“Santaló Summer School 2012: Open Problems”

(Dedicated to J.-P. Dedieu)

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July, 2012

Jean–Pierre Dedieu (1949-2012)

State of the Art:

* [Beltrán, P., 09–11]: $\mathbf{ZPP}_{\mathbb{C}}$ In av. time $O(N^2)$.
* [Bürgisser–Cucker, 12]: Deterministic in av. time $N^{O(\log\log N)}$...

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Conjectures and Trends:
* **Shub–Smale Conjecture (1996):** Use as initial pair the point $e_0 := (1 : 0 : \cdots : 0)$ as zero of the system:

$$g := \begin{cases} d_1 X_0^{d_1-1} X_1 \\ \vdots \\ d_n X_0^{d_n-1} X_n. \end{cases}$$

What about equi–distribution of roots under this solver.?

Problem (Carlos and also Mike's Talks)

Is there a system f_0, easy to solve and such that

$$\max\{\mu_{\mathrm{norm,F}}(f_0,\zeta)\ :\ \zeta\in\mathbb{P}_n(\mathbb{C}), f(\zeta)=0\}\leq nN?.$$

* [Armentano-Shub, 12]: Deformations based on $f_t := f - tK(f(\zeta))$.

Problem (Mike: A Problem "for Carlos and Luismi")

Understand $\theta(h, \eta)$ or, at least, $I(h)$.

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* [Borges-P., 08], [Berhomieu-P., 12]: Complexity is reduced to know the behavior of a Radon transform of the square condition number.

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Is it possible to find real solutions of real systems in probabilistic (or deterministic) average polynomial time?

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Problem (Gregorio's Talk)

What about numerical counting of real solutions in simply exponential time?.

State of the Art (K. Meer):

* [Meer, 05]: $\mathbf{NP}_{\mathbb{R}} = \mathbf{PCP}_{\mathbb{R}}[\mathtt{poly}, 1]$

* [Baartse–Meer, 12]: $\mathbf{NP}_{\mathbb{R}} = \mathbf{PCP}_{\mathbb{R}}[\mathtt{log}, 1]$, $\mathbf{NP}_{\mathbb{C}} = \mathbf{PCP}_{\mathbb{C}}[\mathtt{log}, 1]$!!!!

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Problem (Arora et al. PCP Theorem over $\mathbb{R}$ or $\mathbb{C}$?)

Can the PCP theorem be proved along the lines of the first classical proof by Arora et al?, what else?

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In Klaus talk, [Meer, 12]: $\mathbf{NP}_{\mathbb{R}} = \mathbf{PCP}_{\mathbb{R}}[\mathtt{log}, \mathtt{poly} - \mathtt{log}]$

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Software engineering model + geometrically robust outputs, $\Longrightarrow$ exponential lower complexity bounds for Elimination Theory.

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Problem (Joos Talk: Software Engineering in Semi-Algebraic Solving?)

Is it possible to transfer the notions and models of "software engineering" + "geometric robustness" in Numerical Analysis approach to solving and prove similar lower bounds?, what about Semi-algebraic solving?

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Problem (Jean-Claude: Numerical Analysis and Multiplicities?)

Is it possible to find an efficient numerical method that deals with multiple zeros in efficient average polynomial time?

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* [Bank-Giusti-Heintz-Lehmann-Mbakop-P., 01-12]: Systematic use of Polar,Bi-Polar and Dual varieties to deal with solving of real systems of equations. Efficient complexity time: polynomial in terms of the degree of the polar variety.

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Problem (Marc's Talks: Better control of the degree of the polar)

Is it possible to have sharp bounds on the degree δ^* *of the polar, bi-polar and dual varieties?.*

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Problem (Marc's Talks: Better control of the degree of the polar)

Is it possible to have sharp bounds on the degree δ^* of the polar, bi-polar and dual varieties?.

Is it possible to use these techniques not only for solving but also for deciding, counting, distances etc.? $\cdots\cdots$

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What is known about the convexity along geodesics in the non–linear case with the condition number metric μ_F?

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Problem (Computing geodesics or, maybe, almost geodesics?)

If geodesics in condition number metric were computable in reasonable amount of time, then numerical non-universal methods speed-up till having hyper-fast solvers... Could it be possible to compute geodesics (or maybe "almost" geodesics) in the condition number metric in reasonable amount of time?.

* Numerical Solving in the over-determined case.
* Is it possible speed-up of the homotopy?.
* Numerical Solving and straight-line program encoding of inputs?.
* Ladner's Problem: $\mathbf{P}_{\mathbb{R}} \neq \mathbf{NP}_{\mathbb{R}}$, then $\mathbf{NPI}_{\mathbb{R}} \neq \emptyset$?
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* Better implementations of all these algorithms.
* And more, much more...

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* Thanks to RSME and UIMP, for making this week possible.