Construction of real algebraic plane nodal curves with given topology

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Abstract. We study a constructive method to find an algebraic curve in the real projective plane with a (possibly singular) topological type given in advance. Our method works if the topological model T to be realized has only double singularities. In that case, it gives an algebraic curve of degree $4N + 2K$ or $4N + 2K - 1$, where N and K are the numbers of double points and connected components of T.

The construction is based on a preliminar topological manipulation of the topological model and then some perturbation techniques to obtain the polynomial defining the algebraic curve. Some algorithmic remarks are given.

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1. Introduction.

Throughout this paper we will use the term *algebraic curve* or simply *curve* as an abbreviation for *real projective plane algebraic curve* meaning by this a real homogeneous polynomial $f \in \mathbb{R}[X, Y, Z]$ in three variables, considered up to a constant factor. Sometimes, by abuse of language, we will call curve the zero set $V(f)$ of such a polynomial in the real projective plane $\mathbb{RP}^2$. Two subsets V and W are said to have the same *topological type* if there exists a global homeomorphism of the projective plane into itself sending V to W. Note that this condition is stronger than V and W being homeomorphic.

We want to find an algebraic curve f whose zero set has the same topological type of a certain given $T \subseteq \mathbb{RP}^2$. The conditions that T must satisfy for this to be possible are contained in the following definition.

Definition 1.1 Let T be a subset of $\mathbb{RP}^2$. We say that T is a *topological model* for an algebraic curve if it has the topological type of some algebraic curve. This is equivalent to T being homeomorphic to a graph with an even (possibly zero) number of edges incident to each vertex.

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We say that an algebraic curve f *realizes* T if its zero set $V(f)$ has the same topological type as T.

The equivalence between the two definitions of a topological model can be found, for example, in [**BCR**]. There, the second one is stated locally. The translation to our global statement is easy because of compactness of the projective plane. We want to remark that such a characterization of the topology of real algebraic sets is far from trivial in higher dimensional cases (cf. again [**BCR**], or [**AK**]) and that, even in the plane, the usual proofs of it use polynomial approximation of C^∞ functions and thus say nothing about the degree needed to *realize* a given topological model with an algebraic curve. Our method would give a new, constructive proof of the characterization.

Nevertheless, we are only able to work out the case of double singularities (see the remark just before Proposition 2.2 for details). For this reason our topological models will always be supposed to have only order 2 singular points, where we define the *order* of a point P in T to be half the number of edges incident to P if P is a vertex of the graph and 1 if P lies on an edge. A *singular point* is a point of order at least 2. Note that, here, "singular" has just a topological meaning. To state our main result (Theorem 1.3, which is a paraphrase of Theorem 3.7) let us first introduce some definitions.

Definition 1.2 Let l be an embedded circle in $\mathbb{RP}^2$. Then, l has the topological type of either a line or a circle and is called a *pseudo-line* or an *oval*, respectively.

Let T be a topological model in $\mathbb{RP}^2$ and let l be a pseudo-line transversal to T. We will say that T is *even* (resp. *odd*) if T and l have an even (resp. odd) number of intersections. The definition does not depend on the choice of l and is a topological type invariant of T.

A topological model T in $\mathbb{RP}^2$ is called *orientable* if it does not contain any pseudo-line or, equivalently, if there exists a pseudo-line not intersecting it.

Theorem (cf. 3.7) *Let T be a topological model in $\mathbb{RP}^2$ with only singular points of order 2. Let N be the number of singular points of T and K its number of connected components. Then, T can be realized with an algebraic curve of degree $4N + 2K$ if T is even, or $4N + 2K - 1$ if T is odd.* □

The first question that arises is how good the degree we obtain is. For non-singular curves our theorem gives degree $2K$ or $2K - 1$, which is trivial (for we can construct any non singular model as a product of K circles, or may be $K - 1$ circles and a line) but also optimal in the worst case (if the model consists on K nested ovals it can not be realized with an algebraic curve of degree lower than $2K$).

For singular curves, figure 1 shows a connected topological model with 3 double points that cannot be algebraically realized with degree lower than 8. The example easily generalizes to any number of double points and to non connected models (inserting connected components one inside another), giving:

For any given sequence of numbers $n_1, \ldots, n_K$ there exists a topological model with K connected components having $n_1, \ldots, n_K$ double points on each that cannot be algebraically realized with degree lower than $d = 2\sum_1^K n_i + 2K$.

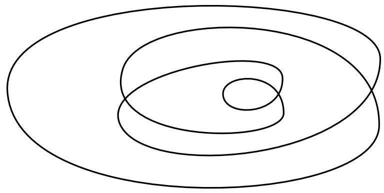


Figure 1

Thus, the degree we obtain is quite good (at most twice the optimal). We have reasons to think that the degree with which any topological model with only double points can be realized is actually this degree $2N + 2K$. In fact, in **[GS]** we presented (jointly with A.G. Corbalán) a draft of another method for costructing algebraic curves with given topology (in the affine plane instead of the projective one), that gave exactly that reachable bound. That construction had a conjectural step that we could not prove to work, but we are currently studying how to use some of its ideas in a different way.

Let us remark that, on the other hand, Bezout's and Harnack's theorems give lower bounds for the degree needed to algebraically realize *any* topological model. This bounds go, roughly speaking, with the square roots of N and K. Moreover, they are optimal for *some* topological models of any degree.

Our construction of an algebraic curve from a topological model consists of two parts, respectively detailed in sections 2 and 3. We sketch here the contents of each. Section 4 is devoted to study some algorithmical aspects of the construction.

First, the problem is easily reduced to the connected case. We then make some topological desingularization of the connected topological model T to obtain what we call a *skeleton* of T. The skeleton can be put in a *standard form* that consists of a circle (or a straight line, depending on the parity of T), that we call the *core* of the skeleton, and some segments or arcs of conics –to be called *bonds*– joining different pairs of points on it. See figure 2 for an example (bonds are the lighter, grey lines). Bonds are disjoint and represent the places where there was a singular point of T. The topological type of T can be recovered from its skeleton by substituting all bonds by double crossings.

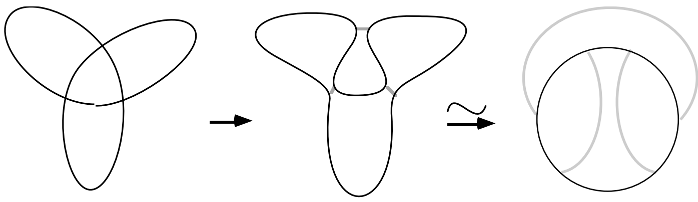


Figure 2

In the second part we introduce an eight shaped curve, or *lemniscata*, along each bond. These lemniscatae are placed so not to intersect to one another and tangent to the core at the ends of the bonds (see figure 3.a). Each lemniscata is constructed with degree 4. Thus, the product of the core and the lemniscatae is an algebraic curve of degree $4N+2$ (or $4N+1$, depending on the parity of T) which has "almost" the same topological type as T. A slight perturbation of this algebraic curve will give us an algebraic curve with the same topology as T (as in figure 3.b). The perturbation techniques used here (cf. Theorem 3.3) are similar to some classically used in the constructive part of Hilbert's 16^{th} problem (cf. [**Vi**]) and are based in some results by Gudkov (cf. [**Gu**]).

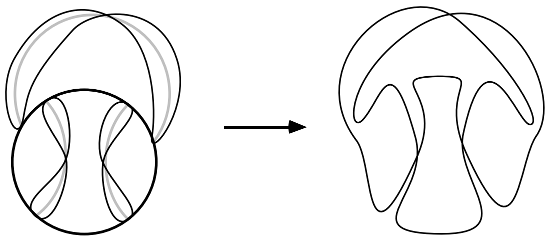


Figure 3

We would like to thank the referee for his attentive reading and useful comments on the first version of this paper.

2. Topological manipulation of the model.

Throughout this section T will be a topological model for an algebraic curve in $\mathbb{RP}^2$ with only order 2 singular points. Let N denote the number of singular points of T. Next lemma will allow us to assume, moreover, that T is connected.

Lemma 2.1 *Let T be a topological model in $\mathbb{RP}^2$ and suppose that each of its connected components $T_1, \ldots, T_K$ is realizable with an algebraic curve f_i of degree d_i. Suppose, moreover, that whenever T_i is orientable the corresponding f_i does not intersect the infinity line of the projective plane.*

Then, the whole model T can be realized algebraically with a curve f which is the product of a scaled translation of each f_i. Thus, f has degree $\sum d_i$.

Proof. First of all, it is easy to check that any topological model has at most one non-orientable connected component. This is so because any non-orientable connected component contains a pseudo-line and two different pseudo-lines always intersect to one another.

Now, the condition that the orientable components are realized without intersecting the infinity line (i.e. in the affine part of the projective plane) implies that we can make these realizations as small as we want, just contracting the affine part of $\mathbb{RP}^2$. We can afterwards translate this contracted curves anywhere in $\mathbb{RP}^2$ by a projective translation, and none of these two operations will change the degree of the curve.

To realize the whole model T we realize first the non-orientable component (if any) and then place the realization of each orientable one, sufficiently reduced, in the appropiate place to have a curve with the topological type of T. This curve –the product of the curves realizing each component– will have degree $\sum d_i$. □

Thus, our aim is to realize T (supposed connected) with degree $4N + 2$ if it is even and $4N + 1$ if it is odd. This will be achieved in Theorem 3.6.

To start, let as assume that T is not a single point (which can trivially be realized with the curve $X^2 + Y^2 = 0$). The condition that T is a connected graph with even number of edges at every vertex implies that it has an eulerian cycle, i.e., a cycle passing once through each edge. Moreover, the next lemma permits to assume that the eulerian cycle has no proper self-crossings. By this we mean that whenever the cycle passes through a vertex P, the ingoing and outgoing edges of the cycle are consecutive in the circular ordering of the edges incident to P.

Incidentally, let us remark that Proposition 2.2 is not true if the topological model has multiple points of order higher than 2. (The reader can convince himself of this by just considering a model consisting on three circles tangent to one another at the same point). In our opinion, this is the main reason why our method cannot be applied for points of higher order. Of course, the algebraic constructions to be described in section 3 would also be more complicated in this case.

Proposition 2.2 *Let T be a topological model connected and with only double points. Then, T has an eulerian cycle that does not properly cross itself.*

Proof. Suppose that we have an eulerian cycle C on T with some proper crossings. Choose one of these crossings, at a vertex P. Call a, b, c and d the four edges incident to P, in their circular ordering (i.e. a opposite to c and b opposite to d). Without loss of generality the eulerian cycle on T is $C = ac\delta bd\gamma$, where δ and γ represent lists of edges in T. Then, the cycle $C' = ab\delta^{-1}cd\gamma$ is also eulerian and contains one proper crossing less than C. Repeating the process at every proper crossing of C we will obtain an eulerian cycle with no proper crossings. □

Once we have an eulerian cycle on T with only non-proper crossings, we can slightly perturb the cycle at each vertex in order to *desingularize* T. In this way we obtain a topological model T' which is very close to T, but non-singular, and still connected. Let us clarify this with an example. Let $T \subseteq \mathbb{RP}^2$ be the topological model in figure 4.a and consider the eulerian cycle $C = abcfed$ which has no proper crossings. In figure 4.b we show the nonsingular model T' obtained perturbing T according to C.

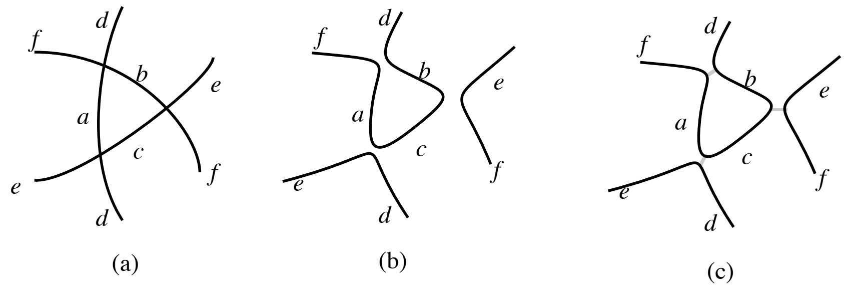


Figure 4

In order to be able to recover the topology of T from T' we introduce small lines joining each two points of T' that were identified in T. The final result T^* (as shown in figure 4.c) will be called a *skeleton* of the topological model T. The non-singular model T' obtained separating the double crossings of T will be called the *core* of the skeleton and the lines joining separated points will be called *bonds.* In all our figures bonds will be represented as lighter, grey lines.

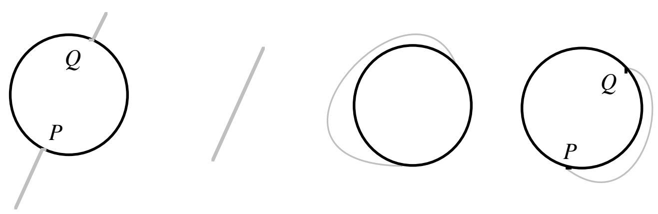
The definition of topological type for skeletons is the obvious one. Two skeletons T_1^* and T_2^* have the same *topological type* if there is an homeomorphism of the projective plane in itself sending the core and bonds of T_1^* respectively to the core and bonds of T_2^*.

Our next result says how to obtain from the skeleton T^* another skeleton with the same topological type than T^*, but made of a circle or a straight line for the core and line segments or arcs of conics for the bonds. This will make the following steps of our algorithm (the algebraic constructions) easier. A skeleton in the conditions of Proposition 2.3 will be said to be in *standard form.* In figure 6 we have some examples of skeletons in standard form and the data below the figures are combinatorial descriptions of their topology. We will come back to this point in section 4.

Proposition 2.3 *Let T be a connected topological model with only double points. Then, any skeleton of T has the same topological type than a skeleton T^* whose core is either the X-axis or the unit circle (depending on the parity of T) and such that:*

i) if T is even, each bond of T^* is either a straight line segment joining two opposite points in the unit circle and passing through infinity, or an arc of circle joining two points of the unit circle (either through the outside or the inside) and perpendicular to it.

ii) if T is odd, each bound of T_0^* is either a half-circle or an arc of hyperbola joining two different points of the straight line.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