Triangulations with very few geometric bistellar neighbors *

Francisco Santos [†]

Depto. de Matemáticas, Estadística y Computación
Universidad de Cantabria
E-39071, Santander, SPAIN.

September 23, 1998

Abstract

We are interested in a notion of elementary change between triangulations of a point configuration, the so-called *bistellar flips*, introduced by Gel'fand, Kapranov and Zelevinski. We construct sequences of triangulations of point configurations in dimension 3 with $n^2 + 2n + 2$ vertices and only $4n - 3$ geometric bistellar flips (for every even integer n), and of point configurations in dimension 4 with arbitrarily many vertices and a bounded number of flips. This drastically improves previous examples and seems to be evidence against the conjecture that any two triangulations of a point configuration can be joined by a sequence of flips.

Introduction

Given a finite point configuration $\mathcal{A}$ in the Euclidean space $\mathbf{R}^d$ of dimension d we call *triangulations of* $\mathcal{A}$ all the geometrically realized simplicial complexes which cover the convex hull of $\mathcal{A}$ and which have their sets of vertices contained in $\mathcal{A}$. In this paper we are interested in a notion of vicinity or elementary change between triangulations of a given point configuration $\mathcal{A}$ known as a *geometric bistellar flip* (or *flip*, for short).

This notion naturally arises in the theory of secondary and fiber polytopes [2], [9, Chapter 7], [14], [21, Lecture 10]. In this theory, given a point configuration of dimension d with n points, the *regular* [2, 12, 21] (also called *coherent* [9] or sometimes *convex*) triangulations of $\mathcal{A}$ are defined as those which coincide with an orthogonal projection of the lower facets of a simplicial polytope of dimension $d + 1$. The *secondary polytope* of $\mathcal{A}$, introduced by Gel'fand, Kapranov and Zelevinski (see [9]) is a $(n - d - 1)$-polytope whose vertices are in bijection with *regular* triangulations of $\mathcal{A}$ and whose edges are in correspondence with bistellar flips between regular triangulations (here n is the cardinality of $\mathcal{A}$. The number $n - d - 1$ will be called the *corank* of $\mathcal{A}$).

[†]`santos@matesco.unican.es, http://matsun1.unican.es/~santos`.

These properties of the secondary polytope imply that the collection of regular triangulations of $\mathcal{A}$ is connected under bistellar flips and that any regular triangulation has at least $n - d - 1$ flips. Non-regular triangulations with less than $n - d - 1$ flips have been constructed by de Loera et al. in [6, 8]. We will call such triangulations *flip-deficient*. For example, a construction from [8] gives, for each integer n, triangulations in $\mathbf{R}^3$ with $5n$ points and only $3n - 2$ bistellar flips. In the following sections we drastically improve these results by showing constructions of:

- For any positive even integer n, a triangulation of a point configuration in $\mathbf{R}^3$ with $n^2 + 2n + 2$ vertices and $4n - 3$ flips (Theorem 4).

- Triangulations of point configurations in $\mathbf{R}^4$ with arbitrarily large number of vertices and bounded number of flips (Theorem 10). The number of flips obtained is easier to analyze in a vector configuration instead of point configuration setting. For any positive integer n we construct a simplicial fan covering the vector space $\mathbf{R}^5$, with $15n + 2$ rank-1 cones (vertices) and only 9 bistellar flips (Theorem 12). For a point configuration in $\mathbf{R}^4$ we can achieve a number of 21 flips, and arbitrarily large number of vertices (Remark 13).

We are interested in triangulations with few flips as an approach to the question of whether any two triangulations of a point configuration can be connected by a sequence of geometric bistellar flips. The answer is known to be positive in the cases of dimension at most 2 ($d \leq 2$) or corank at most 3 ($n \leq d + 4$). In these cases it is also known that no flip-deficient triangulations exist. Different proofs of connectivity in the case $d \leq 2$ exist, the oldest written one is probably in [11]. That no flip-deficiency exists in this case is proved in [8]. Connectivity and no-flip-deficiency in the case $n \leq d + 3$ follow from the fact, proved by Lee [12], that in this case all the triangulations are *regular*. The case $n = d + 4$ is a recent result of Azaola and Santos [1], whose proof is based in the notion of *virtual chamber* introduced in [7].

Another interesting case is that of cyclic polytopes. Rambau [14] has proved that the set of triangulations is connected by flips when $\mathcal{A}$ is the collection of vertices of a cyclic polytope $C(n, d)$. On the other hand, there are triangulations of $C(11, 5)$ with only 4 flips, instead of 5 [16].

No negative example to the connectivity question is known, and the question itself is a weak version of the so-called Baues problem for triangulations, posed by Billera et al. [4] (see also [15, 17, 18]). The Baues problem asks whether the refinement poset of all subdivisions of a point configuration $\mathcal{A}$ with n points in dimension d is homotopy equivalent to an $(n - d - 2)$-sphere. The question has connections to oriented matroid theory [5, Section 9.6], zonotopal tilings (via the Bohne-Dress Theorem [5, 21]) and combinatorial differential geometry, as introduced by MacPherson [13].

The fact that we can construct triangulations with very few geometric bistellar flips seems to be evidence in favour of the existence of a negative example to the Baues question, for three reasons: Firstly, a triangulation with no flips at all would be an isolated element in the refinement poset, thus providing such

an example; secondly, triangulations with very few flips are nodes of very low order in the graph of triangulations of a point configuration, which increase the chances of the graph being disconnected; thirdly, the cases mentioned above in which the graph of triangulations is known to be connected are more or less the same ones for which triangulations are known not to have flip-deficiency. Exceptions to this rule are the case of cyclic polytopes already mentioned and that de Loera et al. [8] have proved that triangulations in dimension 3 with all the vertices in convex position have no flip-deficiency, while the connectivity question is not settled in this case.

Let us mention also that another paper by the author [20] shows that when the notions of triangulation and bistellar flip are generalized in the natural way to triangulations of oriented matroids, there is a triangulation of a non-realizable acyclic polytopal rank 34 oriented matroid on 38 elements with no flips. This can be considered a combinatorial analogue of a triangulation with no flips of a 33-dimensional polytope with 38 vertices. This uses a construction of Richter-Gebert [19].

1 Flips and vector configurations

Throughout the paper we will work in the framework of *simplicial fans of vector configurations* which is more general than that of triangulations of point configurations. This is essentially the same approach as in [3].

Let $\mathcal{A}$ be a finite subset of the real vector space $\mathbf{R}^{d+1}$ (this is what we call a *vector configuration* of dimension d). A cone of $\mathcal{A}$ is the positive span $pos(\sigma)$ of a subset σ of $\mathcal{A}$. If σ is linearly independent we say that the cone is *simplicial*. A *face* of a simplicial cone $pos(\sigma)$ is the simplicial cone $pos(\tau)$ generated by any subset τ of σ. A *facet* of $pos(\sigma)$ is a maximal proper face of it. A *simplicial fan* of $\mathcal{A}$ is a collection T of full-rank simplicial cones of $\mathcal{A}$ such that the intersection $pos(\sigma_1) \cap pos(\sigma_2)$ of any two simplicial cones of T is a face of both (possibly the face $\{0\} = pos(\emptyset)$) and that $\cup_{pos(\sigma) \in T} pos(\sigma) = pos(\mathcal{A})$.

Scaling the vectors of a configuration by positive scalars does not affect what simplicial fans can be obtained. Hence we can consider all the vectors to lie in the unit sphere $S^d \subset \mathbf{R}^{d+1}$. In this setting, simplicial fans become simplicial complexes geometrically realized in the sphere by geodesic simplices which use the points of $\mathcal{A}$ as vertices and cover the convex hull of $\mathcal{A}$ (where the convex hull of $\mathcal{A}$ in the sphere is taken in an obvious natural way: it is the intersection of the sphere with the positive span of $\mathcal{A}$). For this reason, and in order to unify the nomenclature, throughout this paper we will use the term *triangulation* of a vector configuration meaning *simplicial fan* and *convex hull* meaning *positive span*. We will also apply the terms *link*, *star* or *join* in a triangulation of a vector configuration with the meaning they would have in the associated simplicial complex (see [10]).

We say that $\mathcal{A}$ is *totally cyclic* (or *complete*) if it positively spans $\mathbf{R}^{d+1}$. We say that $\mathcal{A}$ is *acyclic* (or *pointed*) if there is a linear functional h which is positive on every point of $\mathcal{A}$. In this case the vector configuration $\mathcal{A}$ is equivalent for the purpose of triangulations to the configuration $\{a/h(a) \mid a \in \mathcal{A}\}$, which is a point

configuration of dimension d in the affine plane $h^{-1}(1)$. Reciprocally, a point configuration $\mathcal{A} \subset \mathbf{R}^d$ can be regarded as a vector configuration $\mathcal{A}' \in \mathbf{R}^{d+1}$, identifying the affine space $\mathbf{R}^d$ with an affine non-linear hyperplane in $\mathbf{R}^{d+1}$. In this sense the concept of triangulation (simplicial fan) of a vector configuration is a generalization of the concept of triangulation of a point configuration.

Following the terminology of matroid theory, we call a minimal linearly dependent subset of $\mathcal{A}$ a *circuit* (see [5] or [21] for details). The unique (up to a scalar factor) dependence equation in a circuit divides its elements into two parts $Z = Z_+ \cup Z_-$ containing respectively the elements with positive and negative coefficient in the equation. The pair (Z_+, Z_-) is called an *oriented circuit*; of course, if (Z_+, Z_-) is an oriented circuit, then so is (Z_-, Z_+), and the two of them are the only orientations of $Z = Z_+ \cup Z_-$. Since our interest will always be in oriented circuits we will use the word circuit assuming they have an orientation and will call the underlying unoriented circuit Z the *support* of (Z_+, Z_-).

We say that a circuit is *acyclic* if both Z_+ and Z_- are non-empty (equivalently if its support Z is an acyclic vector configuration). In a point configuration, all the circuits are acyclic and they coincide with the so-called *minimal Radon partitions*; that is to say, minimal pairs (Z_+, Z_-) such that the relative interiors of $conv(Z_+)$ and $conv(Z_-)$ intersect in a point. The support Z of an acyclic circuit (Z_+, Z_-) can be triangulated in exactly two ways:

$$T_+(Z) := \{conv(Z - \{p\}) | p \in Z_+\} \qquad T_-(Z) = \{conv(Z - \{p\}) | p \in Z_-\}.$$

Definition 1 Let T be a triangulation of $\mathcal{A}$ and $(Z_+, Z_-) \subset \mathcal{A}$ a circuit of $\mathcal{A}$. Suppose that the following conditions are satisfied:

(i) The triangulation $T_+(Z)$ is a subcomplex of T.

(ii) All the maximum-rank simplices of $T_+(Z)$ have the same link L in T. In particular, $T_+(Z) * L$ is a subcomplex of T. Here and in what follows we denote by $A * B$ the *join* of two simplicial complexes A and B [10].

In these conditions we can obtain a new triangulation T' of $\mathcal{A}$ by replacing the subcomplex $T_+(Z) * L$ of T with the complex $T_-(Z) * L$. This operation of changing the triangulation is called a *geometric bistellar flip* (or a *flip*, for short) supported on the circuit (Z_+, Z_-). We say that T and T' are *geometric bistellar neighbors*. We call the flip of type (k, l) if Z_+ and Z_- have k and l elements respectively.

Observe that our definition of flip supported on a circuit explicitly assumes that the circuit is oriented so that the star of the *negative part* of the circuit is "flipped out" and the *positive part* is "flipped in". This convention (which is not present in other definitions, see [9, page 231]) will be important in our exposition.

When counting the number of flips of a triangulation in Sections 3 and 4 the properties that we state as lemmas below will be helpful. Let $conv(\sigma_1)$ and $conv(\sigma_2)$ be two maximal simplices of a triangulation T which share a facet

$conv(\tau)$. Let a_1 and a_2 be the two vertices joined to τ; that is to say, let $\sigma_1 = \tau \cup \{a_1\}$ and $\sigma_2 = \tau \cup \{a_2\}$. Then, $\rho = \tau \cup \{a_1, a_2\}$ is a spanning set of $\mathcal{A}$ with $d + 1$ elements, and there is a unique circuit (Z_+, Z_-) with support contained in ρ, oriented so that $a_1, a_2 \in Z_+$. We say that the facet τ is *flippable* if there is a flip supported on that circuit. If we assume that all the vectors/points of $\mathcal{A}$ are used as vertices of T, then no flips of type $(1, k)$ are possible and every flip of T has one or more flippable facets. Reciprocally, a flippable facet can only correspond to one flip. That is to say:

Lemma 2 *In a triangulation which uses all the elements as vertices, an exhaustive search of flippable facets gives all the possible flips.*

Given a vector configuration $\mathcal{A} \in \mathbf{R}^{d+1}$ and a non-zero vector $a \in \mathcal{A}$, we call *contraction* of $\mathcal{A}$ at a_1 and denote $\mathcal{A}/a_1$ the vector configuration $\{v - \frac{a_1 \cdot v}{a_1 \cdot a_1} a_1 \mid v \in \mathcal{A} \setminus \{a_1\}\}$. $\mathcal{A}/a_1$ can be considered a vector configuration in one dimension less since all its vectors are orthogonal to a_1. For a point configuration $\mathcal{A} \in \mathbf{R}^d$, the contraction at a point $a_1 \in \mathcal{A}$ is the vector configuration $\mathcal{A}/a_1 = \{b - a_1 \mid b \in \mathcal{A} \setminus \{a_1\}\} \subset \mathbf{R}^d$.

Lemma 3 *Let T be a triangulation of a point or vector configuration $\mathcal{A}$. Let a_1 be a vertex of one of the simplices of T which are removed by a flip on a certain circuit (Z_+, Z_-). Then,*

(i) *$(Z_+ \setminus \{a_1\}, Z_- \setminus \{a_1\})$ is a circuit of $\mathcal{A}/a_1$.*

(ii) *$link(a_1)$ is a triangulation of the vector configuration $\mathcal{A}/a_1$.*

(iii) *$link(a_1)$ has a flip supported on the circuit $(Z_+ \setminus \{a_1\}, Z_- \setminus \{a_1\})$.*

Proof: Easy. Left to the reader. □

2 A construction in dimension three

Let n be an even positive integer. Throughout this section we will call $\mathcal{A}_n$ the following point configuration with $n^2 + 2n + 2$ points in $\mathbf{R}^3$:

- The $n^2 + 2$ points $(x, y, 0) \in \mathbf{R}^3$ in the plane $z = 0$ whose coordinates x and y are integer and satisfy

$$|x| \le 2y \le 2n + 1 - |x|$$

$$2y \ne |x| + 1, \qquad 2y \ne 2n - |x|.$$

  We will call these points the *planar part of $\mathcal{A}_n$*. Figure 1 shows a triangulation of the planar part of $\mathcal{A}_n$ (for $n = 12$) which will be used later.

- The n points $P_i := (0, i, 1)$, for $i = 1, \ldots, n$, "above" the planar part.

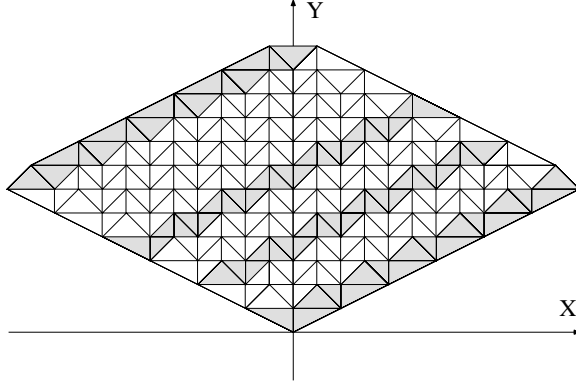- The n points $Q_i := (0, i, -1)$, for $i = 1, \ldots, n$, "below" the planar part.

Figure 1: A triangulation of the planar part of $\mathcal{A}_{12}$.

The triangulation in Figure 1 is obtained by considering all the possible triangles in the configuration of the form $[(x, y, 0), (x + 1, y + 1, 0), (x, y + 1, 0)]$ and $[(x, y, 0), (x + 1, y + 1, 0), (x + 1, y, 0)]$ for even x, all the triangles of the form $[(x, y, 0), (x + 1, y, 0), (x, y + 1, 0)]$ and $[(x + 1, y + 1, 0), (x, y + 1, 0), (x + 1, y, 0)]$ for odd x, and then completing with the introduction of the segments $[(0, 0, 0), (0, 1, 0)]$, $[(n - 2, n/2, 0), (n - 1, n/2 + 1, 0)]$ and $[(-n + 2, n/2, 0), (-n + 1, n/2 + 1, 0)]$ and the segments $[(x - 1, y, 0), (x + 1, y, 0)]$ for every "missing" integer point $(x, y, 0)$ with $2y = |x| + 1$ or $2y = 2n - |x|$. Our goal is to construct a triangulation of $\mathcal{A}_n$ which extends this one and show that it has $O(n)$ flips.

We proceed as follows: for each $i = 1, \ldots, n$, we join P_i to all the triangles in the planar part which have their vertices in the diagonal strip $x + 2i - 2 \leq 2y \leq x + 2i + 1$. The four shaded regions in Figure 1 show the triangles to be joined to P_1, P_4, P_7 and P_{12} respectively. Then, we insert tetrahedra of the form $[P_i, P_{i+1}, P, Q]$ were $[P, Q]$ are certain edges of the triangulation of the planar part, in order to 'fill in the gaps'. This can be done in a unique way to provide a triangulation of the upper half $\mathcal{A}_n \setminus \{Q_1, \ldots, Q_n\}$ of $\mathcal{A}_n$.

Finally, we triangulate the lower half by applying the symmetry transformation $[x \rightarrow -x; z \rightarrow -z]$. In other words, we consider diagonal strips in the planar part in the other possible direction and join them to the points $Q_1, \ldots, Q_n$. Let T_n be the triangulation of $\mathcal{A}_n$ obtained in this way.

In order to clarify the construction Figure 2 shows what we get as the link of each P_i, $i = 1, \ldots, n$ (for $n = 6$). Upward arrows in the link of P_i represent edges going to P_{i+1} and downward arrows represent edges going to P_{i-1}. In particular, there will be an edge $[P_i, P_{i+1}]$ for $i = 1, \ldots, n - 1$. The link of Q_i is obtained from that of P_i by the symmetry $[x \rightarrow -x; z \rightarrow -z]$. The links of P_i and Q_i, $i = 1, \ldots, n$ completely characterize the triangulation, since every tetrahedron contains one of those points as a vertex. Thick edges in the figure separate parts of the link of a point P_i which are joined to different points among the Q_j's. The shaded parts indicate some of the flips of the triangulation, to be discussed now:
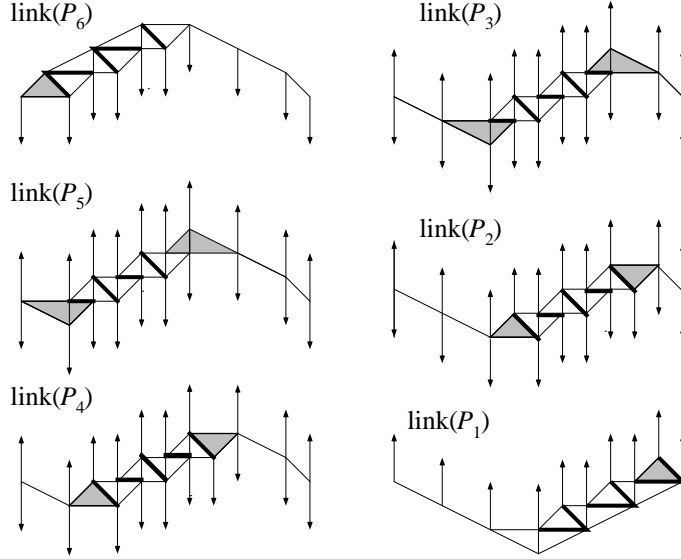
6

Figure 2: Links of the points P_i in the triangulation of $\mathcal{A}_6$.

(i) Suppose that a flip is supported in a circuit fully contained in the planar part. This has to correspond to a flip in Figure 1 but, moreover, all the triangles of the planar part which disappear by the flip have to be joined to the same points P_i and Q_j. In other words, the planar flip has to be contained in one of the links of Figure 2 and the triangles involved not be separated by a thick edge. A quick look at Figure 2 shows that there are no flips of this type.

(ii) Suppose that a flip is supported in a circuit C that contains a point P_i and no Q_j. Since the circuits of type $(\{P_{i-1}, P_{i+1}\}, \{P_i\})$ are clearly non-flippable, C must contain exactly two consecutive points P_i and P_{i+1} and either two or three points in the planar part.

If C contains two points in the planar part, they are of the form $(x, y, 0)$ and $(x, y + 1, 0)$. In this case the flip corresponds to moving the two triangles incident to the edge $[(x, y, 0), (x, y + 1, 0)]$ up (from $link(P_i)$ to $link(P_{i+1})$) or down (from $link(P_{i+1})$ to $link(P_i)$).

For example, the two shaded triangles in the right-top corner of $link(P_3)$ in Figure 2 represent a flip in the circuit $(\{P_3, (3, 5, 0)\}, \{P_4, (3, 4, 0)\})$. The triangles $[P_3, P_4, (3, 4, 0)]$ and $[(3, 5, 0), P_4, (3, 4, 0)]$ are both present in the triangulation and have the same link, consisting of the points $(2, 4, 0)$ and $(5, 4, 0)$. After performing the flip in this circuit, the links drawn in Figure 2 would change in that the two shaded triangles $[(3, 5, 0), (3, 4, 0), (2, 4, 0)]$ and $[(3, 5, 0), (3, 4, 0), (5, 4, 0)]$ will now appear in $link(P_4)$ instead of $link(P_3)$ (with the corresponding changes in the upward and downward arrows). This is what we call "moving up" two triangles and a similar "moving

7

down" will be possible with the two shaded triangles in the bottom-left corner of $link(P_3)$.

Looking at Figure 2 one checks that there are exactly $n-2$ flips of this type, two in $link(P_i)$ for each odd $i = 3, 5, \ldots, n-1$.

If C contains three points in the planar part, then these three points are the vertices of a triangle and the flip corresponds to moving this triangle up or down in a similar fashion. There are n flips of this type: two in $link(P_i)$ for each even $i = 2, 4, \ldots, n-2$ plus one in $link(P_1)$ and one in $link(P_n)$.

(iii) By symmetry, we conclude that there are also $n - 2 + n = 2n - 2$ flips in circuits containing one of the points Q_j and none of the P_i.

(iv) Let (Z_+, Z_-) be a circuit which supports a flip and which contains at least one P_i and one Q_j. Since $Z_+ \cup Z_- \setminus \{a\}$ is a simplex in the triangulation for every $a \in Z_+$ and since P_i and Q_j are not joined by an edge, $Z_+ \subset \{P_i, Q_j\}$. In fact, $Z_+ = \{P_i, Q_j\}$ because otherwise the unique point in Z_+ should lie in the relative interior of the convex hull of the simplex $Z_- \in T$, and thus the point in Z_+ would not be used in the triangulation, which is not the case. Since $Z_- \cup \{P_i\}$ and $Z_- \cup \{Q_j\}$ are simplices in the triangulation, Z_- has to be the set of vertices of a simplex in the planar part of T, and the relative interior of this simplex intersect the segment $[P_i, Q_j]$. In particular, the point $(0, \frac{i+j}{2}, 0)$ must lie in the relative interior of $conv(Z_-)$. Then:

- If $i + j$ is odd, let us call $k = \frac{i+j+1}{2}$. We must have $Z_- = \{(0, k-1, 0), (0, k, 0)\}$. Since the segment $[(0, k-1, 0), (0, k, 0)]$ is joined in T_n to the points P_k and Q_k and to no other of the P_l's or Q_l's, we conclude that $i = j = k$, which contradicts the fact that $i + j$ is odd.
- If $i + j$ is even, let us call $k = \frac{i+j}{2}$. Then, $Z_- = \{(0, k, 0)\}$. This point is joined to P_k, P_{k+1}, Q_k and Q_{k+1}, and to no other P_l or Q_l. Thus, in order to have $i + j = 2k$ we must have $i = j = k$ and the circuit be $(\{P_k, Q_k\}, \{(0, k, 0)\})$. For this circuit to be flippable, the segments $[P_k, (0, k, 0)]$ and $[Q_k, (0, k, 0)]$ need to have the same link, and this only happens for $k = n$. Thus, there is only one flip of this type.

Summarizing, we conclude that:

Theorem 4 *The point configuration $\mathcal{A}_n \subset \mathbf{R}^3$ has $n^2 + 2n + 2$ points. The triangulation T_n of $\mathcal{A}_n$ has $4n - 3$ flips.*

3 Stacking layers of prisms

Prisms

Our construction in dimension four will be based on the strategy of stacking several layers of triangulated prisms in a concentric way without increasing

the number of flips. A similar technique has implicitly been used in the last construction in [8].

The standard *d-dimensional prism* is the point configuration $\{e_1, \ldots e_d, e_1 + e_0, \ldots, e_d + e_0\} \subset \mathbf{R}^{d+1}$, where $e_0, e_1, \ldots, e_d$ is an affine basis in $\mathbf{R}^d$. We say that a (vector or point) configuration $\mathcal{A}$ is a *d-prism* if it consists of $2d + 2$ points and there is a bijective correspondence between them and the points in the standard prism which preserves circuits (in other words, if $\mathcal{A}$ has the same oriented matroid as the standard prism). That is to say, the configuration $\mathcal{A} = \{0, \ldots, d, \overline{0}, \ldots, \overline{d}\}$ is a prism if its circuits are the pairs of the form $(\{i, \overline{j}\}, \{j, \overline{i}\})$, for $i \neq j$. Any prism is projectively equivalent to the standard one.

The subsets $\{0, \ldots, d\}$ and $\{\overline{0}, \ldots, \overline{d}\}$ of a prism are the vertices of two simplicial facets of the prism. We call these facets the *floor* and the *ceiling* of the prism respectively, although which of them is the floor and which the ceiling is an arbitrary choice. Triangulations of a d-dimensional prism are in bijective correspondence with permutations of the symbols $[0, \ldots, d]$ via the following rule: to the permutation $[i_0, \ldots, i_d]$ we associate the triangulation:

$$\{conv(\{i_0, \ldots, i_l, \overline{i_l}, \ldots, \overline{i_d}\}) \mid l = 0, \ldots, d\}.$$

Moreover, two such triangulations differ by a bistellar flip if and only if the corresponding permutations differ by a transposition. This correspondence is well known (see [9, Section 7.3]) and makes the secondary polytope of the prism a *permutahedron*.

On the other hand, permutations of $[0, \ldots, d]$ correspond in an obvious bijective way with acyclic orientations of the complete graph K_{d+1} on $d + 1$ vertices: we identify the nodes of K_{d+1} with the $d + 1$ symbols $0, \ldots, d$ and orient the edges in K_{d+1} according to the total order in the vertices given by the permutation. Transpositions in the permutation correspond to reversing a single edge in the graph. Summing up:

Lemma 5 *Let $\tau \cup \overline{\tau}$ be a prism with floor τ and ceiling $\overline{\tau}$. Then, the triangulations of the prism are in bijective correspondence with acyclic orientations of the 1-skeleton of $conv(\tau)$ and two triangulations differ by a bistellar flip if and only if the corresponding orientations of the 1-skeleton differ by reorientation of a single edge.* □

A further observation will be used later on. In the conditions of the statement, let $i_0, i_d \in \tau$ respectively be the unique source and sink in a given acyclic orientation of the 1-skeleton of $conv(\tau)$. In the triangulation of the prism corresponding to that orientation, the floor $conv(\tau)$ of the prism is joined to the vertex $\overline{i_0}$ which is above the source i_0 and the ceiling $\overline{\tau}$ is joined to the sink i_d. Also, all the facets of the prism other than the ceiling and the floor are prisms over facets of $conv(\tau)$ and the restriction of T to them is obtained by restricting the orientation chosen in the 1-skeleton of $conv(\tau)$ to that facet.

Triangulating layers of prisms

Let T be a triangulation of a point or vector configuration $\mathcal{A}$ and let $O \in \mathcal{A}$ be a distinguished vertex. Let $\mathcal{B}$ be the subconfiguration of $\mathcal{A}$ consisting of the

vertices in the link of O and $\mathcal{C}$ be the subconfiguration of vertices not in the star of O (so that $\mathcal{A}$ is the disjoint union of $\{O\}$, $\mathcal{B}$ and $\mathcal{C}$). In the case of a vector configuration, we assume further that $\{O\} \cup \mathcal{B}$ is acyclic and without loss of generality we consider $\{O\} \cup \mathcal{B}$ to lie in an affine hyperplane.

Let c be a positive real number smaller than 1 (we will typically take c to be very close to 1). For each integer n, consider the following (point or vector) configurations:

$$\mathcal{B}_n := \{c^n P + (1 - c^n)O; P \in \mathcal{B}\}, \qquad \mathcal{A}_n := \{O\} \cup \mathcal{B}_0 \cup \cdots \cup \mathcal{B}_n \cup \mathcal{C}.$$

That is, $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{B}_n$ is a copy of $\mathcal{B}_{n-1}$ contracted towards O; $\mathcal{A}_n$ is obtained adding the n contracted copies of $\mathcal{B}$ to $\mathcal{A}$. We will denote by $c^n \tau$ and $c^n a$ the images in $\mathcal{B}_n$ of any subset τ or element a of $\mathcal{B}$. This is a slight abuse of notation unless O is taken as origin of coordinates (which can be done for a point configuration but not for a vector configuration).

In these conditions, the following is a polytopal subdivision (a fan in the standard terminology for vector configurations) of $\mathcal{A}_n$: the antistar of O in T together with, for each simplex $conv(\tau)$ of T in the link of O, the simplex $conv(\{O\} \cup c^n \tau)$ and the cells $conv(c^k \tau \cup c^{k-1} \tau)$ for each $k = 1, \ldots, n$. Observe that what we do is contracting the star of O in T and inserting the cells $conv(c^k \tau \cup c^{k-1} \tau)$ in between the star and the anti-star. Moreover, the cells $conv(c^k \tau \cup c^{k-1} \tau)$ that we have introduced are (convex hulls of) prisms, whose floor and ceiling are contracted copies of the simplices in the link of O in T. Figure 3 shows an example of the construction.
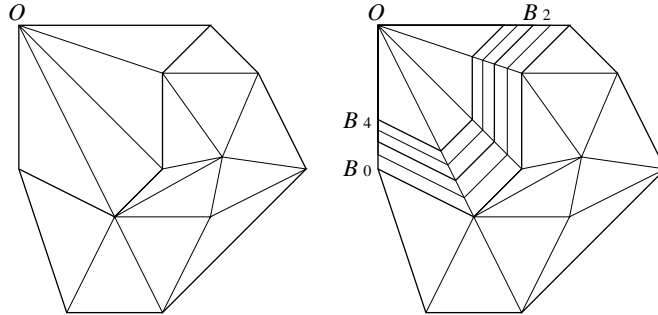

Figure 3: Insertion of 4 layers of prisms in a planar triangulation.

The subdivision S_n described above will be called the subdivision *obtained by inserting n layers of prisms around O in T with parameter c.* We call k-th *layer* of S_n, for $k = 1, \ldots, n$, the union of the prisms $conv(c^k \tau \cup c^{k-1} \tau)$.

If we want to refine S_n into a triangulation, we just need to triangulate each of the prisms in the layers. Let us concentrate in a layer, with index k. The prisms in the layer have a simplicial facet in $\mathcal{B}_{k-1}$ and another one in $\mathcal{B}_k$. We consider the first one to be the floor of the prism. In order to specify a triangulation of the layer we just need to give an orientation to all the edges of

10

S_n which have their vertices in $\mathcal{B}_{k-1}$ in such a way that each simplex is oriented acyclically. Figure 4 shows how to triangulate the third layer in the example of Figure 3.
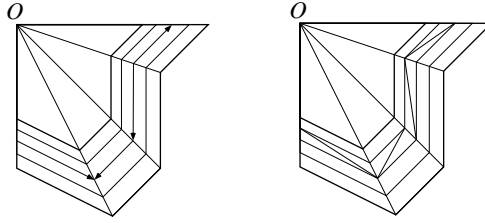

Figure 4: Orienting the edges in $\mathcal{B}_2$ we triangulate a layer of Figure 3.

Since the floor of each layer is a contracted version of the link of O in the original triangulation T, we conclude that:

Lemma 6 *Let S_n be the subdivision obtained by inserting n-layers of prisms around the vertex O in a certain triangulation T. Let G be the 1-skeleton of the link of O in T. Then, in order to refine S_n into a triangulation it is sufficient to choose n oriented copies $G_1, \ldots, G_n$ of the graph G, all of them acyclic on every simplex of the link, and use G_i to triangulate the i-th layer, for $i = 1, \ldots, n$.*

Inserting layers with no addition of flips

Remember that flips in triangulations of a prism correspond to edges which can be reversed in the orientation of the floor; then, it will be interesting to us that the orientations $G_1, \ldots, G_n$ which appear in Lemma 6 be "rigid" in the following sense:

Definition 7 Let K be a simplicial complex and let G be its 1-skeleton. We say that an orientation of G is *rigid* (with respect to K) if it is acyclic on every simplex of K but reversing the orientation of any single edge makes it cyclic in some simplex. Following standard graph theory terminology we say that the orientation *has no sources or sinks* if no vertex of G has all the orientations of its incident edges in-going or out-going.

For a directed graph G, $-G$ denotes the opposite orientation of G.

Theorem 8 *Let T be a triangulation of a configuration $\mathcal{A}$ which uses all the elements. Let $O \in \mathcal{A}$ be one of the vertices of the triangulation and let L be the link of O in T. Suppose that no two adjacent maximal simplices in L lie in the same hyperplane (we will say that the link of O is* generic *when it satisfies this condition). Let S_n be the subdivision obtained by inserting n layers around O, with the parameter c which appears in the construction sufficiently close to 1.*

Let G be a rigid orientation of the 1-skeleton of L with no sources or sinks. Consider the refinement T_n of S_n obtained as in Lemma 6 using $G_k = G$ for even k and $G_k = -G$ for odd k. Then,

(i) T_n *has exactly the same number of flips as* T_1, *for every* n.

(ii) *For every flip of* T_n, *either all the flippable facets contain* O *or some flippable facet has its vertices contained in* $\mathcal{B}_0 \cup \mathcal{C}$.

Proof: Let us recall that we can count flips of T_n by saying which facets of T_n are flippable, as in Lemma 2 (we call facets of T_n the facets of the maximal cells of T_n).

Let τ be a facet in T_n. By construction, either τ contains O, or is contained in $\mathcal{B}_0 \cup \mathcal{C}$, or is contained in $\mathcal{B}_k \cup \mathcal{B}_{k-1}$, for some $k \in \{1, \ldots, n\}$. The following cases (a) to (d) cover all possibilities except for the case $\tau \subset \mathcal{B}_0 \cup \mathcal{C}$ which we will not need to care about. The four cases are illustrated in parts (a) to (d) of Figure 5, where the facets under study are drawn thicker:
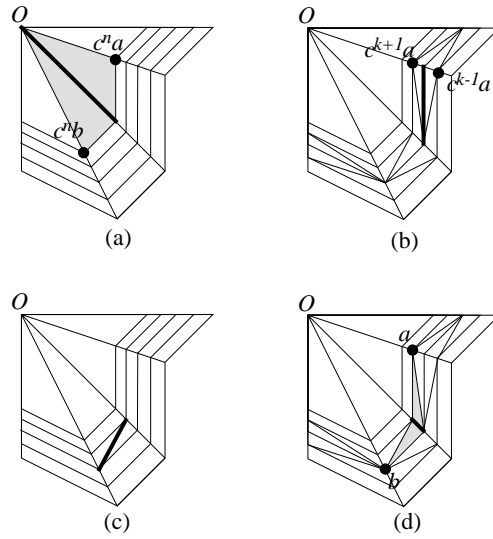

Figure 5: Different types of facets in the proof of Theorem 8

(a) Let $conv(c^n\tau \cup \{O\})$ be a facet of T_n which has O as a vertex, and suppose that it is flippable. By construction of T_n, $conv(c^n\tau \cup \{O\})$ is the contracted version of a certain facet $conv(\tau \cup \{O\})$ of T with $\tau \subset \mathcal{B}_0 = \mathcal{B}$, as is implicit in our notation. The facet $conv(\tau)$ is joined in T to at most two (perhaps one, but this case is treated in an analogous way) points a and b in $\mathcal{B}$, so that $conv(c^n\tau \cup \{O\})$ is joined to two points $c^n a$ and $c^n b$.

Since the link of O in T is generic, O is in the support of the circuit contained in $\{c^n a, c^n b, O\} \cup c^n\tau$. Moreover, $\{c^n a, c^n b\} \cup c^n\tau$ is not a simplex in T_n (since $\{c^n a\} \cup c^n\tau$ and $\{c^n b\} \cup c^n\tau$ are the ceilings of two different prisms in the n-th layer). Then, for the circuit to be flippable (with our sign convention) it is necessary that O be in the negative part of the circuit. This implies that all the maximal simplices to be removed by the flip (the shaded triangles in part (a) of Figure 5) have O as a vertex. In

12

this case it is clear that the same flip can be performed in T_1 (actually, in T as well), changing $c^n a$, $c^n b$ and $c^n \tau$ for $c^1 a$, $c^1 b$ and $c^1 \tau$.

(b) Let us now see that no facet $conv(c^k \tau)$ with $c^k \tau \subset \mathcal{B}_k$ ($k \in \{1, \ldots, n\}$) is flippable. The facet $conv(c^k \tau)$ is joined to a vertex $c^{k-1} a$ in $c^{k-1} \tau$ and either to O (if $k = n$) or to an element $c^{k+1} b$ in $c^{n+1} \tau$, if $k < n$. More precisely, a is the sink of τ in the orientation G_k of the 1-skeleton of L and, if $k < n$, b is the source of τ in G_{k+1}. Since G_k and G_{k+1} are opposite to one another, $a = b$. If $conv(c^k \tau)$ is flippable, the circuit in which the flip is supported is $(\{O, c^{n-1} a\}, \{c^n a\})$ if $k = n$ and $(\{c^{k-1} a, c^{k+1} a\}, \{c^k a\})$ if $k < n$. Thus, the link of $conv(\{c^{k-1} a, c^k a\})$ in T_n has to coincide with that of $conv(\{c^{k+1} a, c^k a\})$ (or $conv(\{O, c^n a\})$, if $k = n$) which implies that all the vertices in the link of $conv(\{c^{k-1} a, c^k a\})$ belong to $\mathcal{B}_k$. But this would imply a to be a sink of the orientation G_k of L. Since $G_k = \pm G$ and G has no sources or sinks, G_k has no sinks.

(c) Also, no facet interior to a prism $c^{k-1} \tau \cup c^k \tau$ can be flippable, because then the circuit involved must be one of the circuits of a prism. Performing the flip is equivalent to reversing the orientation of a certain edge in the graph G_k, which is impossible because G_k is rigid.

(d) Finally, let $conv(\tau)$ be a facet having vertices both in $\mathcal{B}_k$ and $\mathcal{B}_{k-1}$ for some $k \in \{1, \ldots, n\}$, but not interior to a prism. That is to say, the two maximal simplices $conv(\tau \cup a)$ and $conv(\tau \cup b)$ sharing $conv(\tau)$ belong to different prisms of the same layer. If the facet is flippable, the flip is supported in the unique circuit $Z = (Z_+, Z_-)$ contained in $\tau \cup \{a, b\}$, oriented so that $\{a, b\} \subset Z_+$. The fact that no two facets in the link of O in T are coplanar implies that, if c is sufficiently close to 1, the segment $conv(\{a, b\})$ is not contained in the k-th layer. We assume c to be so close to 1 that this happens. Then:

(d.1) If $Z_+ = \{a, b\}$, then this together with $Z_- \subset \tau$ imply that $conv(\tau \cup \{a\})$ and $conv(\tau \cup \{b\})$ (which are both contained in the k-th layer) cover $conv(Z_+ \cup Z_-)$. This contradicts the assumption that $conv(\{a, b\})$ is not contained in the k-th layer.

(d.2) If there is an element $p \in Z_+$ other than a and b, then $\sigma = Z_+ \cup Z_- \setminus \{p\}$ is a simplex of T_n which contains a and b. In particular, a and b are joined by an edge. The fact that $conv(\{a, b\})$ is not contained in the k-th layer implies that this edge has not been introduced by the insertion of layers. Thus, the only possibility is that $k = 1$, a and b are in $\mathcal{B}_0$ and σ is a simplex outside the layers. In particular, σ only contains vertices in $\mathcal{B}_0 \cup \mathcal{C}$. The facets $conv(\sigma \setminus \{a\})$ and $conv(\sigma \setminus \{b\})$ of σ are flippable by the flip, so that the flip is in the conditions of (ii).

The above case study implies that all the flips either are in the conditions of (a) or (d.2) or all their flippable facets lie in $\mathcal{B}_0 \cup \mathcal{C}$. This proves part (ii) of the statement. The flips studied in (a) appear in T_1 as mentioned above. The flips mentioned in (d.2) and those with flippable facets in $\mathcal{B}_0 \cup \mathcal{C}$ only remove

simplices from the first layer or outside the layers. Thus, they are flips in T_1 as well. This proves (i). □

Remark 9 Among the hypotheses in Theorem 8, the only one which is difficult to fulfill in a construction is a rigid orientation without sources or sinks of the 1-skeleton of L. If $\mathcal{A}$ is a configuration in dimension d, the link L of a point is a triangulation of the $(d-1)$-sphere or the $(d-1)$-ball. It is clear that no orientation in a triangulation of the 1-sphere or 1-ball is rigid. For the 2-sphere or the 2-ball the same is not obvious, but still can be proved as follows.

Let T be a triangulation of the 2-sphere or the 2-ball with v vertices, e edges and t triangles. Let an orientation be given to each edge in such a way that every triangle is acyclic. Then, each triangle of T prevents one of its edges to be reversed, which implies that the number of edges whose orientations cannot be changed is at most t. Thus, in order for the orientation to be rigid we would need to have $t \geq e$. This is impossible since $t - e = \chi - v < 0$ (where $\chi = 1$ for the 2-ball and $\chi = 2$ for the 2-sphere).

Still, a version of our construction can serve to insert layers of prisms in a 3-dimensional triangulation adding less flips per layer than vertices. The idea is to orient the 1-skeleton of a triangulated 2-sphere or 2-ball in a way acyclic on each triangle and with few reversible edges (edges whose reversal preserve acyclicness in triangles). The best possible ratio of reversible edges versus number of vertices is $3/5$, obtained by orienting the 1-skeleton of a triangular bipyramid (5 vertices, 6 triangles) with only 3 reversible edges. Although it is impossible to do this without sinks or sources, this idea is used in [8] to obtain triangulations in dimension 3 with $5n$ vertices and $3n - 2$ flips.

4 A construction in dimension four

A rigid orientation of a triangulation of the 3-sphere

We are going to show a rigid orientation with no sinks or sources of the 1-skeleton of a triangulation of the unit 3-sphere $S^3 \subset \mathbf{R}^4$, in order to apply Theorem 8 to it.

Our triangulation will have 15 vertices that we denote $t_{i,j}$, h_i and v_j for $i, j = 0, 1, 2$ (indices will be regarded modulo 3 in what follows), with coordinates

$$t_{ij} = \frac{1}{\sqrt{2}} \left(\cos\left(\frac{2\pi i}{3}\right), \sin\left(\frac{2\pi i}{3}\right), \cos\left(\frac{2\pi j}{3}\right), \sin\left(\frac{2\pi j}{3}\right) \right),$$

$$h_i = \left(\cos\left(\frac{2\pi i}{3}\right), \sin\left(\frac{2\pi i}{3}\right), 0, 0 \right), \quad v_j = \left(0, 0, \cos\left(\frac{2\pi j}{3}\right), \sin\left(\frac{2\pi j}{3}\right) \right)$$

(we use the letters t, v and h because if we think of S^3 in its stereographic projection Π into $\mathbf{R}^3$ given by $\Pi(x, y, z, t) = (\frac{x}{t-1}, \frac{y}{t-1}, \frac{z}{t-1})$, then the points h_i, $t_{i,j}$ and v_j lie, respectively, on a horizontal circle, a torus around it and its vertical rotation axis).

The six points h_i and v_j are the vertices of the join $\Delta_2 * \Delta_2$ of two triangles. The nine points $t_{i,j}$ are the vertices of the product $\Delta_2 \times \Delta_2$ of two triangles.

The common refinement of these two polytopes has eighteen facets, all of which are triangular prisms: each of the six facets of $\Delta_2 \times \Delta_2$ (which are 3-prisms themselves) is divided into three subprisms by its medial axis and each of the nine facets of $\Delta_2 * \Delta_2$ (which are tetrahedra) is divided into two prisms by a quadrilateral with vertices in the mid-points of four edges.

Our triangulation T of S^3 consists of the following 54 tetrahedra. For each $i = 0, 1, 2$ and $j = 0, 1, 2$:

$$[h_i, t_{i,j}, t_{i,j+1}, t_{i+1,j+1}], \quad [h_i, t_{i,j}, t_{i+1,j}, t_{i+1,j+1}], \quad [h_i, h_{i+1}, t_{i+1,j}, t_{i+1,j+1}],$$

$$[v_j, t_{i,j}, t_{i+1,j}, t_{i+1,j+1}], \quad [v_j, t_{i,j}, t_{i,j+1}, t_{i+1,j+1}], \quad [v_j, v_{j+1}, t_{i,j+1}, t_{i+1,j+1}].$$

Each of the two rows of tetrahedra above triangulates one of the 18 prisms mentioned, for each choice of indices i and j. Since there are nine possible choices, this gives the total of 18 triangulated prisms.

The permutations $[t_{i,j} \to t_{i+1,j} \; ; \; h_i \to h_{i+1} \; ; \; v_j \to v_j]$ and $[t_{i,j} \to t_{j,i} \; ; \; h_i \to h_i \; ; \; v_j \to v_j]$ produce symmetries of the triangulation T. They generate the symmetry group of the triangulation, which acts transitively and with trivial stabilizer over the 18 subprisms. Thus, the symmetry group has 18 elements. The first three tetrahedra in the list above are representatives for the three orbits of simplices.

There are two orbits of vertices in T, one containing all the $t_{i,j}$ and the other containing the h_i and v_j. There are five orbits of edges. We show an orientation for a representative of each and let the others be oriented by the action of the symmetry group: we orient $[h_i \to h_{i+1}]$ (6 edges in the orbit), $[t_{i,j} \to h_i]$ (18 edges), $[h_i \to t_{i+1,j}]$ (18 edges), $[t_{i,j} \to t_{i+1,j}]$ (18 edges) and $[t_{i,j} \to t_{i+1,j+1}]$ (9 edges). Then,

Theorem 10 *The above is a rigid orientation of a triangulation of the 3-sphere with no sources or sinks. Thus, there are triangulations of point configurations in $\mathbf{R}^4$ and of vector configurations in $\mathbf{R}^5$ with arbitrarily large number of vertices and a bounded number of flips.*

Proof: The orientations shown for the edges produce the following acyclic orientations on the three representatives of simplices: $[t_{i,j} \to t_{i,j+1} \to h_i \to t_{i+1,j+1}]$, $[t_{i,j} \to h_i \to t_{i+1,j} \to t_{i+1,j+1}]$ and $[h_i \to t_{i+1,j} \to t_{i+1,j+1} \to h_{i+1}]$. Reorienting any of the five representatives of edges would produce a 3-cycle in a triangle of the triangulation; for example, the five triangles are: $[h_i \to t_{i+1,j} \to h_{i+1}]$, $[t_{i,j} \to t_{i,j+1} \to h_i]$, $[h_i \to t_{i+1,j-1} \to t_{i+1,j}]$, $[t_{i,j} \to h_i \to t_{i+1,j}]$ and $[t_{i,j} \to t_{i,j+1} \to t_{i+1,j+1}]$. Thus, the orientation is rigid. From the oriented cycles $[h_0 \to h_1 \to h_2 \to h_0]$, $[v_0 \to v_1 \to v_2 \to v_0]$ and $[t_{i,0} \to t_{i,1} \to t_{i,2} \to t_{i,0}]$, $i = 0, 1, 2$, it follows that no vertex is a source or a sink.

For the final statement, we consider the 15 vectors in $S^3 \subset \mathbf{R}^4$ as points in $\mathbf{R}^4$ and consider the origin as an extra point. This gives a point configuration $\mathcal{A}$ with 16 points. For each tetrahedra in the sphere with vertices $\{a, b, c, d\}$ we consider the simplex with vertices $\{O, a, b, c, d\}$ in $\mathbf{R}^4$. This produces a collection of simplices of $\mathcal{A}$; we perturb a little the coordinates of every point so that the link of the origin O is generic. This collection of simplices may not

be a triangulation, if it does not cover the whole convex hull of $\mathcal{A}$. However, it can always be completed to a triangulation, perhaps with the addition of extra vertices. Thus, we can construct a triangulation of a point configuration $\mathcal{A}'$ in $\mathbf{R}^4$ such that the link of one of the vertices is generic and admits a rigid orientation of edges without sinks or sources. Theorem 8 tells us how to construct triangulations with arbitrarily many vertices and bounded number of flips from that. □

Counting flips

In what follows we look more closely at the above construction to compute an actual bound for the number of flips mentioned in the statement of Theorem 10. Let $\mathcal{A}' \subset \mathbf{R}^5$ be the following vector configuration: $\mathcal{A}' := \{O, -O, t_{i,j}, h_i, v_j \mid i, j = 1, 2, 3\}$, where $O = (1, 0, 0, 0, 0)$, $-O = (-1, 0, 0, 0, 0)$,

$$ t_{ij} = \frac{\alpha_{i,j}}{\sqrt{2}} \left(\frac{\sqrt{2}}{\alpha_{i,j}}, \cos\left(\frac{2\pi i}{3}\right), \sin\left(\frac{2\pi i}{3}\right), \cos\left(\frac{2\pi j}{3}\right), \sin\left(\frac{2\pi j}{3}\right) \right), $$

$$ h_i = \left(1, \alpha_i \cos\left(\frac{2\pi i}{3}\right), \alpha_i \sin\left(\frac{2\pi i}{3}\right), 0, 0 \right), $$

$$ v_j = \left(1, 0, 0, \alpha_j \cos\left(\frac{2\pi j}{3}\right), \alpha_j \sin\left(\frac{2\pi j}{3}\right) \right), $$

for some positive coefficients $\alpha_{i,j}$, α_i and α_j ($i, j = 0, 1, 2$).

The configuration $\mathcal{A}'$ is obtained from $\mathcal{A}$ by scaling the 15 vectors of $\mathcal{A}$ with the coefficients α, embedding $\mathbf{R}^4$ in an affine hyperplane of $\mathbf{R}^5$ with a central vector O and including the opposite vector $-O$. Observe that $\mathcal{A}$ is (up to scaling) the contraction of $\mathcal{A}'$ at O (and at $-O$ as well).

Let T' be the triangulation of $\mathcal{A}'$ obtained joining the triangulation T of $\mathcal{A}$ both to O and to $-O$, so that $link(O) = link(-O) = T$. The coefficients $\alpha_{i,j}$, α_i and α_j are chosen sufficiently generic for the link of O in T' to satisfy the genericity condition required in the statement of Theorem 8.

Proposition 11 *The triangulation* $T = link(O) = link(-O)$ *has exactly 9 flips, supported on the 9 circuits of the form* $(\{t_{i+1,j}, t_{i,j+1}\}, \{t_{i,j}, t_{i+1,j+1}\})$.

Proof: From the three representatives of simplices we conclude that there are six orbits of facets, with the following being representatives of them:

$$ conv(h_i, t_{i,j}, t_{i+1,j+1}), \qquad conv(h_i, t_{i,j}, t_{i+1,j}), $$

$$ conv(h_i, t_{i,j}, t_{i,j+1}), \qquad conv(h_i, t_{i+1,j}, t_{i+1,j+1}), $$

$$ conv(h_i, h_{i+1}, t_{i+1,j}), \qquad conv(t_{i,j}, t_{i,j+1}, t_{i+1,j+1}) $$

Let us see that the last five facets are not flippable:
- the two vertices joined to $conv(h_i, t_{i,j}, t_{i+1,j})$ are $t_{i+1,j+1}$ and $t_{i,j-1}$. Since $2t_{i,j} + t_{i+1,j+1} + t_{i,j-1} = 2\sqrt{2}h_i + t_{i+1,j}$, the associated circuit is $(\{t_{i,j}, t_{i+1,j+1}, t_{i,j-1}\},$

$\{h_i, t_{i+1,j}\}$). The circuit is not flippable, since $conv(h_i, t_{i+1,j+1}, t_{i,j-1}, t_{i+1,j})$ is not a simplex in the triangulation.

- the vertices joined to $conv(h_i, t_{i,j}, t_{i,j+1})$ are h_{i-1} and $t_{i+1,j+1}$. Since $t_{i,j+1} = t_{i+1,j+1} + \frac{1}{\sqrt{2}}h_{i-1} + \frac{2}{\sqrt{2}}h_i$, the associated circuit is $(\{t_{i,j+1}\}, \{t_{i+1,j+1}, h_{i-1}, h_i\})$. It is not flippable since $conv(t_{i,j+1}, t_{i+1,j+1}, h_{i-1})$ is not a simplex in the triangulation.

- the vertices joined to $conv(h_i, t_{i+1,j}, t_{i+1,j+1})$ are h_{i+1} and $t_{i,j}$. Since $h_{i+1} + \sqrt{2}t_{i,j} = h_i + \sqrt{2}t_{i+1,j}$ the associated circuit is $(\{t_{i,j}, h_{i+1}\}, \{h_i, t_{i+1,j}\})$, and it is not flippable since the links of $conv(t_{i,j}, h_i, t_{i+1,j})$ and $conv(h_{i+1}, h_i, t_{i+1,j})$ are not the same; the first one consists of the vertices $t_{i+1,j+1}$ and $t_{i,j-1}$ and the second one of $t_{i+1,j+1}$ and $t_{i+1,j-1}$.

- the two vertices joined to $conv(h_i, h_{i+1}, t_{i+1,j})$ are $t_{i+1,j+1}$ and $t_{i+1,j+2}$. The associated circuit is $(\{t_{i+1,j}, t_{i+1,j+1}, t_{i+1,j+2}\}, \{h_{i+1}\})$, since $t_{i+1,j} + t_{i+1,j+1} + t_{i+1,j+2} = \frac{3}{\sqrt{2}}h_{i+1}$. The link of $conv(t_{i+1,j}, t_{i+1,j+1}, h_{i+1})$ contains $t_{i+2,j+1}$ while the link of $conv(t_{i+1,j}, t_{i+1,j+2}, h_{i+1})$ contains $t_{i+2,j}$. Thus, the facet is not flippable.

- the vertices joined to $conv(t_{i,j}, t_{i,j+1}, t_{i+1,j+1})$ are h_i and v_j. The associated circuit is $(\{t_{i,j}\}, \{h_i, v_j\})$ since $\sqrt{2}t_{i,j} = h_i + v_j$. It is not flippable since the links of $conv(h_i, t_{i,j})$ and $conv(v_j, t_{i,j})$ do not coincide; h_{i-1} is a vertex in one but not in the other.

Thus, the only possible flippable facets are $conv(h_i, t_{i,j}, t_{i+1,j+1})$ and its images under the symmetry group. The vertices joined to this facet are $t_{i,j+1}$ and $t_{i+1,j}$ and the circuit is $(\{t_{i,j+1}, t_{i+1,j}\}, \{t_{i,j}, t_{i+1,j+1}\})$ since $t_{i,j+1} + t_{i+1,j} = t_{i,j} + t_{i+1,j+1}$. The circuit supports a flip since the facets $conv(t_{i,j}, t_{i+1,j}, t_{i+1,j+1})$ and $conv(t_{i,j}, t_{i,j+1}, t_{i+1,j+1})$ are joined to the same vertices, namely h_i and v_j. $\square$

Theorem 12 *Inserting n layers in the triangulation T' with a parameter c sufficiently close to 1 gives a triangulation T'_n of a vector configuration in $\mathbf{R}^5$ with $15(n + 1) + 2$ vertices and 9 flips.*

Proof: By part (ii) of Theorem 8, all the flips of T'_n have a flippable facet containing O or with its vertices in $\mathcal{B}_0 \cup \{-O\}$.

Let C be a circuit which produces a flip in which a flippable facet contains O. Genericity of the link of O implies that O is in the support of the circuit and that the rest of the elements in the circuit lie in the link of O, that is, they are in $\mathcal{B}_n$. In other words, C is of the form $(c^n C_+ \cup \{O\}, c^n C_-)$ or $(c^n C_+, c^n C_- \cup \{O\})$. The first case is impossible since then $conv(c^n C_+ \cup c^n C_-)$ would have to be a simplex in our triangulation T'_n, which is not the case. Thus, the flippable circuit is of the form $(c^n C_+, c^n C_- \cup \{O\})$. Lemma 3 implies that $(c^n C_+, c^n C_-)$ has to be a flippable circuit in $link(O)$, considered as a triangulation of a vector configuration in $\mathbf{R}^4$. That is, $(c^n C_+, c^n C_-)$ is the copy in $\mathcal{B}_n$ of one of the 9 circuits of the configuration $\mathcal{A}$ which supports a flip of T. Thus, we conclude that the only circuits which can produce flips with flippable facets containing O are those of the form $(c^n C_+, c^n C_- \cup \{O\})$, where (C_+, C_-) is a flippable circuit of T. There are 9 possibilities for (C_+, C_-).

17

No flippable facet can have all its vertices in $\mathcal{B}_0$: If $conv(c^0\tau)$ is a flippable facet with its vertices in $\mathcal{B}_0$, then one of the vertices joined to $conv(c^0\tau_0)$ is $-O$ and the other one is one of the vertices c^1a of $c^1\tau$. Thus, the corresponding circuit is $(\{c^1a, -O\}, \{c^0a\})$. A flip supported on that circuit would imply that a is a sink of the oriented graph used to construct the triangulation of the first layer, which is not the case.

Then, part (ii) of Theorem 8 tells us that the remaining flips have a flippable facet containing $-O$. Genericity of the link of O implies genericity of the link of $-O$. Thus, $-O$ is in the circuit which supports the flip and the rest of the points in the circuit are in $\mathcal{B}_0$. Let $(c^0C_+ \cup \{-O\}, c^0C_-)$ or $(c^0C_+, c^0C_- \cup \{-O\})$ be that circuit. Similar arguments as in the star of O prove that (C_+, C_-) is the copy in $\mathcal{B}_0$ of one of the 9 circuits of $\mathcal{A}$ which supports a flip of T and that $-O$ cannot be in the positive part of the circuit. Thus, the circuit is of the form $(c^0C_+, c^0C_- \cup \{-O\})$, for one of the 9 flippable circuits (C_+, C_-) of T.

We conclude that all the flippable circuits are of the form $(c^0C_+, c^0C_- \cup \{-O\})$ or $(c^nC_+, c^nC_- \cup \{O\})$, where (C_+, C_-) is a flippable circuit of T. This, in principle, gives 18 possibilities. However, for each circuit (C_+, C_-) of T, $(c^0C_+, c^0C_- \cup \{O\})$ is a circuit of $\mathcal{A}_n$ if and only if $(c^nC_+, c^nC_- \cup \{O\})$ is a circuit as well (this follows easily from the construction of the layers). The later is a circuit if and only if $(c^nC_+ \cup \{-O\}, c^nC_-)$ is a circuit (since O and $-O$ are opposite vectors). Finally, only one of $(c^nC_+ \cup \{-O\}, c^nC_-)$ and $(c^nC_+, c^nC_- \cup \{-O\})$ can be a circuit. This means that each flippable circuit of T produces (at most) one, instead of two flippable circuits in T'_n. Although we will not prove it it is easy to show that each flippable circuit of T indeed produces a flippable circuit of T'_n, or otherwise the link of O could not be generic. □

Remark 13 One may ask how many flips can be obtained in an acyclic version of the last construction. This amounts to removing the star of $-O$ and completing to a triangulation, perhaps with additional vertices, in Theorem 12. This case is more difficult to analyze because the simplices to be added in order to complete to a triangulation depend on the values chosen for the parameters α which perturb the points. The following argument shows that a number of flips lower or equal to 21 can be obtained. We think that a value of 9 flips can be obtained with a more careful analysis.

Consider the point configuration $\mathcal{A}$ in $\mathbf{R}^4$ consisting on the origin O and the following 15 points:

$$t_{ij} = \frac{1}{\sqrt{2}}\left(\cos\left(\frac{2\pi i}{3} - \beta\right), \sin\left(\frac{2\pi i}{3} - \beta\right), \cos\left(\frac{2\pi j}{3} - \beta\right), \sin\left(\frac{2\pi j}{3} - \beta\right)\right),$$

$$h_i = \left(\cos\left(\frac{2\pi i}{3}\right), \sin\left(\frac{2\pi i}{3}\right), 0, 0\right), \quad v_j = \left(0, 0, \cos\left(\frac{2\pi j}{3}\right), \sin\left(\frac{2\pi j}{3}\right)\right)$$

These are the same points of Section 4 except for the parameter β in the points t_{ij}, which we take to be positive and smaller than $\pi/12$. In these conditions the following 18 tetrahedra and 18 square pyramids are the facets of $conv(\mathcal{A})$. For each $i = 0, 1, 2$ and $j = 0, 1, 2$:

$$[h_i, t_{i,j}, t_{i,j+1}, t_{i+1,j}, t_{i+1,j+1}], \quad [h_i, h_{i+1}, t_{i+1,j}, t_{i+1,j+1}],$$

$$[v_j, t_{i,j}, t_{i,j+1}, t_{i+1,j}, t_{i+1,j+1}], \quad [v_j, v_{j+1}, t_{i,j+1}, t_{i+1,j+1}].$$

In other words, each of the 18 subprisms mentioned in Section 4 is broken into a square pyramid and a tetrahedron by the introduction of the parameter β. The only minimal non-simplicial faces of $conv(\mathcal{A})$ are the nine quadrilaterals $[t_{i,j}, t_{i+1,j}, t_{i,j+1}, t_{i+1,j+1}]$. The triangulation T of Section 4 refines $conv(\mathcal{A})$ and, thus, joining T to O we obtain a triangulation of $\mathcal{A}$, that we still call T.

The link of O has not the genericity property required in Theorem 8. We can get this by changing the three points t_{ii} to $\alpha O + (1 - \alpha)t_{ii}$, for a small positive constant α (we are moving these three points slightly towards O). This perturbation breaks the coplanarity of the former nine quadrilaterals but makes the union of the simplices that we have so far non-convex. The non-convex parts can be filled in with the following six 4-simplices. For each $i = 0, 1, 2$:

$$[h_i, t_{i,i}, t_{i,i+1}, t_{i+1,i}, t_{i+1,i+1}] \qquad [v_i, t_{i,i}, t_{i,i+1}, t_{i+1,i}, t_{i+1,i+1}]$$

Now we can count the number of flips in this triangulation with the same arguments as in the proof of Theorem 12. The six quadrilaterals that have been perturbed convex produce six flips in the star of the central point O, and they are the only flips containing O in a flippable facet. Part (ii) of Theorem 8 tells us that the other possible flips will have a flippable facet in one of the six extra 4-simplices added to the triangulation. Each of those simplices has three interior facets: the complements of h_i (or v_i), $t_{i+1,i}$ and $t_{i,i+1}$ and one of them is common to two 4-simplices. This gives a total of 15 possible flippable facets and, together with the six flips at O, a total of 21 possible flips.

References

[1] M. AZAOLA AND F. SANTOS, The graph of triangulations of $d + 4$ points is 3-connected, manuscript 1998.

[2] L. BILLERA, P. FILLIMAN AND B. STURMFELS, Constructions and complexity of secondary polytopes, *Adv. Math.* **83** (1990), 155–179.

[3] L. BILLERA, I.M. GEL'FAND AND B. STURMFELS, Duality and minors of secondary polyhedra, *J. Comb. Theory Ser. B* **57** (1993), 258–268.

[4] L. BILLERA, M. KAPRANOV AND B. STURMFELS, Cellular Strings on Polytopes, *Proc. Amer. Math. Soc.* **122**(2) (1994), 549–555.

[5] A. BJÖRNER, M. LAS VERGNAS, B. STURMFELS, N. WHITE AND G. ZIEGLER *Oriented Matroids*, Cambridge University Press, Cambridge 1992.

[6] J.A. DE LOERA, *Triangulations of Polytopes and Computational Algebra*, Ph.D. thesis, Cornell University, 1995.

[7] J.A. DE LOERA, S. HOŞTEN, F. SANTOS AND B. STURMFELS, The polytope of all triangulations of a point configuration, *Doc. Math. J. DMV.* **1** (1996), 103–119.

[8] J.A. DE LOERA, F. SANTOS AND J. URRUTIA, The number of geometric bistellar neighbors of a triangulation, *Discrete Comput. Geom.*, to appear.

[9] I.M. GEL'FAND, M.M. KAPRANOV AND A.V. ZELEVINSKY, *Multidimensional Determinants, Discriminants and Resultants*, Birkhäuser, Boston, 1994.

[10] L. GLASER, *Geometrical Combinatorial Topology*, Van Nostrand Reinhold, New York, 1970.

[11] C.L LAWSON, Software for C^1-interpolation, in *Mathematical Software III* (John Rice ed.), Academic Press, New York, 1977.

[12] C.W. LEE., Regular triangulations of convex polytopes, in *Applied Geometry and Discrete Mathematics–The Victor Klee Festschrift* (P. Gritzmann and B. Sturmfels eds.), 443-456, DIMACS series in Discrete Math. and Theoretical Comp. Science, 4, 1991.

[13] R. MACPHERSON, Combinatorial Differential Manifolds, in *Topological Methods in Modern Mathematics: a Symposium in Honor of John Milnor's Sixtieth Birthday*, Publish or Perish, 1993.

[14] J. RAMBAU, Triangulations of cyclic polytopes and higher Bruhat orders, *Mathematika* **44** (1997), 162–194.

[15] J. RAMBAU, *Polyhedral subdivisions and projections of polytopes*, Ph.D. Thesis, Fachbereich Mathematik, TU-Berlin, Shaker-Verlag, Aachen, 1996.

[16] J. RAMBAU AND F. SANTOS, *The generalized Baues problem for cyclic polytopes*, manuscript, 1998.

[17] J. RAMBAU AND G.M. ZIEGLER, Projections of polytopes and the generalized Baues conjecture, *Discrete Comput. Geom.* (3) **16** (1996), 215–237.

[18] V. REINER, *The generalized Baues problem*, manuscript 1998.

[19] J. RICHTER-GEBERT, Oriented Matroids with few mutations, *Discrete Comput. Geom.* **10** (1993), 251–269.

[20] F. SANTOS Triangulations of Oriented Matroids, preprint, 1997, 77 pages. http://matsun1.unican.es/ santos/Articulos/OMtri.ps.gz

[21] G. ZIEGLER, *Lectures on Polytopes*, Springer-Verlag, New York, 1994.