Parametrization of semialgebraic sets

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Abstract: In this paper we consider the problem of the algorithmic parametrization of a d-dimensional semialgebraic subset S of $\mathbb{R}^n$ ($n > d$) by a semialgebraic and continuous mapping from a subset of $\mathbb{R}^d$. Using the Cylindrical Algebraic Decomposition algorithm we easily obtain semialgebraic, bijective parametrizations of any given semialgebraic set, but in this way some topological properties of S (such as being connected) do not necessarily hold on the domain of the so constructed parametrization. If the set S is connected and of dimension one, then the Euler condition on the associated graph characterizes the existence of an almost everywhere injective, finite-to-one parametrization of S with connected domain. On the other hand, for any locally closed semialgebraic set S of dimension $d > 1$ and connected in dimension l (i.e. such that there exists an l-dimensional path among any two points in S) we can always algorithmically obtain a bijective parametrization of S with connected in dimension l domain (theorem 6). Our techniques are mainly combinatorial, relying on the algorithmic triangulation of semialgebraic sets.

Keywords: algorithmic real algebraic geometry, semialgebraic sets, triangulation.

1 Introduction

A *semialgebraic* set S in $\mathbb{R}^n$ is a subset of points in $\mathbb{R}^n$ verifying a finite number of polynomial equalities and inequalities; a subset $S \subset \mathbb{R}^n$ is *connected in dimension l* or *l-connected* (Definition 3) when, for any two points x and y in S, there is a continuous injection from the compact l-ball B^l into S whose image contains x and y.

It is known that, if S is a compact semialgebraic set in $\mathbb{R}^n$, there is a doubly exponential (in the number n of variables describing S) algorithm triangulating S (c.f. [1] Ch.9, §2, and [6]). Thus, semialgebraic compact sets can be considered as finite simplicial complexes, but we remark that the known algorithm could produce a doubly exponential number of simplexes. If S is not compact but locally closed then it can be compactified adding one point, and this compactification obviously preserves the property of being connected in dimension l. The compactification method described in [1], (Ch. 2, Prop. 5.9) can be easily transformed into an algorithmic procedure. When regarding (eventually after compactification) a semialgebraic (locally closed) set S as a finite compact simplicial complex K, some interesting constructions over S can be expressed in a simpler way as combinatorial constructions on K. Our aim in this paper is to present a combinatorial algorithm to cut a

finite, connected, compact, d-dimensional simplicial complex in $\mathbb{R}^n$ and make it "flat", deforming it by means of semialgebraic functions. More precisely, we show that any d-dimensional simplicial complex K in $\mathbb{R}^n$ is the continuous bijective semialgebraic image of a connected subset P in $\mathbb{R}^d$, when $d > 1$; moreover, if the simplicial complex is l-connected, the connected subset in $\mathbb{R}^d$ can be chosen l-connected too (Theorem 6). We can even preserve the dimension of connectivity between each pair of points, but in this case our parametrization will not be bijective: every d-dimensional simplicial complex K is the almost everywhere injective, finite to one, image of a compact subset P of $\mathbb{R}^d$, having the property that for every two points in K which are connected in dimension r ($1 < r < d$) there exist two corresponding points in P connected in dimension r. The whole procedure can be combinatorially carried over. For the case $d = 1$, we remark that a one dimensional simplicial complex in $\mathbb{R}^n$ cannot be, in general, the bijective continuous image of a connected subset of $\mathbb{R}$. Making flat (i.e. like a line) a one dimensional complex can be regarded as the construction of an Eulerian tour in a graph, which is possible if and only if every vertex (except maybe two of them) belongs to an even number of edges (c.f., for instance, [5]); even in this case the parametrization cannot be, in general, bijective but, at least, finite to one and almost everywhere injective (Proposition 2).

The semialgebraic formulation of the above stated

results for simplicial complexes is made in Corollary 7. For example, a locally closed semialgebraic set S of dimension $d > 1$ and connected in dimension l can be bijectively parametrized by a semialgebraic, connected in dimension l, subset of $\mathbb{R}^d$ (Corollary 7a): if S is compact then, after triangulation, we are in the above situation; if it is not compact, we parametrize its compactification and, treating carefully the new added point, we can assure that, deleting it, the restricted parametrization of S keeps the l-connectivity properties of S. In particular, if $l = d$, the domain of the parametrization is semialgebraically homeomorphic to a subset of the closed d-ball, containing the open d-ball (Corollary 8).

Semialgebraic sets arise quite naturally in some applied topics, where the parametrization technique could be useful. For example, in Geometric Modelling and Computer Aided Geometric Design, semialgebraic sets can represent a very general category of sets (for instance, obtained by the manipulations of Constructive Solid Geometry or as boundaries of some solids). Parametrizing such semialgebraic sets could be a technique for rendering these objects. In robotics, semialgebraic sets represent the configuration space of robots, and motion planning deals with answering several questions about the geometrical structure of these sets. The idea here behind the application of the parametrization technique is that of replacing the (usually high) number of variables describing the set, by a much smaller number equal to its dimension. Even if the parametrization algorithm is of high complexity in the given number of variables, it could turn out that, after parametrizing the set, several different problems could be solved with complexity depending only upon the dimension of the set. Unfortunately, with the current algorithms for triangulation of semialgebraic sets, this idea is still far from beeing practical, as we have no control on the complexity of the description (number of simplexes) of the "flattened" set.

2 Injective parametrizations of semialgebraic sets with arbitrary domain

Definition 1 Let S be a semialgebraic subset of $\mathbb{R}^n$ of dimension d. A *(semialgebraic) parametrization* of S is a pair (P, ψ), where $P \subset \mathbb{R}^d$ is a semialgebraic set of dimension d and $\psi : P \longrightarrow S$ is a semialgebraic, continuous, surjective and finite-to-one mapping. □

It is possible to obtain algorithmically a parametrization (P, ψ) of any semialgebraic set S of dimension d such that ψ is, moreover, injective: the Cylindrical Algebraic Decomposition algorithm (CAD) provides a partition of S in cells, each one of them semialgebraically homeomorphic to $(0, 1)^{r_i}$, $r_i \leq d$. We

can consider the hypercubes $(0, 1)^{r_i}$ with $r_i \leq d$ included in $\mathbb{R}^d$. The union with disjoint closure of all the $(0, 1)^{r_i}$ hypercubes in $\mathbb{R}^d$ provides the domain P of the parametrization and the mapping ψ corresponds to the semialgebraic homeomorphism between the hypercubes and the cells. See [2] and [4] for the details. For example, the two dimensional sphere in $\mathbb{R}^3$, $S = \{(x, y, z) \in \mathbb{R}^2 / x^2 + y^2 + z^2 = 1\}$, would be injectively parametrized with the following non connected domain in $\mathbb{R}^2$:
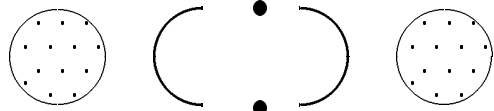
We remark that:

1.- In general, the CAD method outlined above gives the mapping ψ by means of its graph, which is a semialgebraic set.

2.- It seems desirable that the domain of the parametrization has a more "compact" shape, but we find several obstructions to this possibility: a circumference cannot be obtained as the continuous bijective image of a semialgebraic, compact and connected set in $\mathbb{R}$. The same difficulty raises if we only require the domains to be connected: the following semialgebraic subset S of $\mathbb{R}^2$ cannot be the continuous, bijective image of a connected set U in $\mathbb{R}$:
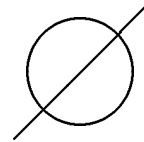
3 Connected domain, one dimensional case

Let us suppose that $S \subset \mathbb{R}^n$ is a semialgebraic, connected, compact set of dimension 1, which implies, in this particular case, that its dimension is one everywhere. Let us consider a triangulation of S:

$$\Phi : K := \cup_{i=1}^p \sigma_i \longrightarrow S$$

Such a triangulation with semialgebraic mappings can be algorithmically computed, as in [1], from the algebraic description of the set S, and, obviously, a parametrization of K provides a parametrization of S with the same properties. We will denote $\mathcal{G}_S$ the graph whose vertices are the 0-dimensional triangles and whose edges are the one dimensional triangles in K. Remark that an Eulerian tour in the graph $\mathcal{G}_S$ is a continuous mapping from $[0, 1]$ onto S which is one-to-one everywhere except for a finite number of points of S (the vertices of $\mathcal{G}_S$), where it is finite-to-one. Now, it is a well known result in graph theory

that graphs with Eulerian tours have either zero or two odd-degree vertices (c.f. [5]). It is easy to see that, over semialgebraic sets, Eulerian tours can be chosen to be semialgebraic mappings. We summarize these comments in the following proposition (cf. [3] for details).

Proposition 2 *If $S \subset \mathbb{R}^n$ is a semialgebraic, connected and compact set of dimension 1, then the following statements are equivalent:*

(i) *There exists a parametrization mapping*

$$\psi : [0, 1] \longrightarrow S$$

almost everywhere injective, i.e. such that

$$cl\{t \in P/ \psi^{-1}\psi(t) = \{t\}\} = P.$$

(ii) *Euler condition: The number of vertices of the graph $\mathcal{G}_S$ in which there is an odd number of edges is zero or two.*

Moreover, if the Euler condition holds, the mapping in (i) can be algorithmically constructed from the algebraic description of S. □

If S is not compact, the Euler condition has no sense as, in general, we cannot triangulate S. Still we can ask if there exists a semialgebraic, connected set $P \subset \mathbb{R}$ and a semialgebraic, continuous, surjective and almost everywhere injective mapping from P to S. If we compactify S with a point $\{s\}$ (which is possible because every 1-dimensional semialgebraic set is locally closed) obtaining a set S', it is easy to check that the posed question has an affirmative answer if and only if one of the following statements holds:

(a) in $\mathcal{G}_{S'}$ the vertex corresponding to $\{s\}$ has two edges and all the other vertices have an even number of edges,

(b) in $\mathcal{G}_{S'}$ the vertex corresponding to $\{s\}$ has one edge and there is exactly one other vertex with an odd number of edges.

On the other hand, if we are interested in parametrizing S by means of a mapping with some smoothness requirements but still computable, for instance if we search for a Nash (analytic-algebraic) mapping $\psi : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $\psi(\mathbb{R}) = S$, we must consider the following result from M. Shiota (private communication):

> *Let S be a connected, semialgebraic set in $\mathbb{R}^n$, everywhere of dimension d; let S^{reg} denote the set of d-dimensional, C^w-regular points of S. Then S is the image of some Nash map $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ if and only if there exists an analytic curve $\alpha : \mathbb{R} \longrightarrow S$ which meets every connected component of S^{reg}.*

For the case $d = 1$, this result implies that, in general, it is not possible to parametrize a set S with a Nash mapping, even if the Euler condition holds: for example, if S is a triangle (dimension 1) in $\mathbb{R}^2$ then it verifies the Euler condition, but S^{reg} has three connected components which are the three sides of the triangle (excluding the vertices) and any curve $\alpha : \mathbb{R} \to S$ that meets all of them must contain at least one vertex of the triangle, therefore it cannot be analytic. If S is non singular and connected, by the result above, it is the image of some Nash mapping. In the general case of dimension one the only result we can state is the equivalence between the Euler condition and the existence of a piecewise Nash parametrization.

4   Connected domain and dimension $d > 1$

Definition 3 A semialgebraic set $S \subset \mathbb{R}^n$ is *connected in dimension l or l-connected* if for all $x, y \in S$ there exists a semialgebraic, continuous, injective mapping $\phi_{x,y} : B^l \longrightarrow S$ whose image contains x and y, where B^l denotes the closed unit ball in $\mathbb{R}^l$. We say that $\phi_{x,y}$ is an *l-dimensional path* between x and y and that $x, y \in S$ are *connected in dimension l or x is l-connected to y.* □

Remark that the l-connectivity relation is not, in general, transitive; nevertheless, if y is l-connected to x and z in S and there exists some l-ball centered at y contained in S, then x is l-connected to z.

Lemma 4 *Let Δ_1 and Δ_2 be two d-dimensional simplexes, and let $\Omega_1 \subset \Delta_1$ and $\Omega_2 \subset \Delta_2$ be two l-faces, with $l \geq 1$. Let Ω_1' be another l-simplex properly contained in Ω_1 (i.e. such that $\Omega_1' \subset \Omega_1^o$; here it is essential the condition $l \neq 0$) and suppose that we are given a semialgebraic homeomorphism ϕ between Ω_1' and Ω_2. Then, this homeomorphism can be extended to a semialgebraic homeomorphism between Δ_1 and Δ_2.*

Proof: Let us denote $\Delta_1 = [a_0, \ldots, a_d] :=$

$$:= \left\{ \sum_{i=0}^d \lambda_i a_i / \sum_{i=0}^d \lambda_i = 1, \ 0 \leq \lambda_i, \ i = 0, \ldots, d \right\}$$

and we can suppose $\Omega_1 = [a_0, \ldots, a_l]$. We reduce the lemma to obtaining a semialgebraic homeomorphism ψ between Δ_1 and a new d-simplex Δ_3, such that $\psi(\Omega_1')$ is an l-face Ω_3 in Δ_3: in fact, the mapping $\chi := \phi \circ (\psi^{-1} |_{\Omega_3})$ will be then a semialgebraic homeomorphism between two l-faces, therefore χ can be extended to a semialgebraic homeomorphism $\tilde{\chi} : \Delta_3 \to \Delta_2$, because a simplex is a semialgebraic cone over every face. The mapping $\tilde{\chi} \circ \psi$ is the searched extension, see the following figure:
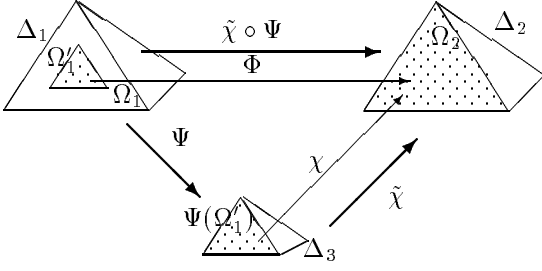
In order to obtain ψ, it suffices to obtain a semialgebraic homeomorphism, which will be also called ψ, between the boundaries of Δ_1 and Δ_3 (such that $\psi(\Omega_1') = \Omega_3$ will be an l-face), because mappings between the boundaries of simplexes can be semialgebraically extended to the interior.

Case 1. $l = d - 1$

We suppose, without loss of generality, that $\Delta_1 = \Delta_3$ and that Ω_1' is centered in the $(d-1)$-face $\Omega_1 = [a_0, \dots, a_{d-1}]$ of Δ_1, i.e. $\Omega_1' =$

$$\left\{ \sum_{i=0}^{d-1} \lambda_i a_i \; / \; \sum_{i=0}^{d-1} \lambda_i = 1, \quad \lambda_i \geq 0, \quad \min(\lambda_i) \leq \frac{1}{2d} \right\}$$

For any point $[a_0, \dots, a_d]$ in Δ_1 we denote $m := \min(\lambda_0, \dots, \lambda_{d-1})$, $\alpha := \dfrac{1 + 2dm}{2(1 - dm)}$, $\beta := \dfrac{1 -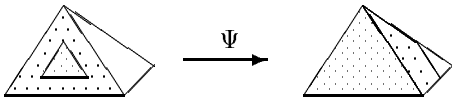 2dm}{2}$, and define the semialgebraic homeomorphism ψ : $Bd\Delta_1 \to Bd\Delta_1$ as follows: $\psi\left(\sum_{j=0}^{d} \lambda_j a_j\right) :=$

$$\begin{cases} \sum_{j=0}^{d-1} \dfrac{\lambda_j}{2} a_j + \dfrac{\lambda_d + 1}{2} a_d, & \text{if } \lambda_d > 0, \\ 2\sum_{j=0}^{d-1} \left(\lambda_j - \dfrac{1}{2d}\right) a_j, & \text{if } \lambda_d = 0, \; m \geq \dfrac{1}{2d}, \\ \alpha \sum_{j=0}^{d-1} (\lambda_j - m) a_j + \beta a_d, & \text{if } \lambda_d = 0, \; m \leq \dfrac{1}{2d}. \end{cases}$$

where $Bd-$ denotes the boundary, i.e. $Bd\Delta_1 =$

$$\left\{ \sum_{i=0}^{d} \lambda_i a_i \; / \; \sum_{i=0}^{d} \lambda_i = 1, \quad \lambda_i \geq 0, \quad \min(\lambda_i) = 0 \right\}$$

The next picture illustrates this mapping for a 3-simplex.

Case 2. $l < d - 1$

We choose $(l+1)$-faces $\Delta_1' \subset \Delta_1$ containing Ω_1' and $\Delta_3' \subset \Delta_3$; applying case 1 to Δ_1' and Δ_3' we obtain a semialgebraic homeomorphism $\tilde{\psi} : \Delta_1' \to \Delta_3'$. Now,

semialgebraic homeomorphisms between two faces can be trivially extended to the whole d-simplexes. $\square$

In what follows, when we refer to a simplicial complex $K := \Delta_1 \cup \dots \cup \Delta_k$ we suppose that no Δ_i is contained in any other Δ_j, i.e. we do not consider as members of a simplicial complex the faces of its simplexes. This is the contrary of the usual notational rule, but is more convenient to our purposes.

Theorem 5 *5. Let $K := \Delta_1 \cup \dots \cup \Delta_k$ be a finite simplicial complex in $\mathbb{R}^n$ of dimension $d > 1$. There exists a parametrization (P, ψ) of K such that:*

(i) *ψ is finite to one and bijective in the interior of each simplex Δ_i (in particular, ψ is almost everywhere injective).*

(ii) *The domain $P \subset \mathbb{R}^d$ of the parametrization is compact.*

(iii) *For every $x, y \in K$ and for every l, $1 \leq l \leq d$, x, y are connected in dimension l if and only if there exist $a \in \psi^{-1}(x)$ and $b \in \psi^{-1}(y)$ connected in dimension l in P.*

Proof: We will prove the theorem adding the simplexes in K one by one to the parametrization; in step i we will have parametrized the subset $K_i := \Delta_1 \cup \dots \cup \Delta_i \subset K$ by means of a semialgebraic map $\psi_i : P_i \subset \mathbb{R}^d \to K_i$ with all the required properties, plus an additional one that will be needed for the recursive process: that for every face Ω in K_i (of dimension r) there exists an r-dimensional piece Q in the boundary of the domain P_i such that $\psi_i(Q) \subset \Omega$, and $\psi_i |_Q$ is a semialgebraic homeomorphism. The map ψ_i will not be properly speaking a parametrization unless K_i is of the same dimension d than K, but the final P_k will.

Suppose, without loss of generality, that the simplexes $\Delta_1, \dots, \Delta_k$ in K are numbered according to the following rule: let Δ_1 be arbitrary, and for every $i \in \{2, \dots, k\}$ let Δ_i be one of the simplexes not still numbered, and sharing a face of the maximum possible dimension with $\Delta_1 \cup \dots \cup \Delta_{i-1}$.

In these conditions we have the following property: for every $x, y \in K_{i-1}$, x and y are l-connected in K_i if and only if they are l-connected in K_{i-1}: if it was not so, there would be two simplexes Δ_{j_1} and Δ_{j_2}, $j_1, j_2 < i$, l-connected to Δ_i in K_i, but not l-connected to one another in K_{i-1}. Take j_1' and j_2' the smallest numbers such that Δ_{j_1} is l-connected to $\Delta_{j_1'}$ in K_{i-1}, and Δ_{j_2} to $\Delta_{j_2'}$. Maybe $j_1 = j_1'$ or $j_2 = j_2'$, but anyway $j_2' \neq j_1'$, and we may assume $j_1' < j_2'$. Then, we have that $\Delta_{j_2'}$ does not share a $(l-1)$-face with any simplex in $K_{j_2'-1}$, but is l-connected to $\Delta_{j_1'}$ in K_i. This gives a contradiction, because then there must be some simplex not in $K_{j_2'-1}$, but sharing an $(l-1)$-face with it, and this simplex should have been numbered before than $\Delta_{j_2'}$. This property will

be used to assure that our parametrization verifies property (iii).

The first step in the parametrization is trivial: Δ_1 is a simplex of dimension $d_1 \leq d$, and thus there exists a semialgebraic homeomorphism ψ_1 from a d_1-simplex $\Delta_1' \subset \mathbb{R}^{d_1}$ onto Δ_1, and we can consider $\Delta_1' \subset \mathbb{R}^d$. This homeomorphism verifies all the required properties, including the extra one that every r-face of Δ_1 has an r-dimensional piece in the boundary of $P_1 := \Delta_1'$.

In step i, suppose that we have parametrized $K_{i-1} := \Delta_1 \cup \ldots \cup \Delta_{i-1}$, by a semialgebraic map $\psi_{i-1} : P_{i-1} \subset \mathbb{R}^d \to K_{i-1}$ in the required conditions, and let us add the simplex Δ_i of dimension $d_i \leq d$. Call Ω one of the faces of the maximum dimension shared by Δ_i and K_{i-1}, and r its dimension (we suppose $r \geq 1$, $r = 0$ is trivial). Take a smaller simplex Ω_1 contained in Ω, such that $\Omega' = \psi_{i-1}(Q)$, where Q is some r-dimensional simplex in the boundary of P_{i-1} (Q exists because of the additional property). Then, we can attach to Q a d_i-dimensional simplex in $\mathbb{R}^d$ Δ_i', in such a way that Q is a face of Δ_i', and $\Delta_i' \cap P_{i-1} = Q$ (this is possible because of Q being in the boundary of P_{i-1}, and P_{i-1} being compact). Now, by lemma 4, the semialgebraic homeomorphism $\psi_{i-1} \mid_Q$ from Q to Ω' can be extended to a semialgebraic homeomorphism $\tilde{\psi} : \Delta_i' \longrightarrow \Delta_i$. We finally define ψ_i as follows

$$P_i := P_{i-1} \cup \Delta_i' \xrightarrow{\psi_i} K_i$$
$$\underline{x} \longmapsto \begin{cases} \psi_{i-1}(\underline{x}), & \text{if } \underline{x} \in P_{i-1}, \\ \tilde{\psi}(\underline{x}), & \text{if } \underline{x} \in \Delta_i'. \end{cases}$$

In the following figure, we consider that the simplex Δ_1 is parametrized by Δ_1', choose the 1-simplex Ω' properly contained in the common 1-face Ω, and attach a new simplex Δ_2' to the inverse image Q of Ω'.
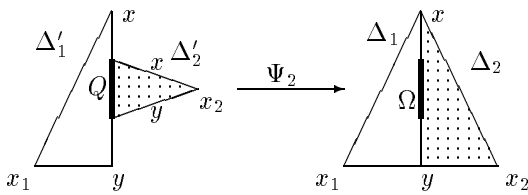

This map ψ_i is in fact a semialgebraic parametrization of K_i, and verfies all the required properties; we shall only discuss property (iii): the 'if' part comes from the fact that ψ_i is continuous and finite to one, and for the 'only if' let x and y be two points in K_i which are l-connected. If $x, y \in K_{i-1}$, then x and y are also l-connected in K_{i-1} (by choice of the numbering of the simplexes) and, by hypothesis, there are $a \in \psi_{i-1}^{-1}(x)$ and $b \in \psi_{i-1}^{-1}(y)$ such that a and b are connected in dimension l in P_{i-1}, and thus also in P_i. If $x, y \in \Delta_i$ and not both are in Ω, then they are d_i-connected in K_i, and $\tilde{\psi}$ being a homeomorphism between Δ_i' and Δ_i, there are $a \in \tilde{\psi}^{-1}(x)$ and

$b \in \tilde{\psi}^{-1}(y)$, which are d_i-connected in Δ_i' and thus in P_i. Finally, if $x \in K_{i-1} \setminus \Omega$ and $y \in \Delta_i \setminus \Omega$, then it must be $l \leq r+1$. Let Δ_j' be the simplex in P_{i-1} that contains the r-dimensional piece Q, and let z be a point in the interior of $\Delta_j = \psi_{i-1}(\Delta_j')$. Then, z is $(r+1)$-connected to y in K_i, and thus also l-connected to x. Now, as in the first case, x and z are l-connected in K_{i-1}, and thus there is $a \in \psi_i^{-1}(x)$ l-connected in P_i to the only point $c = \psi_i^{-1}(z)$. Besides, by choice of z, $c \in \Delta_j'$ and thus c is $(r+1)$-connected to the only point $b \in \psi_i^{-1}(y) \cap \Delta_i'$, so a and b are l-connected in P_i. □

The construction in Theorem 5 does not give a bijective parametrization because faces that are shared by several simplices appear several times in the domain P. Nevertheless, we can consider bijective parametrizations if we delete in the domain P the repeated faces, but then neither P will be compact nor the parametrization will verify condition (iii). These are theoretical obstructions because, for example:

1) the semialgebraic, compact set $SO(3) \subset \mathbb{R}^9$ consisting on the proper orthogonal matrices (which is 3-dimensional) cannot be the continuous bijective image of a compact subset of $\mathbb{R}^3$ (in fact, a continuous bijective map with compact domain would be an homeomorphism, and $SO(3)$ is not homeomorphic to any subset of $\mathbb{R}^3$).

2) in the set shown in next figure (a Moebius strip with 3 'holes') the six marked points are 2-connected to one another, and no bijective parametrization from $\mathbb{R}^2$ onto it keeps this property.
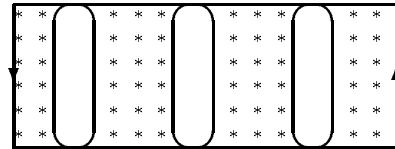
However, if the semialgebraic is globally connected in some dimension l this connectivity can be preserved by a bijective parametrization:

Theorem 6 *Let K be a finite simplicial complex in $\mathbb{R}^n$ of dimension $d > 1$, connected in dimension l. There exists a parametrization (P, ψ) of K such that ψ is bijective and P is connected in dimension l.*

Proof: K being l-connected implies that each simplex Δ_i is of dimension $d_i \geq l$, and that each Δ_i shares a face of dimension at least $l-1$ with $K_{i-1} = \Delta_1 \cup \ldots \cup \Delta_{i-1}$.

We make the same construction as in Theorem 5 but, now, when adding each simplex Δ_i' to the domain P_{i-1} of the parametrization, we delete in Δ_i' the pieces corresponding to faces of Δ_i that were already parametrized because of being shared by some Δ_j, $j < i$ (except, of course, for the piece Q in which we attach Δ_i'). In this way we obtain a bijective parametrization, and it is easy to see that P_i is l-connected:

in fact, P_1 is exactly the first simplex Δ'_1, which is of dimension $d_1 \geq l$, and thus is l-connected. Suppose that P_{i-1} is l-connected. The piece Q of its boundary is of dimension at least $l - 1$, and thus $P_{i-1} \cup \Delta'_i$ is l-connected. Now, as we only delete pieces in the boundary of Δ'_i, P_i is still l-connected. $\square$

Theorems 5 and 6 can be rewriten for semialgebraic sets:

Corollary 7 *Let S be a semialgebraic locally closed set in $\mathbb{R}^n$ of dimension $d > 1$.*

(a) *There is a semialgebraic parametrization (P, ψ) of S, with $P \subset \mathbb{R}^d$ and $\psi : P \longrightarrow S$ finite to one and almost everywhere injective, such that for every two l-connected points $x, y \in S$, there are $a \in \psi^{-1}(x)$ and $b \in \psi^{-1}(y)$ l-connected in P.*

(b) *If S is connected in dimension l, there is a semialgebraic bijective parametrization (P, ψ) of S with P connected in dimension l.*

Proof: We suppose S connected, because in any case it has a finite number of connected components, and the corollary can be applied to each one of them independently.

Let $\tilde{S}$ be a one-point compactification of S, which is a compact semialgebraic set in $\mathbb{R}^{n+1}$ (cf. [1], Ch. 2, Prop. 5.9). We remark that $\tilde{S}$ is also of dimension d. Let K be a triangulation of $\tilde{S}$ (cf. [1], Ch. 9, §2) and suppose that the added point s is a vertex of K; the numbering of the simplexes Δ_i in K can be made without considering the new 1-connectivity (perhaps) introduced by s. Obviously, any parametrization of K gives us a parametrization of $\tilde{S}$, with the same topological properties and, thus, theorems 5 and 6 give, respectively, parametrizations of $\tilde{S}$ in the conditions of (a) and (b); we remark that, in the constructions of Theorems 5 and 6, the inverse image(-s) of a vertex always belong to the boundary of the domain $\tilde{P}$, and deleting them does not affect connectivity in $\tilde{P}$. Thus, the parametrizations of $\tilde{S}$, restricted to the set $P := \tilde{P} \setminus \tilde{\psi}^{-1}(s)$, give parametrizations of S with the required properties. $\square$

Corollary 8 *Let S be a locally closed semialgebraic set in $\mathbb{R}^n$ of dimension $d > 1$, connected in dimension d. There are a semialgebraic subset P of the closed d-ball containing the open d-ball, and a semialgebraic mapping $\psi : P \longrightarrow S$, which is continuous and bijective.*

Proof: Let $\tilde{S}$ be the compactification of S (as in the previous result), and K a triangulation of $\tilde{S}$. Now every Δ_i in K is d-dimensional, and shares with $\Delta_1 \cup \ldots \Delta_{i-1}$ at least a $d - 1$-face. With the same construction as in Theorem 6, P_1 is a d-simplex, thus homeomorphic to the closed d-ball, and if in each step we assume P_{i-1} homeomorphic to some subset

of the closed d-ball containing the open d-ball, then the P_i will also be, because it is obtained from P_{i-1} pasting to it a d-simplex in a $d - 1$-dimensional piece of its boundary, and eventually deleting some boundary pieces. This gives a parametrization of $\tilde{S}$, and to parametrize S it suffices to delete some point that will be also in the boundary, as remarked in Corollary 7. $\square$

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