Real Number PCPs

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Santaló's Summer School, Part 3, July, 2012

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(work supported by DFG, GZ:ME 1424/7-1)

Outline

1. Introduction

First two talks:

- Real number complexity with important complexity class $\mathrm{NP}_{\mathbb{R}}$
- Probabilistically checkable proofs PCPs in Turing model and PCP theorem to characterize Turing class NP

Today: PCPs in real number complexity

Example (Quadratic Polynomial Systems QPS)

Input: $n, m \in \mathbb{N}$, real polynomials in n variables

$p_1, \ldots, p_m \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most 2; each p_i depending on at most 3 variables;

Do the p_i's have a common real zero?

$\mathrm{NP}_{\mathbb{R}}$ verification for solvability of system

$$p_1(x) = 0 \ , \ \ldots, p_m(x) = 0$$

guesses solution $y^* \in \mathbb{R}^n$ and plugs it into all p_i's ; obviously all components of y^* have to be inspected

PCP question makes perfect sense:

Can we stabilize a verification proof, e.g., for QPS, and detect faults with high probability by inspecting constantly many (real) components only?

Real verifiers: particular probabilistic BSS machines running in polynomial time

Definition (Real verifiers)

$r, q : \mathbb{N} \mapsto \mathbb{N}$; a real verifier $V(r, q)$ is a polynomial time probabilistic BSS machine working as follows: V gets as input vectors $x \in \mathbb{R}^n$ (the problem instance) and $y \in \mathbb{R}^s$ (the verification proof)

i) V generates non-adaptively $r(n)$ random bits;

ii) from x and the $r(n)$ random bits V determines $q(n)$ many components of y;

iii) using x, the $r(n)$ random bits and the $q(n)$ components of y V deterministically produces its result (accept or reject)

Acceptance condition for a language $L \subseteq \mathbb{R}^*$:

A real verifier V accepts a language L iff

- for all $x \in L$ there is a guess y such that

$$\Pr_{\rho \in \{0,1\}^{r(n)}} \{V(x, y, \rho) = \text{'accept'}\} = 1$$

- for all $x \notin L$ and for all y

$$\Pr_{\rho \in \{0,1\}^{r(n)}} \{V(x, y, \rho) = \text{'reject'}\} \geq \frac{1}{2}$$

Important: probability aspects still refer to discrete probabilities.

Real verifiers as well produce random bits.

Definition

$\mathcal{R}, \mathcal{Q}$ function classes.

$L \in \mathrm{PCP}_{\mathbb{R}}(\mathcal{R}, \mathcal{Q})$: L is accepted by a real verifier $V(r, q)$ with

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Example

$\mathrm{NP}_{\mathbb{R}} = \mathrm{PCP}_{\mathbb{R}}(0, poly)$

$\mathrm{NP}_{\mathbb{R}} \supseteq \mathrm{PCP}_{\mathbb{R}}(O(\log n), O(1))$

$\mathrm{PCP}_{\mathbb{R}}(O(\log n), 1)$: leads to questions about zeros of univariate polynomials given by straight line program

Goal: Characterizations of $\mathrm{NP}_{\mathbb{R}}$ via real number PCPs

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- both need long transparent proofs for NP, i.e.,

$$\mathrm{NP} \subseteq PCP(poly(n), O(1))$$

- Arora et al. prove existence of short, almost transparent proofs:

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Here: more on first and third item

2. Long transparent proofs for $\mathrm{NP}_{\mathbb{R}}$

Theorem

$\mathrm{NP}_{\mathbb{R}} \subseteq PCP_{\mathbb{R}}(O(f(n)), O(1))$, *where* f *is superpolynomial*

i.e., $\mathrm{NP}_{\mathbb{R}}$ has long transparent proofs.

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As by-product proof gives generalization of results by Rubinfeld & Sudan on self-testing and -correcting linear functions on finite subsets of $\mathbb{R}^n$: property testing

Problem setting; new difficulties

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Consider QPS input $p_1, \ldots, p_m$, guess $y \in \mathbb{R}^n$; for $r \in \{0,1\}^m$ define

$$P(y, r) := \sum_{i=1}^{m} p_i(y) \cdot r_i$$

Observations:

- if $a \in \mathbb{R}^n$ is a zero, then $P(a, r) = 0 \ \forall \ r$;
- if $a \in \mathbb{R}^n$ is no zero, then $\Pr_r[P(a, r) > 0] \geq \frac{1}{2}$

Minor technical difference to classical setting: structure of P

Important: Separate dependence of P on guessed zero a from that on real coefficients of p_i's

Lemma

There are real linear functions A, B of n, n^2 variables, respectively, depending on a only, as well as linear functions

$$L_A, L_B : \{0,1\}^m \mapsto \mathbb{R}^n, \mathbb{R}^{n^2} \text{ and } C : \{0,1\}^m \mapsto \mathbb{R}$$

such that

$$P(a, r) = C(r) + A \circ L_A(r) + B \circ L_B(r)$$

Moreover, L_A, L_B and C depend on the coefficients of the p_i's.

More precisely: A, B depend on a guessed zero $a \in \mathbb{R}^n$ as follows:

$$A : \mathbb{R}^n \mapsto \mathbb{R}, A(w_1, \ldots, w_n) = \sum_{i=1}^{n} a_i \cdot w_i$$

$$B : \mathbb{R}^{n^2} \mapsto \mathbb{R}, B(w_{11}, \ldots, w_{nn}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \cdot a_j \cdot w_{ij}$$

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Important: In order to evaluate

$$P(a, r) = C(r) + A \circ L_A(r) + B \circ L_B(r)$$

one needs to know only two values, one for A and one for B.

Example

$$p_1(a) = \pi + 1 \cdot a_1 + 2 \cdot a_2, \qquad p_2(a) = 3 \cdot a_1 a_3 - 1 \cdot a_4^2,$$

$$p_3(a) = 1 + \pi a_1 + 7 a_2 a_3$$

$$\begin{aligned} P(a,r) := \sum_{i=1}^{3} p_i \cdot r_i \;\; = \;\; & \pi r_1 + 1 \cdot r_3 + \\ & + a_1 \cdot (1 \cdot r_1 + \pi r_3) + a_2 \cdot 2 r_1 + \\ & + a_1 a_3 \cdot 3 r_2 + a_4^2 \cdot 1 \cdot r_2 + a_2 \cdot a_3 \cdot 7 r_3 \end{aligned}$$

results in

$$C(r) \;=\; (\pi, 0, 1) \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \quad , \quad L_A(r) \;=\; \begin{pmatrix} 1 & 0 & \pi \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

Similarly for L_B!

Idea to stabilize verification proofs (classical):

Instead of potential zero a guess function tables for A, B

Then probabilistically check that with high probability

- functions are linear (linearity test)
- functions do result from the same a (consistency test)
- this a is a zero (satisfiability test)

Problems with this idea

Main problems occur because of new domains to be considered:

Turing model:

domains where to guess function values obvious: $\mathbb{Z}_2^n$

each evaluation like $A(a+b)$, $a, b \in \mathbb{Z}_2^n$ remains in $\mathbb{Z}_2^n$;

BSS model:

domains for A, B not obvious: L_A, L_B give real values for each $r \in \mathbb{Z}_2^n$ (and different ones for each new input); evaluations like $A \circ L_A(r_1 + r_2)$ once more enlarge domain

uniform distribution over $\mathbb{Z}_2^n$

invariant under shifts;

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no constants beside $0, 1$

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arbitrary reals as constants, so linearity check also requires $A(\lambda \cdot x) = \lambda \cdot A(x)$. Again: what domain for λ's?

Sketch of what verifier will do:

i) Expect verification proof to contain function tables for A, B on appropriate domains; tables will have an double exponential size

ii) Check: both functions linear on their domains with high probability;

iii) Check: both functions arise from same $a \in \mathbb{R}^n$ with high probability;

iv) Check: a is a zero of input polynomials with high probability.

Appropriate domain for map A (I)

Outline of construction:

test domain: $\mathcal{X}_1$, proof should provide $A(x)$ for all $x \in \mathcal{X}_1 \oplus \mathcal{X}_1$

safe domain: $\mathcal{X}_0 \subset \mathcal{X}_1$ and Λ; if all tests succeed function A is almost surely linear on $\mathcal{X}_0$ with scalar factors from Λ

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Goal for defining domains: obtain as far as possible shift-invariance

- for fixed $x \in \mathcal{X}_0$ it is $\Pr_{y \in \mathcal{X}_1}(x + y \in \mathcal{X}_1) \geq 1 - \epsilon$
- for fixed $\lambda \in \Lambda$ it is $\Pr_{y \in \mathcal{X}_1}(\lambda y \in \mathcal{X}_1) \geq 1 - \epsilon$

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Clear: $\mathcal{X}_0$ must contain $L_A(\mathbb{Z}_2^m)$

Appropriate domain for map A (II)

$\Lambda := \{\lambda_1, \ldots, \lambda_K\} \subset \mathbb{R}$ multiset of entries in $L_A, K := O(n)$

(w.l.o.g. $m = O(n)$) and as safe domain

$\mathcal{X}_0 := \{\sum_{i=1}^{K} s_i \cdot \lambda_i \mid s_i \in \{0,1\}\}^n$.

Then

- $\mathbb{Z}_2^n \subseteq \mathcal{X}_0$ (i.e., a basis of $\mathbb{R}^n$) and
- $L_A(\mathbb{Z}_2^m) \subseteq \mathcal{X}_0$

Appropriate domain for map A (III)

The test domain then is

$$\mathcal{X}_1 := \left\{ \frac{1}{\alpha} \sum_{\beta \in M^+} s_\beta \cdot \beta \mid s_\beta \in \{0, \ldots, n^3\}, \alpha \in M \right\}^n,$$

where $M := \{\prod_{i=1}^K \lambda_i^{t_i} | t_i \in \{0, \ldots, n^2\}\}$, and

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Lemma

$\mathcal{X}_1$ is almost invariant under additive shifts with fixed $x \in \mathcal{X}_0$ and multiplicative shifts with fixed $\lambda \in \Lambda$. It has doubly exponential cardinality in n.

Testing linearity

Test Linearity

Choose $k \in \mathbb{N}$ large enough; perform k rounds of the following:

- uniformly and independently choose random x, y from $\mathcal{X}_1$ and random α, β from M;
- check if $A(x+y) = \frac{1}{\alpha}A(\alpha x) + \frac{1}{\beta}A(\beta y)$?

If all k checks were correct accept, otherwise reject.

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In k rounds linearity test requires to read $3k$ proof components.

Theorem

If A passes k rounds of linearity test for large enough k, then there is a function f_A such that

a) f_A is defined via

$$f_A(x) := \text{majority}_{y \in \mathcal{X}_1, \alpha \in M} \quad \frac{1}{\alpha} \cdot (A(\alpha(x+y)) - A(\alpha x))\}$$

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b) f_A *is the unique linear function close to A, i.e., both differ in at most a given arbitarily small fraction $0 < \epsilon < \frac{1}{2}$ of points from $\mathcal{X}_1$;*

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c) A can be self-corrected, i.e., for any $x \in \mathcal{X}_0$ the correct value $f_A(x)$ can be computed with high probability from finitely many entries in the table for A.

Consistency, solvability

Do the above as well for function table B and f_B;

suppose both functions are linear with high probability, then:

- Check consistency, i.e., whether f_A, f_B result from a single assignment a; uses self-correction and easy test to check whether coefficient vector $\{b_{ij}\}$ of f_B satisfies $b_{ij} = a_i \cdot a_j$;
- check solvability: see beginning of talk

Theorem (Existence of long transparent proofs)

$\mathrm{NP}_{\mathbb{R}} \subseteq \mathrm{PCP}_{\mathbb{R}}(f(n), O(1))$, *where* $f = n^{O(n)}$.

The same holds for BSS model over complex numbers.

Proof.

Tests use constantly many values stored in doubly exponentially large tables.

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IMPORTANT: The theorem is applied in the full PCP theorem in a situation where n is constant; so size of $f(n)$ does not matter; more crucial: structure of verification proof!

3. The PCP theorem over $\mathbb{R}$

Goal: Adaption of Dinur's proof to show real PCP theorem

Changed viewpoint of QPS problem

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Example

Way we considered QPS instances so far, i.e., m single equations of quadratic polynomials, each with at most 3 variables, can be changed easily to

Example (cntd.)

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QPS$(\tilde{m}, 1, 2, 3)$-instances: arrays have dimension 3, original constraints depend on 1 such array, consistency between components expressed in further constraints depending on 2 arrays.

Below we always try to work with $q = 2$ arrays per constraint in order to define constraint graphs between arrays: edge between two arrays represents constraint depending on those arrays.

Example

$p_1(x_1, x_2, x_3) = 0, p_2(x_2, x_3, x_4) = 0, p_3(x_4, x_5) = 0$

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Variable arrays of dimension 3:

$\chi^{(1)} = (z_1, z_2, z_3), \chi^{(2)} = (z_4, z_5, z_6), \chi^{(3)} = (z_7, z_8)$

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Consistency constraints depending on two arrays:

$$\left.\begin{array}{l} z_2 - z_4 = 0 \\ z_3 - z_5 = 0 \end{array}\right\} \text{ consistency constraint for } (\chi^{(1)}, \chi^{(2)})$$

$z_6 - z_7 = 0$ consistency constraint for $(\chi^{(2)}, \chi^{(3)})$

Crucial: gap-reduction between QPS-instances, i.e., polynomial time transformation of QPS(m, k, q, s)-instance ϕ into QPS(m', k', q, s)-instance ψ such that:

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- if ϕ is satisfiable so is ψ ;
- if ϕ is unsatisfiable, then at least a fraction of $\epsilon > 0$ constraints in ψ is violated by each assignment.

  Here ϵ is fixed constant

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Proof.

Verifier for instance ϕ computes reduction result ψ and expects proof to provide satisfying assignment for ψ. Randomly choose a constraint in ψ and evaluate. If ϕ is unsatisfiable the chosen constraint is violated with probability $\geq \epsilon$. Verifier reads *qs* proof components. Finitely many repetitions increase probability sufficiently. □

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3. Dimension reduction scales parameters q and s down again at price of small lost in unsatisfiability ratio.

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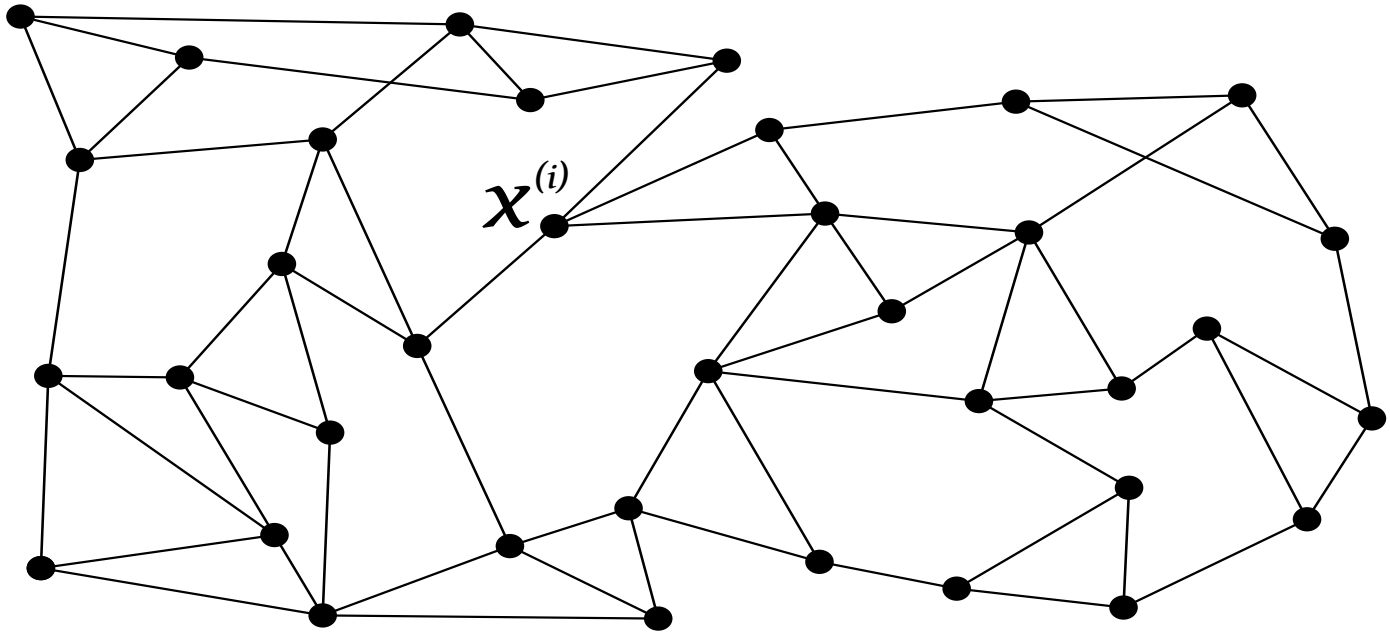
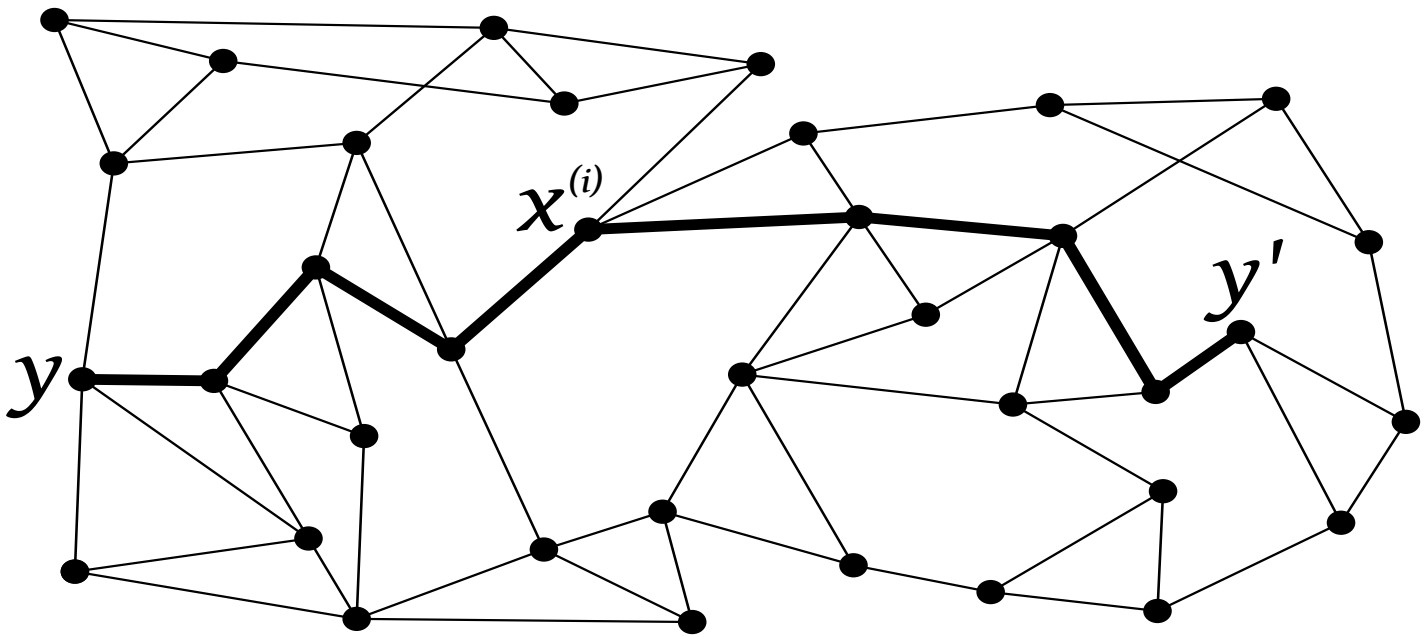
Basic idea is to transform QPS$(m, k, 2, s)$-instance to new one such that violated constraints in old instance occur in significantly more constraints of new instance and violate it;

this is achieved using random walks of constant length t on constraint graph; new constraints are made of paths in old graph; due to expander structure violated constraints = edges occur in many paths.

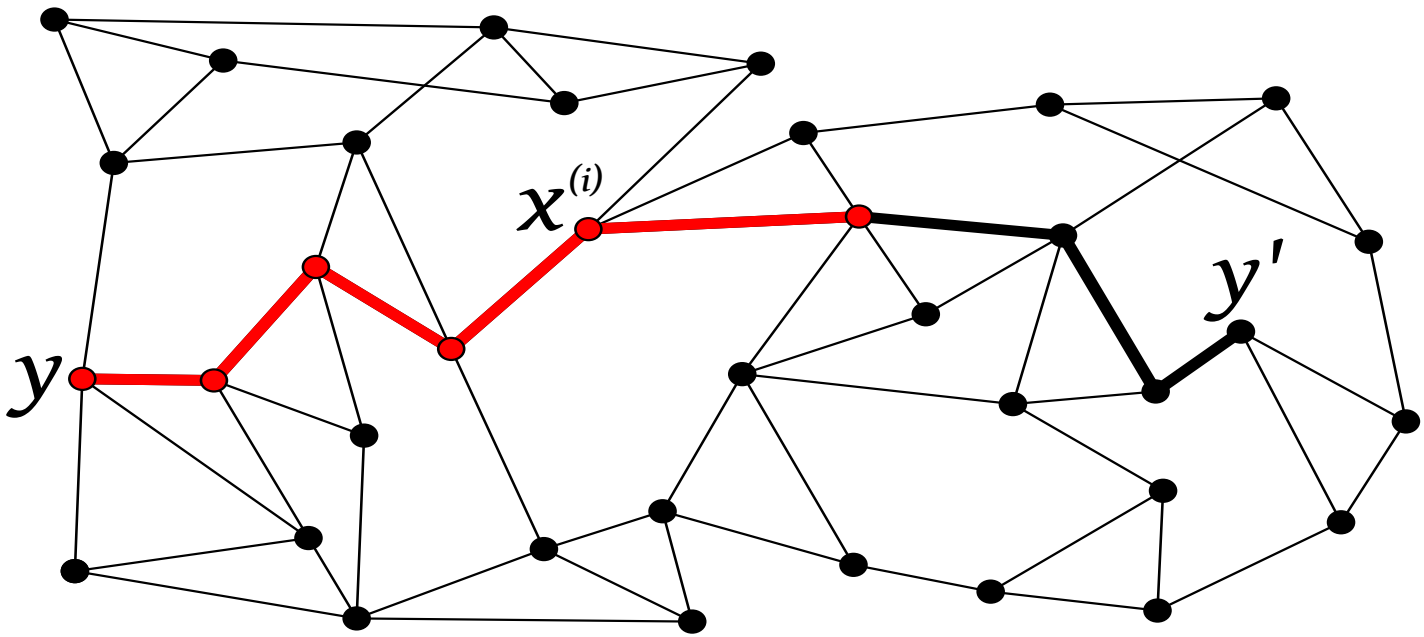


Old constraint graph and variable arrays $x^{(i)} \in \mathbb{R}^s$

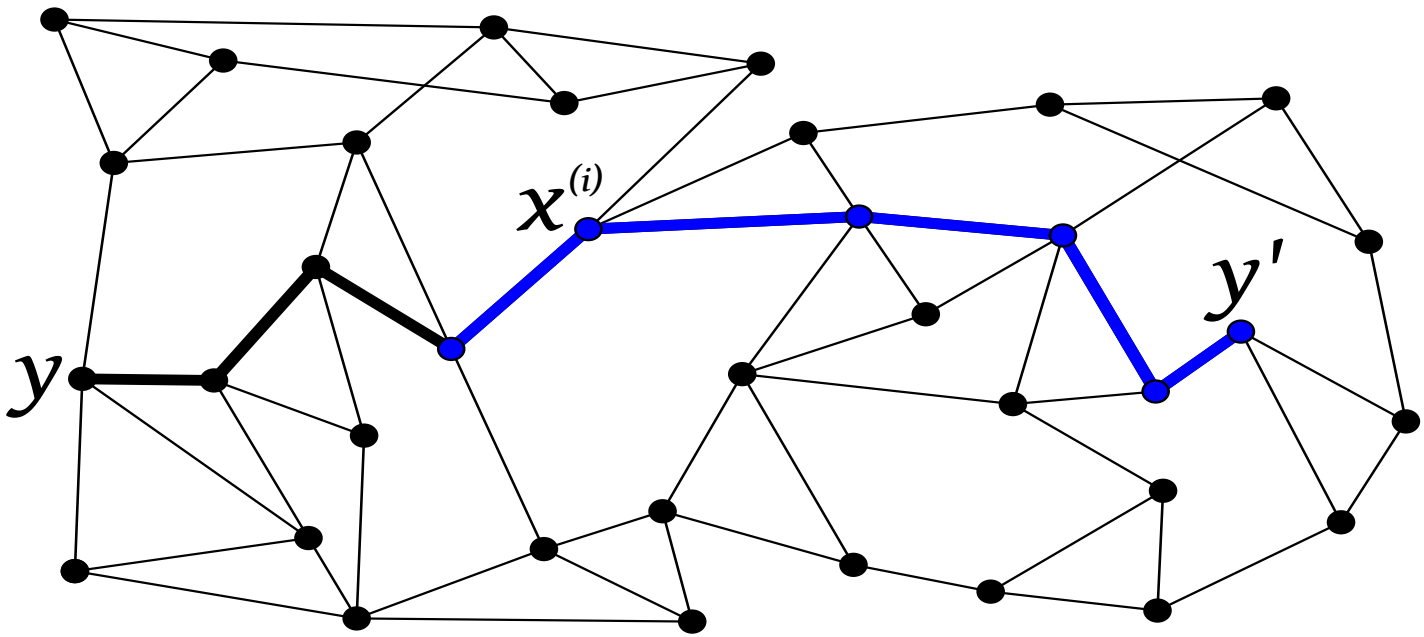


New constraint for each path of length $2t$, t constant;

new variable arrays $y \in \mathbb{R}^{s(t)}$ with $s(t) \leq d^{t+\sqrt{t}+1} \cdot s$



New arrays y claim values on old arrays in t-neighborhood



similarly for y'

Theorem

There exists a polynomial time algorithm that maps a (preprocessed) QPS$(m, k, 2, s)$ instance ψ to a QPS$(d^{2t}m, 2\sqrt{t}k + (2\sqrt{t}+1)s, 2, d^{t+\sqrt{t}+1}s)$-instance ψ^t and has the following properties:

- *If ψ is satisfiable, then ψ^t is satisfiable.*
- *If ψ is not satisfiable and UNSAT$(\psi) < \frac{1}{d\sqrt{t}}$, then UNSAT$(\psi^t) \geq \frac{\sqrt{t}}{3520d} \cdot$ UNSAT(ψ).*

We choose t, d such that $\frac{\sqrt{t}}{3520d} \geq 2$.

Note: if initially ψ has m constraints, then $UNSAT(\psi) \geq \frac{1}{m}$,

thus $O(\log m)$ rounds of amplification increase gap to a constant.

Why not done?

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Why not done?

Problem: array size will not be constant any longer, and so will either query complexity

Step 3: Dimension reduction

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while not decreasing gap too much;

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use long transparent proofs to replace C by

$\mathrm{QPS}(\widehat{m}(t), K, Q, 1)$-instance, where K is constant and Q is the constant query complexity of a long transparent proof

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Reduction in gap factor is harmless!

Theorem (Baartse & M.)

The PCP theorem holds for the real Blum-Shub-Smale model, i.e.,

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The same is true for the complex BSS model:

$$NP_{\mathbb{C}} = PCP_{\mathbb{C}}(O(\log n), O(1))$$

Final remarks

Theorem implies non-approximability result for following optimization problem:

Given a system of polynomial equations over $\mathbb{R}$, find the maximum number of equations that commonly can be satisfied.

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Existence of gap-reduction implies:

Unless $\mathrm{P}_{\mathbb{R}} = \mathrm{NP}_{\mathbb{R}}$ there is no polynomial time algorithm (in the system's size) which, given the system and an $\epsilon > 0$, approximates the above maximum within a factor $1 + \epsilon$.

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Theorem (M.)

$$\mathrm{NP}_{\mathbb{R}} = PCP_{\mathbb{R}}(O(\log n), poly \log n)$$

Classical proof constructs final verifier by composing long transparent proofs with low-degree proofs; needs better structure than the one sufficient to show above theorem; existence over $\mathbb{R}$ unclear.

Thanks for your audience and

thanks again to L.M. Pardo and P. Montaña!