5. THE NUMBER OF BISTELLAR NEIGHBORS (EXTENDED VERSION)

We will now provide a triangulation of $C(11,5)$ with flip deficiency, i.e., fewer flips than the dimension of the corresponding secondary polytope. This example was found (together with the others mentioned in Theorem 1.2) while enumerating the set of all triangulations of $C(11,5)$ and $C(12,5)$ by a special C++ computer program. The algorithm makes full use of the fact that the set of triangulations of a cyclic polytope forms a bounded poset [10]. Modulo implementation details, the algorithm is straightforward; thus we do not discuss it here. Table 1 contains the resulting numbers of triangulations. This same table appears in [2].

number of points:	3	4	5	6	7	8	9	10	11	12
dimension 2	1	2	5	14	42	132	429	1,430	4,862	16,796
dimension 3		1	2	6	25	138	972	8,477	89,405	1,119,280
dimension 4			1	2	7	40	357	4,824	96,426	2,800,212
dimension 5				1	2	8	67	1,233	51,676	5,049,932
dimension 6					1	2	9	102	3,278	340,560
dimension 7						1	2	10	165	12,589
dimension 8							1	2	11	244
dimension 9								1	2	12
dimension 10									1	2

TABLE 1. The number of triangulations of $C(n, d)$ for $n \leq 12$.

Example 5.1. Throughout this section, T will be the following collection of 36 simplices in $C(11,5)$. We give it in five pieces which we call T_3, T_9, T_6, T_- and T_+, since the vertices 3, 6 and 9 play a special role in them and in the proofs. All the simplices in T contain either 3 or 9. The parts T_3, T_9 and T_6 consist respectively of those simplices not containing 3, not containing 9 and containing both 3 and 9 but not 6. Then, T_- and T_+ consist of the simplices containing 3, 6 and 9, divided into two groups according to whether they contain two elements in $\{1, 2, 4, 5\}$ and one in $\{7, 8, 10, 11\}$ or vice versa. T is symmetric under the reversal of the indices.

$$T_3 := \{\{1,2,6,7,8,9\}, \{1,2,6,7,9,11\}, \{1,2,7,8,9,11\},$$
$$\{1,6,7,9,10,11\}, \{1,7,8,9,10,11\}, \{4,5,6,7,9,11\},$$
$$\{4,5,6,9,10,11\}, \{4,5,7,8,9,11\}, \{5,6,7,8,9,11\}\}$$
$$T_9 := \{\{3,4,5,6,10,11\}, \{1,3,5,6,10,11\}, \{1,3,4,5,10,11\},$$
$$\{1,2,3,5,6,11\}, \{1,2,3,4,5,11\}, \{1,3,5,6,7,8\},$$
$$\{1,2,3,6,7,8\}, \{1,3,4,5,7,8\}, \{1,3,4,5,6,7\}\}$$
$$T_6 := \{\{1,2,3,9,10,11\}, \{1,3,4,5,8,9\}, \{1,3,4,5,9,10\},$$
$$\{2,3,7,8,9,11\}, \{3,4,5,7,8,9\}, \{3,4,7,8,9,11\}\}$$
$$T_- := \{\{1,2,3,6,8,9\}, \{1,2,3,6,9,11\}, \{1,3,5,6,8,9\},$$
$$\{1,3,5,6,9,10\}, \{3,4,5,6,9,10\}, \{3,4,5,6,7,9\}\}$$
$$T_+ := \{\{3,4,6,9,10,11\}, \{1,3,6,9,10,11\}, \{3,4,6,7,9,11\},$$
$$\{2,3,6,7,9,11\}, \{2,3,6,7,8,9\}, \{3,5,6,7,8,9\}\}.$$

We will prove that T is a triangulation and has only the following four bistellar flips: Two upward ones supported on

$$\{1,2,3,6,7,8,9\}, \{3,4,5,6,9,10,11\},$$

and two downward ones, supported on

$$\{1,2,3,6,9,10,11\}, \{3,4,5,6,7,8,9\}.$$

The fact that the example above is a triangulation and has only the claimed flips is computationally straightforward once the example is in hand. It was checked by the maple program PUNTOS [14] which studies triangulations of arbitrary point configurations and by two other independent maple routines.

However, we will provide a computer-free proof. [1] The proof is essentially a long but transparent case study. It is incorporated here for two reasons: first we want to show how a computer free proof can be organized at all. Just going along the definitions would certainly result in a much longer proof than the one presented below. Secondly, the computer programs we used are not available to the public. This is mainly due to the fact that the necessary documentation for an official software publication has not been and probably will never be accomplished.

Theorem 5.2. *The collection* T *of simplices of Example 5.1 is a triangulation of* $\mathsf{C}(11,5)$.

Proof. For proving this we produce Tables 2 and 3 below. The first five numbers in each row represent a codimension one simplex τ of T. It is followed by one number (in Table 2) or two numbers (in Table 3) in bold, each representing a vertex v to which τ is joined, so that $\tau \cup \{v\}$ is a simplex in T. The final information in each row says in which of the subsets T_3, T_9, T_6, T_+ or T_- of T the simplex in question can be found. The reader can verify the following facts:

1	2	3	4	5	**11**	(T_9)	1	2	3	4	11	**5**	(T_9)
1	2	3	5	6	**11**	(T_9)	1	2	4	5	11	**3**	(T_9)
1	2	3	6	7	**8**	(T_9)	1	2	5	6	11	**3**	(T_9)
1	2	3	7	8	**6**	(T_9)	1	2	6	7	11	**9**	(T_3)
1	2	3	8	9	**6**	(T_-)	1	2	7	8	11	**9**	(T_3)
1	2	3	9	10	**11**	(T_6)	1	2	8	9	11	**7**	(T_3)
1	2	3	10	11	**9**	(T_6)	1	2	9	10	11	**3**	(T_6)
1	3	4	5	6	**7**	(T_9)	2	3	4	5	11	**1**	(T_9)
1	3	4	6	7	**5**	(T_9)	2	3	5	6	11	**1**	(T_9)
1	3	4	7	8	**5**	(T_9)	2	3	6	7	11	**9**	(T_+)
1	3	4	8	9	**5**	(T_6)	2	3	7	8	11	**9**	(T_6)
1	3	4	9	10	**5**	(T_6)	2	3	8	9	11	**7**	(T_6)
1	3	4	10	11	**5**	(T_9)	2	3	9	10	11	**1**	(T_6)
1	4	5	6	7	**3**	(T_9)	3	4	5	6	11	**10**	(T_9)
1	4	5	7	8	**3**	(T_9)	3	4	6	7	11	**9**	(T_+)
1	4	5	8	9	**3**	(T_6)	3	4	7	8	11	**9**	(T_6)
1	4	5	9	10	**3**	(T_6)	3	4	8	9	11	**7**	(T_6)
1	4	5	10	11	**3**	(T_9)	3	4	9	10	11	**6**	(T_+)
1	5	6	7	8	**3**	(T_9)	4	5	6	7	11	**9**	(T_3)
1	5	6	8	9	**3**	(T_-)	4	5	7	8	11	**9**	(T_3)
1	5	6	9	10	**3**	(T_-)	4	5	8	9	11	**7**	(T_3)
1	5	6	10	11	**3**	(T_9)	4	5	9	10	11	**6**	(T_3)
1	6	7	8	9	**2**	(T_3)	5	6	7	8	11	**9**	(T_3)
1	6	7	9	10	**11**	(T_3)	5	6	8	9	11	**7**	(T_3)
1	6	7	10	11	**9**	(T_3)	5	6	9	10	11	**4**	(T_3)
1	7	8	9	10	**11**	(T_3)	6	7	8	9	11	**5**	(T_3)
1	7	8	10	11	**9**	(T_3)	6	7	9	10	11	**1**	(T_3)
1	8	9	10	11	**7**	(T_3)	7	8	9	10	11	**1**	(T_3)

TABLE 2. Simplices of T incident to facets of $\mathsf{C}(11,5)$

1. The 56 codimension one simplices in Table 2 are the facets of $\mathsf{C}(11,5)$, by Gale's evenness criterion (upper facets on the right and lower facets on the left). Joining them to the element in bold in the same row produces a simplex of T.
2. The 80 codimension one simplices in Table 3 are non-facets of $\mathsf{C}(11,5)$. Let τ be any of them and let v_1 and v_2 be the two elements in bold in the same row. Then, $\tau \cup \{v_1\}$ and $\tau \cup \{v_2\}$ are in T and they lie in opposite sides of τ. The latter is equivalent to saying that there are an odd number of elements of τ between v_1 and v_2.

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[1]The final version of the paper does not contain the proofs of theorems 5.2 and 5.3.

Without 3:

1	2	6	7	9	**8**	**11**	(T_3, T_3)
1	2	6	8	9	**3**	**7**	(T_-, T_3)
1	2	6	9	11	**3**	**7**	(T_-, T_3)
1	2	7	8	9	**6**	**11**	(T_3, T_3)
1	2	7	9	11	**6**	**8**	(T_3, T_3)
1	6	7	9	11	**2**	**10**	(T_3, T_3)
1	6	9	10	11	**3**	**7**	(T_+, T_3)
1	7	8	9	11	**2**	**10**	(T_3, T_3)
1	7	9	10	11	**6**	**8**	(T_3, T_3)
2	6	7	8	9	**1**	**3**	(T_3, T_+)
2	6	7	9	11	**1**	**3**	(T_3, T_+)
2	7	8	9	11	**1**	**3**	(T_3, T_6)
4	5	6	7	9	**3**	**11**	(T_-, T_3)
4	5	6	9	10	**3**	**11**	(T_-, T_3)
4	5	6	9	11	**7**	**10**	(T_3, T_3)
4	5	7	8	9	**3**	**11**	(T_6, T_3)
4	5	7	9	11	**6**	**8**	(T_3, T_3)
4	6	7	9	11	**3**	**5**	(T_+, T_3)
4	6	9	10	11	**3**	**5**	(T_+, T_3)
4	7	8	9	11	**3**	**5**	(T_6, T_3)
5	6	7	8	9	**3**	**11**	(T_+, T_3)
5	6	7	9	11	**4**	**8**	(T_3, T_3)
5	7	8	9	11	**4**	**6**	(T_3, T_3)

Without 9:

1	2	3	5	11	**4**	**6**	(T_9, T_9)
1	2	3	6	8	**7**	**9**	(T_9, T_-)
1	2	3	6	11	**5**	**9**	(T_9, T_-)
1	3	4	5	7	**6**	**8**	(T_9, T_9)
1	3	4	5	8	**7**	**9**	(T_9, T_6)
1	3	4	5	10	**9**	**11**	(T_6, T_9)
1	3	4	5	11	**2**	**10**	(T_9, T_9)
1	3	5	6	7	**4**	**8**	(T_9, T_9)
1	3	5	6	8	**7**	**9**	(T_9, T_-)
1	3	5	6	10	**9**	**11**	(T_-, T_9)
1	3	5	6	11	**2**	**10**	(T_9, T_9)
1	3	5	7	8	**4**	**6**	(T_9, T_9)
1	3	5	10	11	**4**	**6**	(T_9, T_9)
1	3	6	7	8	**2**	**5**	(T_9, T_9)
1	3	6	10	11	**5**	**9**	(T_9, T_+)
2	3	6	7	8	**1**	**9**	(T_9, T_+)
3	4	5	6	7	**1**	**9**	(T_9, T_-)
3	4	5	6	10	**9**	**11**	(T_-, T_9)
3	4	5	7	8	**1**	**9**	(T_9, T_6)
3	4	5	10	11	**1**	**6**	(T_9, T_9)
3	4	6	10	11	**5**	**9**	(T_9, T_+)
3	5	6	7	8	**1**	**9**	(T_9, T_+)
3	5	6	10	11	**1**	**4**	(T_9, T_9)

With neither 3 nor 9:

1	2	6	7	8	**3**	**9**	(T_9, T_3)
4	5	6	10	11	**3**	**9**	(T_9, T_3)

With 3 and 9 but not 6:

1	2	3	9	11	**6**	**10**	(T_-, T_6)
1	3	4	5	9	**8**	**10**	(T_6, T_6)
1	3	5	8	9	**4**	**6**	(T_6, T_-)
1	3	5	9	10	**4**	**6**	(T_6, T_-)
1	3	9	10	11	**2**	**6**	(T_6, T_+)
2	3	7	8	9	**6**	**11**	(T_+, T_6)
2	3	7	9	11	**6**	**8**	(T_+, T_6)
3	4	5	7	9	**6**	**8**	(T_-, T_6)
3	4	5	8	9	**1**	**7**	(T_6, T_6)
3	4	5	9	10	**2**	**6**	(T_6, T_-)
3	4	7	8	9	**5**	**11**	(T_6, T_6)
3	4	7	9	11	**6**	**8**	(T_+, T_6)
3	5	7	8	9	**4**	**6**	(T_6, T_+)
3	7	8	9	11	**2**	**4**	(T_6, T_6)

With 3, 6 and 9:

1	2	3	6	9	**8**	**11**	(T_-, T_-)
1	3	5	6	9	**8**	**10**	(T_-, T_-)
1	3	6	8	9	**2**	**5**	(T_-, T_-)
1	3	6	9	10	**5**	**11**	(T_-, T_+)
1	3	6	9	11	**2**	**10**	(T_-, T_+)
2	3	6	7	9	**8**	**11**	(T_+, T_+)
2	3	6	8	9	**1**	**7**	(T_-, T_+)
2	3	6	9	11	**1**	**7**	(T_-, T_+)
3	4	5	6	9	**7**	**10**	(T_-, T_-)
3	4	6	7	9	**5**	**11**	(T_-, T_-)
3	4	6	9	10	**5**	**11**	(T_-, T_+)
3	4	6	9	11	**7**	**10**	(T_+, T_+)
3	5	6	7	9	**4**	**8**	(T_-, T_+)
3	5	6	8	9	**1**	**7**	(T_-, T_+)
3	5	6	9	10	**1**	**4**	(T_-, T_-)
3	6	7	8	9	**2**	**5**	(T_+, T_+)
3	6	7	9	11	**2**	**4**	(T_+, T_+)
3	6	9	10	11	**1**	**4**	(T_+, T_+)

TABLE 3. Codimension 1 interior simplices of T

Once these properties are checked, we can prove that T is a triangulation as follows: a simple counting argument shows that Tables 2 and 3 cover all the codimension 1 simplices in T, since $2 \times 80 + 56 = 36 \times 6$, where 36 is the number of simplices in T. Then, Table 3 shows that every codimension 1 simplex in T lies in precisely two simplices of T, and that these two simplices intersect properly. In other words, T satisfies the *interior cocircuit equations* introduced in [15]. Parts (i) and (ii) of Theorem 1.1 in [15] say that in order to prove that a collection T of simplices which satisfies the interior cocircuit equations is a triangulation it suffices to show that there is a point in the interior of $C(11,5)$ which is covered by exactly one simplex of T. In our case, this holds for every point sufficiently close to a facet of $C(11,5)$ since there is a unique simplex incident to that facet.   □

Theorem 5.3. *Let* $A = \{a_1, \ldots, a_7\}$ *be a circuit of* $C(11,5)$ *which supports a flip of* T. *Then,*

(i) A *contains 3 and 9.*

(ii) A *contains 6.*

(iii) A *contains exactly two elements among 1, 2, 4 and 5 and other two among 7, 8, 10 and 11.*

(iv) A *contains one of the pairs* $\{1,2\}$, $\{4,5\}$ *and one of* $\{7,8\}$, $\{10,11\}$.

Thus, T *has only the four bistellar flips mentioned in Example 5.1.*

Proof. To say that $A = \{a_1, \ldots, a_7\}$ supports a flip of T means that T contains one of the two triangulations of A, which are

$$T_A^e := \{ A \backslash \{a_i\} : i = 2, 4, 6 \} \quad \text{and} \quad T_A^o := \{ A \backslash \{a_i\} : i = 1, 3, 5, 7 \},$$

where we assume $a_1 < \cdots < a_7$. Moreover, the flip supported on A is upward (in the poset structure on the collection of triangulations of $C(11,5)$) if $T_A^o \subset T$ and downward if $T_A^e \subset T$.

If $A = \{a_1, \ldots, a_7\}$ supports a flip, at least three simplices of T have to be contained in A and at least two of them must contain a_i, for each $i = 1, \ldots, 7$. This simple remark is essentially all that is used in the proof of (i), (ii), (iii) and (iv), together with the fact that T is symmetric under reversal of indices.

The conclusion of the Theorem follows from parts (i), (ii), (iii) and (iv) as follows: By (i), (ii) and (iii) A contains 3, 6 and 9 plus two vertices among 1, 2, 4 and 5 and other two among 7, 8, 10 and 11. Then (iv) implies that the only four possibilities for A are those in Example 5.1. That these four circuits actually support flips can be trivially checked by finding among the simplices in T one of the two triangulations T_A^o and T_A^e, for each case. Also, this check tells whether the flip is upwards or downwards.

• For proving part (i) we only need to prove that A contains 3 since then it will follow by symmetry that A contains 9 as well.

Suppose that A does not contain 3. Then one of the two triangulations T_A^e or T_A^o of A is contained in T_3. Since in T_3 only $\{1,2,6,7,8,9\}$ does not contain 11, we have that $a_7 = 11$. Moreover, if the triangulation of A contained in T was T_A^o, then $A \backslash \{11\} = \{1,2,6,7,8,9\}$ and $A = \{1,2,6,7,8,9,11\}$; this case is easily discarded, so we assume $T_A^e \subset T$.

Since 9 is in every simplex of T_3, 9 is in A and equals a_i for an odd i. Thus, $a_5 = 9$ and $a_6 = 10$. With similar arguments one can prove that 7 is in A and $7 = a_i$ for an odd i, so $a_3 = 7$ and $a_4 = 8$. Thus, the simplex $A \backslash \{a_2\} \in T_A^e \subset T$ contains $\{7,8,9,10,11\}$, which implies $A \backslash \{a_2\} = \{1,7,8,9,10,11\}$ and $a_1 = 1$. This is impossible because then $A \backslash \{a_6\}$ contains $\{1,7,8,9,11\}$, which is not contained in any simplex of T_3.

• For part (ii), if A does not contain 6 then one of the two triangulations T_A^e or T_A^o of A is contained in the following twelve simplices, which are those in T and not containing 6. They are the six simplices in T_6, together with three from T_3 and three from T_9:

$$\{\{1,2,3,4,5,11\}, \{1,2,3,9,10,11\}, \{1,2,7,8,9,11\}, \{1,3,4,5,10,11\},$$
$$\{1,3,4,5,7,8\}, \{1,3,4,5,8,9\}, \{1,3,4,5,9,10\}, \{1,7,8,9,10,11\},$$
$$\{2,3,7,8,9,11\}, \{3,4,5,7,8,9\}, \{3,4,7,8,9,11\}, \{4,5,7,8,9,11\}\}.$$

The four simplices in the last row are the only ones not containing 1, but they all contain the consecutive three elements 7, 8 and 9, and they cannot contain a triangulation of a circuit. Thus, $a_1 = 1$ and if $T_A^o \subset T$ then $\{7,8,9\} \subset A$. The same argument on the four simplices which do not contain 11 proves that $a_7 = 11$ and that $T_A^e \subset T$, since $T_A^o \subset T$ would imply that $\{1,3,4,5,7,8,9,11\} \subset A$. Now, $T_A^e \subset T$ implies that T_A^e is contained in the set of simplices of T which contain both 1 and 11, which are:

$$\{\{1,2,3,4,5,11\}, \{1,2,3,9,10,11\}, \{1,2,7,8,9,11\},$$
$$\{1,3,4,5,10,11\}, \{1,7,8,9,10,11\}\}.$$

Only the two in the second row do not contain 2, so we should have $a_2 = 2$ and one of those two simplices equal $A \backslash a_2 \in T_A^e$. But this is impossible because $A \backslash 2$ must contain both 3 and 9, by part (i).

• For part (iii) we will prove that A contains at least two vertices among 1, 2, 4 and 5. With this, symmetry proves the same thing for 7, 8, 10 and 11 and then the fact that A contains 3, 6 and 9 proves the statement.

Since every simplex of T contains at least one of 1, 2, 4 and 5, A contains at least one of them too. If A contains only one of them, then a triangulation of A is contained in the following list of eleven simplices, which are those in T and containing only one of $\{1, 2, 4, 5\}$: The six simplices in T_+ together with three simplices from T_3 and two from T_6:

$$\{\{1,3,6,9,10,11\}, \{1,6,7,9,10,11\}, \{1,7,8,9,10,11\},$$
$$\{2,3,6,7,9,11\}, \{2,3,6,7,8,9\}, \{2,3,7,8,9,11\},$$
$$\{3,4,6,9,10,11\}, \{3,4,6,7,9,11\}, \{3,4,7,8,9,11\},$$
$$\{3,5,6,7,8,9\}, \{5,6,7,8,9,11\}\}.$$

We have displayed them so that the four rows correspond, respectively, to simplices using 1, 2, 4 and 5. If A contains only one of 1, 2, 4 or 5, then the triangulation of A must be contained in one of the rows. This is clearly not the case.

• For part (iv), the statement on 1, 2, 4 and 5 will follow from part (iii) and the fact that A cannot contain exactly one of 1 and 2 and one of 4 and 5, which we now prove. The statement on 7, 8, 10 and 11 follows by symmetry.

If A contains exactly one of 1 and 2 and one of 4 and 5, then $a_1 \in \{1, 2\}$, $a_2 = 3$, $a_3 \in \{4, 5\}$, $a_4 = 6$. We have $T_A^o \subset T$, since $T_A^e \subset T$ would imply that $A \backslash \{3\}$ is in T_3 and contains one of $\{1, 2\}$ and one of $\{4, 5\}$ but no such simplex exists. In particular, $A \backslash \{a_1\} \in T$ and $A \backslash \{a_3\} \in T$. Both must contain 3, 6, 9 and only one of $\{1, 2, 4, 5\}$, so they are in T_+. More precisely, we must have

$$A \backslash \{a_3\} \in \{\{2,3,6,7,9,11\}, \{1,3,6,9,10,11\}, \{2,3,6,7,8,9\}\}$$
$$A \backslash \{a_1\} \in \{\{3,4,6,7,9,11\}, \{3,4,6,9,10,11\}, \{3,5,6,7,8,9\}\}.$$

This gives three possibilities for $A \backslash \{a_1, a_3\}$ and A, namely:

$$A \backslash \{a_1, a_3\} = \{3,6,7,9,11\}, \qquad A = \{2,3,4,6,7,9,11\},$$
$$A \backslash \{a_1, a_3\} = \{3,6,9,10,11\}, \qquad A = \{1,3,4,6,9,10,11\},$$
$$A \backslash \{a_1, a_3\} = \{3,6,7,8,9\}, \qquad A = \{2,3,5,6,7,8,9\}$$

In no case is $T_A^o \subset T$, so the proof is complete. $\qquad\qquad\square$

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