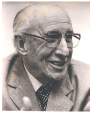

Tracking multiplicities

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The problem

1– $x = (x_1, \dots, x_n) \in \mathbb{C}^n$.

2– a polynomial system $f = (f_1, \dots, f_m) \in \mathbb{C}^m[x]$.

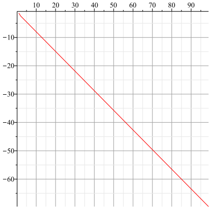
3– w an isolated multiple root of f, i.e. $Df(w)$ has not a full rank.

The quadratic convergence of the Newton method is lost in a neighborhood of w. To recover it

1– determine a regular system at w from the initial.

$$f_i = \sum_{k=1}^{9} x_k + x_i^2 - 2x_i - 8, \quad i = 1:9$$

$$(1, \dots, 1) \; \textit{multiplicity} \; 256 \; (\textit{Dayton} - \textit{Zeng})$$



Contents

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Theorem Let $I =< f_1, \ldots, f_m >$ and $V(I) \subset \mathbb{C}^n$ the associated variety.

1– The dimension of $C[x]/I$ is finite iff the dimension of $V(I)$ is zero.

2– In the finite dimension case one has

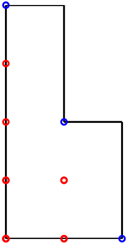
$$dim C[x]/I = \#V(I)$$

where $\#V(I)$ is the number of points of $V(I)$ counted with multiplicities.

Example

$f_1(x,y) = x^2 + x^3, \quad f_2(x,y) = x^3 + y^2$ and $V(I) = \{(0,0),(-1,1)\}$.
A Groebner basis of I is :
$g_1(x,y) = x^0y^4 - y^2, \quad g_2(x,y) = x^1y^2 + y^2, \quad g_3(x,y) = x^2y^0 - y^2,$
We deduce

$$dim C[x]/I = 6$$



Let w an isolated root of $f = (f_1, \ldots, f_m)$ and $I =< f_1, \ldots f_m >$.
Let $\mathbb{C}\{x - w\}$ the local ring of convergent power series in n variables which the maximal ideal is generated by $x_1 - w_1, \ldots x_n - w_n$.
Let $I\mathbb{C}\{x - w\}$ the ideal generated by I in $\mathbb{C}\{x - w\}$
We note $A_w = \mathbb{C}\{x - w\}/I\mathbb{C}\{x - w\}$.

Theorem.

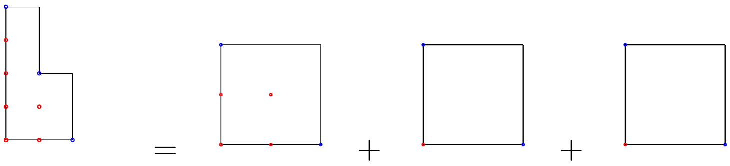
Let $V(I) = \{w_1, w_2, \ldots w_N\}$. Then

1– $\mathbb{C}[x]/I \sim A_{w_1} \times \ldots \times A_{w_N}$.

2– $dim\, \mathbb{C}[x]/I = \displaystyle\sum_{i=1}^{N} dim\, A_{w_i}$.

Example

$f_1(x,y) = x^2 + x^3, \quad f_2(x,y) = x^3 + y^2$ and
$V(I) = \{(0,0), (-1,1), (-1,1)\}$.
1– A standard basis of $I\mathbb{C}\{x,y\}$ is : $g_1 = x^2, \quad g_2 = y^2$.
Hence $dimA_{(0,0)} = 4$.
2– A standard basis of $I\mathbb{C}\{x+1, y-1\}$ (resp. $I\mathbb{C}\{x+1, y+1\}$) is :
$g_1 = 3x, \quad g_2 = 2y$.
Hence $dimA_{(-1,1)} = dimA_{(-1,-1)} = 1$.



Let w a multiple root of f with multiplicity μ. Let $\bar{f}$ a pertubated system of f. In an open neighborhood U_w where the Rouché theorem holds, i.e. if for all $x \in \partial U_w$ we have

$$||\bar{f}(x) - f(x)|| < ||f(x)||$$

then $\bar{f}$ has μ roots in U_w.

$$dim\, A_w^f = \sum_{\bar{w} \in \bar{f}^{-1}(0) \cap U_w} dim\, A_{\bar{w}}^{\bar{f}}.$$

We define

$$\mathcal{D}_w^k(f) = \{L = \sum_{|\alpha| \le k} L_\alpha \partial_\alpha[w] \quad : \quad L(f) = 0$$

$$\textit{and } \phi_i(L) \in \mathcal{D}_w^{k-1}, \forall i = 1 : n\}$$

where the ϕ_i's are the linear anti-differentiation transformations :

$$\phi_i(\partial_\alpha[w]) = \partial_\beta[w] \quad \textit{with} \quad \beta_j = \begin{cases} \beta_j & \text{if } j \neq i \\ \beta_i - 1 & \text{if } j = i \end{cases}$$

Theorem

1– The root w is isolated iff there exists l s.t. $\mathcal{D}_w^{l-1} = \mathcal{D}_w^l$.

2– In this case the dimension of $\mathcal{D}_w^l$ is equal to the multiplicity of w.

We consider the Macaulay matrices

$$S_k = (\partial_\alpha[w]((x-w)^\beta f_i(x))) \quad \text{for } |\beta| \le k-1 \text{ and } i = 1 : m.$$

Theorem D_w^k is isomorphic to the kernel of S_k.

Consequently $D_w^{l-1} = D_w^l$ when $nullity(S_{l-1}) = nullity(S_l)$

Example

$f_1 = x^2 + y^2 - 2,\ f_2 = xy - 1.\ w = (1, 1).$

		∂_{00}	∂_{10}	∂_{01}	∂_{20}	∂_{11}	∂_{02}
S_0	f_1	0	2	2	2	0	2
	f_2	0	1	1	0	1	0
	$- - -$	$-$					
S_1							
	$- - -$	$-$	$-$	$-$			
S_2	$(x-1)f_1$	0	0	0	4	2	0
	$(x-1)f_2$	0	0	0	2	1	0
	$(y-1)f_1$	0	0	0	0	2	4
	$(y-1)f_2$	0	0	0	0	1	2

$rank(S_0) = 0,\ corank(S_0) = 1$
$rank(S_1) = 1,\ corank(S_1) = 2$
$rank(S_2) = 4,\ corank(S_2) = 2.$ Hence $\mu = 2.$

The idea is to determine a sequence of systems which the last is regular at the multiple root of the original system.

Let r the rank of the jacobian matrix J of f at w.
The LVZ method consist to add at each step of deflation $r+1$ equations and unknowns at the initial system.

$$\begin{aligned} & f(x) = 0 \\ & J(x)B\lambda = 0 \qquad B \in \mathbb{C}^{n\times(r+1)} \textit{ random matrix} \\ & \lambda^T h - 1 = 0 \qquad h \in \mathbb{C}^{r+1} \textit{ random vector} \end{aligned}$$

The unknowns of this new system are $(x,\lambda) \in \mathbb{C}^{n+r+1}$.
The corank of the system $(J(x)B\lambda = 0, h^T\lambda - 1 = 0)$ is generalically equal to 1 and the multiplicty of the new system is less than the initial system.
We have added $m+r+1$ equations and $n+r+1$ unknowns.
The number of step of deflations to restore quadratic convergence for the Gauss-Newton method is bounded by the multiplicity of the root.

$$f(x) = 0$$
$$J(x)\lambda = 0$$
$$\lambda^T h - 1 = 0$$

At the end of this of this process we can to add $2^{\mu-1} \times m$ equations and $2^{\mu-1} \times n$ unknowns.

Breadth one : if at each deflation step $\mathit{corank}(\mathit{system}) = 1$.
In this case the second method of DLZ constructs a regular system with at most $\mu \times m$ equations and $\mu \times n$ unknowns.

Step 1- Compute a basis for the dual space of the local quotient ring $A_w : \Lambda = (\Lambda_1, \ldots, \Lambda_\mu)$
Step 2- Compute the system $f^\Lambda := (\Lambda(f_1), \ldots, \Lambda(f_m))$.

Theorem The system f^Λ is regular at w.

Example.
$f_1(x, y) = x^2 + y^2 - 2,\ f_2 = xy - 1.\ w = (1, 1).$
$\Lambda = (1, \partial_1 - \partial_2)$.
$f^\Lambda = (f_1, f_2, x - y)$.

Newton method does not converge quadratically closed to an isolated root. To recover this quadratic convergence the idea is to determine a polynomial system which admits w as regular root, i.e, its jacobian is full rank at w.

To do that we use both numerical and symbolic computation. Two ingredients

1– Deflation based on numerical computation of derivatives.

2– Determination of new polynomials based on computation of numerical ranges.

This way does not require to compute the whole structure of the local qotient algebra which is too huge.

We proceed by two actions :

1– Deflating

2– Kerneling

We do that without to add new variables. The number of equations only increases.

Deflating

We can replace an equation $g(x) = 0$ by the n equations $\frac{\partial g(x)}{\partial x_i} = 0$, $i = 1 : n$ if

$$g(w) = 0, \quad \frac{\partial g(w)}{\partial x_i} = 0, \quad i = 1 : n.$$

We decide of this thanks to a point closed to w.

Deflating. Example

$f_1 = x^3/3 + y^2x + x^2 + 2yx + y^2, \quad f_2 = x^2y + x^2 + 2yx + y^2$
The multiplicity of $(0,0)$ is 6.
We have

$$\begin{array}{cc|cc}
\partial_1 & \partial_2 & \partial_1 & \partial_2 \\
x^2 + y^2 + 2x + 2y & 2yx + 2x + 2y & 2xy + 2x + 2y & x^2 + 2x + 2y
\end{array}$$

$$\begin{array}{cc|cc||cc|cc}
\partial_{11} & \partial_{12} & \partial_{21} & \partial_{22} & \partial_{11} & \partial_{12} & \partial_{11} & \partial_{12} \\
2x + 2 & 2y + 2 & 2y + 2 & 2x + 2 & 2y + 2 & 2x + 2 & 2x + 2 & 2
\end{array}$$

The new polynomial system consists of tree polynomials

$$x^2 + y^2 + 2x + 2y, \quad xy + x + y, \quad x^2 + 2x + 2y.$$

Let us consider a polynomial system $f(x) = 0$ such that $Df(w)$ is rank deficient and $Df_i(w) \neq 0$, $i = 1 : n$. Hence the rank of $Df(w)$ is $r > 0$. If

$$Df(w) = \begin{pmatrix} A(w) & B(w) \\ C(w) & D(w) \end{pmatrix} \in \mathbb{C}^{m \times n}$$

with $A(w)$ invertible matrix of size $r \times r$.
The Jacobian matrix $Df(w)$ has rank r iff the Schur complement is zero

$$D(w) - C(w)A(w)^{-1}B(w) = 0.$$

i.e., w is a root of

$$D(x) - C(x)A(x)^{-1}B(x).$$

Then we add to the initial system at most the $(m-r)\times(n-r)$ equations given by:

$$numer(D(x) - C(x)A^{-1}(x)B(x)) = 0.$$

Alternative : we can also deal with

1– the rational functions $D(x) - C(x)A^{-1}(x)B(x) = 0.$

2– the power series $D(x) - C(x)A_1(x)B(x)$ where $A_1(x)$ is the power series of $A^{-1}(x)$ at w.

$f_1 = x^2 + y^2 + 2x + 2y, \quad f_2 = xy + x + y, \quad f_3 = x^2 + 2x + 2y.$
The root $(0,0)$ has multiplicity 2.

$J = \begin{pmatrix} 2x+2 & 2y+2 \\ y+1 & x+1 \\ 2x+2 & 2y+2 \end{pmatrix}$ and $Df(0,0,0) = \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$ with rank 1.

Then there are two equations which are added to f since :

$$numer(J(2..3,2) - J(2..3,1)J(1,1)^{-1}J(1,2)) = \begin{pmatrix} 2x^2 + 4x - 2y \\ 2y \end{pmatrix}$$

Moreover the rank of the new system is 2.

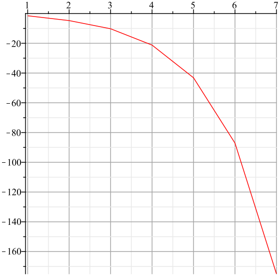
Let $GN_f(x) = x - (Df(x)^T Df(x))^{-1} Df(x)^T f(x)$ and

$$x_{k+1} = GN_f(x_k), \quad k \geq 0.$$

Consider

$f_1 = x^2 + y^2 + 2x + 2y, \quad f_2 = xy + x + y, \quad f_3 = x^2 + 2x + 2y$
$f_4 = x^2 + 2x - y, \quad f_5 = y.$

$$\begin{aligned} e_1 &= 0.22 \\ e_2 &= 0.0088 \\ e_3 &= 3 \times 10^{-5} \\ e_4 &= 6 \times 10^{-10} \\ e_5 &= 1.7 \times 10^{-19} \\ e_6 &= 1.4 \times 10^{-38} \\ e_7 &= 1 \times 10^{-76} \end{aligned}$$



Example from Kobayashi-Suzuki-Sakai

$$f_i = \sum_{k=1}^{9} x_k + x_i^2 - 2x_i - n + 1, \quad i = 1:n. \; w = (1,\dots,1).$$

$$J(x) = \begin{pmatrix} 2x_1 - 1 & 1 & \dots & 1 \\ 1 & 2x_2 - 1 & \dots & 1 \\ & & & \\ 1 & 1 & \dots & 2x_n - 1 \end{pmatrix} \quad J_w = (1)_{i,j}. \text{ Hence } rank(J) = 1.$$

The Shur complement is $\begin{pmatrix} 2x_2 - 1 & 1 & \dots & 1 \\ 1 & 2x_3 - 1 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & 2x_n - 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \frac{1}{2x_1 - 1}(1,\dots,1).$

The deflated system is

$$(f_1, \dots, f_n, x_1 - 1, (2x_1 - 1)(2x_i - 1) - 1, i = 2:n)$$

$$f = \begin{bmatrix} 2\,x + 2\,x^2 + 2\,y + 2\,y^2 + z^2 - 1 \\ (x + y - z - 1)^3 - x^3 \\ (2\,x^3 + 2\,y^2 + 10\,z + 5\,z^2 + 5)^3 - 1000\,x^5 \end{bmatrix}$$

The multiplicity of the root $(0, 0, -1)$ is 18.
The deflated system is :

$$x$$

$$y$$

$$1+z$$

$$y-z-1$$

$$\frac{9}{14}\,x^5+\frac{5}{28}\left(2\,x^3+2\,y^2+10\,z+5\,z^2+5\right)x^2-\frac{625}{126}\,x$$

$$x+y-z-1$$

$$x+x^2+y+y^2+1/2\,z^2-1/2$$

$f = (f_1, \dots f_m)$ and w a multiple root of f.

Theorem Let $f(w) = 0$ and $\nabla f_1(w) = 0$ such that $\nabla \partial_i f_1(w) \neq 0$, $i = 1 : n$. Then

$$multiplicity_w(f) > multiplicity_w(\nabla f_1, f)$$

Theorem Let $f(w) = 0$, $rank(Df(w)) < n$ and $\nabla f_i(w) \neq 0$, $i = 1 : n$. Let $S = \{$ numerators of a Schur complement of $Df(x)\}$. Then

$$multiplicity_w(f) > multiplicity_w(f, S)$$

1- At each step of deflating-kernelling the multiplicity decreases at less by one.
2- The number of steps of deflating-kernelling is bounded by the multiplicity of the root.