and, moreover, forgetting the trivial subdivision they coincide with $\Omega(\mathcal{A})$. If $\epsilon = 0$ the latter is still true for $\Omega_c(\mathcal{A})$ but not for $\Omega_{sc}(\mathcal{A})$. In this case the order complex of $\Omega_{sc}(\mathcal{A})$ has a relatively complicated homotopy type: it is two 2-spheres glued along the equator if we do not consider the trivial subdivision, and a 3-ball glued along the equator to a 2-sphere if we do. In contrast, $\Omega_c(\mathcal{A})$ is a homotopy 2-sphere and contractible, respectively, regardless of ϵ. The same happens with $\Omega(\mathcal{A})$. All this indicates that $\Omega_c(\mathcal{A})$ is more interesting than $\Omega_{sc}(\mathcal{A})$, both for its intrinsic properties and for its relation to the Baues poset.

4.3. **Toric GIT-quotients of the projective space and moduli spaces of stable semiabelic toric pairs.** Here we show the relation of the coherent poset of subdivisions $\Omega_c(\Pi, \mathcal{A})$ and two algebraic schemes considered respectively in [24, Section 4] and [2, Section 2]. We assume that $\mathcal{A}$ is an integer point configuration.

First we sketch an alternative definition of the secondary polytope as a particular case of a fiber polytope [8]. Let Δ be the unit simplex of dimension $|A| - 1$, let $Q = \operatorname{conv}(\mathcal{A})$ and let $\pi : \Delta \to Q$ be the affine projection sending the vertices of Δ to $\mathcal{A}$. The *chamber complex* of $\mathcal{A}$ is the coarsest common refinement of all its triangulations, and it is a polyhedral complex with the property that for any b and b' in the same *chamber* the fibers $\pi^{-1}(b)$ and $\pi^{-1}(b')$ are polytopes with the same normal fan. The secondary polytope of $\mathcal{A}$ equals the Minkowski integral of $\pi^{-1}(b)$ over Q. At the combinatorial level, it coincides with the Minkowski sum of a finite number of $\pi^{-1}(b)$, with one b chosen in each chamber.

The normal toric variety associated to a polytope depends only on its normal fan, so there is no ambiguity in calling F_σ the toric variety of $\pi^{-1}(b)$, where σ is the chamber containing b. If $b \in \sigma$ and $b' \in \tau$ for two chambers with $\tau \subseteq \overline{\sigma}$ then the normal fan of $\pi^{-1}(b)$ refines the normal fan of $\pi^{-1}(b')$, which implies that there is a natural equivariant morphism $f_{\sigma\tau} : F_\sigma \to F_\tau$. Let us denote $\Lambda_{\mathcal{A}} := \varprojlim F_\sigma$ the inverse limit of all the F_σ and morphisms $f_{\sigma\tau}$. It has the following two interpretations:

(i) Let X_Δ be the projective space of dimension $|\mathcal{A}|-1$, which is the toric variety associated with the simplex Δ (what follows is valid for any polytope Δ). The toric varieties F_σ are the different toric GIT-quotients of X_Δ modulo the algebraic sub-torus whose characters are the monomials with exponents in $\mathcal{A}$ [24, Section 3]. $\Lambda_{\mathcal{A}}$ is the inverse limit of all of them, which contains the Chow quotient as an irreducible component [24, Section 4].

(ii) In [2, Section 2.11] a scheme M_{simp} is defined exactly as our $\Lambda_{\mathcal{A}}$. Although there $\mathcal{A}$ is assumed to be the set of all lattice points in Q, the connection of M_{simp} with $\Sigma_c(\mathcal{A})$ carried out (our Theorem 4.8) is independent of this fact.

    The main interest in that paper is in the moduli space M of *stable semi-abelic toric pairs* (see Sections 1.1.A and 1.2.B in [2] for the definitions). The author shows that there is a finite morphism $M \to M_{simp}$ (Corollary 2.11.11) and uses M_{simp} as a simplified model for studying M.

Theorem 4.8 (Alexeev). (i) *For each subdivision* Π *of* $\mathcal{A}$ *there is a morphism* $F_{\Sigma(\Pi,\mathcal{A})} \to M_{simp}$*, where* $F_{\Sigma(\Pi,\mathcal{A})}$ *is the normal toric variety of the generalized* *secondary polytope* $\Sigma(\Pi, \mathcal{A})$ *(See* [2, 2.12.13]*). This induces a natural stratification on* M_{simp} *(and on* M*) whose strata are (perhaps non-normal) toric varieties in 1-to-1 correspondence with subdivisions of* $\mathcal{A}$ *(Corollary 2.11.10).*

(ii) *The gluing of the moment maps in the different strata induces a surjective continuous map* Mom : $M_{simp} \to \varprojlim \Sigma_c(\Pi, \mathcal{A})$ *(Lemma 2.13.2) whose image* $\varprojlim \Sigma_c(\Pi, \mathcal{A})$ *is glued from the generalized secondary polytopes* $\Sigma_c(\Pi, \mathcal{A})$ *via the coherent refinement relation. The same holds for* M*.*

Most probably, the gluing referred to in the second part makes $\varprojlim \Sigma_c(\Pi, \mathcal{A})$ homeomorphic to the order complex of the poset $\Omega_c(\mathcal{A})$. In other words, most probably barycentric subdivisions of the generalized secondary polytopes $\Sigma_c(\Pi, \mathcal{A})$ can be chosen in a way compatible with the gluings induced by the moment maps. Even if this is not the case, it is straightforward that $\varprojlim \Sigma_c(\Pi, \mathcal{A})$ is connected if and only if (the order complex of) $\Omega_c(\mathcal{A})$ is connected. Hence:

Corollary 4.9. *The following properties are equivalent for an integer point configuration* $\mathcal{A} \subset \mathbb{Z}^k$*. Let* $Q = \mathrm{conv}(\mathcal{A})$*.*

(i) *The graph of triangulations of* $\mathcal{A}$ *is connected.*

(ii) *The coherent refinement poset* $\Omega_c(\mathcal{A})$ *is connected.*

(iii) *The inverse limit of all the toric GIT-quotients of the projective space of dimension* $|\mathcal{A}| - 1$ *modulo the subtorus defined taking the elements of* $\mathcal{A}$ *as characters is connected, and*

(iv) *(If* $\mathcal{A} = Q \cap \mathbb{Z}^k$*) the moduli space of all stable semiabelic toric pairs of type* $\leq Q$ *is connected.* $\square$

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Abstract. By the "space of triangulations" of a finite point configuration $\mathcal{A}$ we mean either of the following two objects: the *graph of triangulations* of $\mathcal{A}$, whose vertices are the triangulations of $\mathcal{A}$ and whose edges are the geometric bistellar operations between them or the partially ordered set (*poset*) of all polyhedral subdivisions of $\mathcal{A}$ ordered by *coherent refinement*. The latter is a modification of the more usual *Baues poset* of $\mathcal{A}$. It is explicitly introduced here for the first time and is of of special interest in the theory of toric varieties.

We construct an integer point configuration in dimension 6 and a triangulation of it which admits no geometric bistellar operations. This triangulation is an isolated point in both the graph and the poset, which proves for the first time that these two objects can be not connected.

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