ON THE ZETA MAHLER MEASURE FUNCTION OF THE JACOBIAN DETERMINANT, CONDITION NUMBERS AND THE HEIGHT OF THE GENERIC DISCRIMINANT

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ABSTRACT. In [16] and, later on, in [1] the authors introduced zeta Mahler measure functions for multivariate polynomials ([16] called them "zeta Igusa" functions, but we follow here the terminology of [1]). We generalize this notion by defining a zeta Mahler measure function $\mathcal{Z}_X(\cdot, f) : \mathbb{C} \longrightarrow \mathbb{C}$, where X is a compact probability space and $f : X \longrightarrow \mathbb{C}$ is a function bounded almost everywhere in X. We give sufficient conditions that imply that this function is holomorphic in certain domains. Zeta Mahler measure functions contains big amounts of information about the expected behavior of f on X. This generalization is motivated by the study of several quantities related to numerical methods that solve systems of multivariate polynomial equations. We study the functions $\mathcal{Z}(\cdot, 1/\|\cdot\|_{\rm aff})$, $\mathcal{Z}(\cdot, 1/\mu_{\rm norm})$ and $\mathcal{Z}(\cdot, {\rm JAC})$, respectively associated to the norm of the affine zeros ($\|\cdot\|_{\rm aff}$), the non-linear condition number ($\mu_{\rm norm}$) and the Jacobian determinant (JAC) of complete intersection zero-dimensional projective varieties. We find the exact value of these functions in terms of Gamma functions and we also describe their respective domains of holomorphy in $\mathbb{C}$. With the exact value of these zeta functions we can immediately prove and exhibit expectations of some average properties of zero-dimensional algebraic varieties. For instance, the exact knowledge of $\mathcal{Z}(\cdot, 1/\|\cdot\|_{\rm aff})$ yields as a consequence that the expectation of the mean of the logarithm of the norms of the affine zeros of a random system of polynomial equations is one half of the n-th harmonic number H_n. Other conclusions are exhibited along the manuscript. Using these generalized zeta functions we exhibit the exact value of the arithmetic height of the hyper-surface known as the *discriminant variety* (roughly speaking the variety formed by all systems of equations having a singular zero).

Keywords: Polynomial equation solving, affine and projective varieties, condition number, Discriminant, zeta Mahler measure, Co-area Formula.

1. INTRODUCTION

In [16], J. Cassaigne and V. Maillot introduced zeta Mahler measure function for polynomials, based on [8]. This was re-introduced in the same Journal by H. Akatsuka in his 2009 article [1]. These holomorphic functions have been a useful mathematical concept that have been extensively used in several works on the higher Mahler measures of polynomials (cf. [10, 11, 40] and references therein). In this manuscript we explore several extensions and generalizations of the notion to analyze the average behavior of several quantities related to zero-dimensional projective algebraic varieties and numerical solving.

The natural extension of zeta Mahler measure functions can be stated as follows. Let X be a compact topological space endowed with a measure μ such that $\mu(X) < +\infty$. Let $f : X \longrightarrow \mathbb{C}$ be a measurable function, bounded almost everywhere on X and let $t \in \mathbb{C}$ be a complex number. Let $g : X \longrightarrow \mathbb{C}$ be a continuous function defined on X and assume that g is not identically zero almost everywhere on X. Zeta Mahler measure function associated to f and g over X is the complex function defined by the following identity

$$\mathcal{Z}_X(t, f, g) := \frac{1}{I_X[|g|]} \int_{x \in X} |f(x)|^t |g(x)| d\mu(x),$$

where

Research was partially supported by Spanish Ministry of Science Grant MTM2014-55262-P..

Research was partially supported by Spanish Ministry of Science Grant MTM2014-55262-P and the Program of Pre-doctoral Grants of the University of Cantabria.

$$I_X[|g|] = \int_X |g(x)| d\mu(x) \neq 0, \tag{1.1}$$

provided that these integrals are defined and finite. Note that $\mathcal{Z}_X(t, f, g)$ is the expectation of $|f|^t$ with respect to the probability distribution of X defined by the measure μ and using $|g|$ as probability density function. We will write $\mathcal{Z}_X(t, f)$ in the case $g \equiv 1$. In [16], $X = S^{2n+1} \subseteq \mathbb{C}^{n+1}$ is the complex unit sphere, g is constantly equal to 1 and $f \in \mathbb{C}[X_0, \ldots, X_n]$ is a homogeneous polynomial. In [16] the authors also considered Zeta functions related to the Gaussian distribution on $\mathbb{C}^{n+1}$ which is related to the sphere case (see Identity (1.3) below, for instance). In [1], $X := \prod_{i=1}^n S^1$ is the product of the unit spheres $S^1 \subseteq \mathbb{C}$, g is again constantly equal to 1 and $f \in \mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_n, X_n^{-1}]$ is a Laurent polynomial. In Lemma 2.1 and Lemma 2.2 we exhibit some conditions on f and X that ensure that the function $\mathcal{Z}_X(\cdot, f) : D_{\mathcal{Z}}(f) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ is well-defined and holomorphic in some complex region $D_{\mathcal{Z}}(f) \subseteq \mathbb{C}$.

The relevance of a zeta Mahler measure function like $\mathcal{Z}_X(\cdot, f)$ is the large amount of information that it contains. For instance, for values $t \in i\mathbb{R} := \{ib : b \in \mathbb{R}\}$, the function $\mathcal{Z}_X(t, f)$ is the characteristic function of $\log(|f|)$ as random variable. Additionally, if $p \geq 1$ is a real number, and $p \in D_{\mathcal{Z}}(f)$, the value $\mathcal{Z}_X(p, f) = \|f\|_{L^p}^p$, where $\|\cdot\|_{L^p}$ is the L^p norm defined by the probability distribution induced by μ on X. Moreover, according to Lemma 2.1, if $0 \in \mathbb{C}$ is an interior point of the region $D_{\mathcal{Z}}(f)$, we can easily obtain the moments of $\log|f|$ just differentiating with respect to t in $\mathcal{Z}_X(t, f)$. Namely,

$$E_X[\log^k |f|] = \frac{d^k \mathcal{Z}_X(\cdot, f)}{dt^k}\Big|_{t=0},$$

where $E_X[\log^k |f|]$ denotes the k-th moment of $\log|f|$ with respect to the probability distribution induced on X by the measure μ.

As this zeta function is so rich in information about the function f with respect to the measure μ on X, it is very convenient to have explicit descriptions of these functions (and not merely bounds or approximations).

The goal of this manuscript is to explicitely exhibit several zeta Mahler measure functions, mostly related to properties of zero-dimensional complete intersection algebraic varieties. At the end of the manuscript, we show an example on how this extension of Akatsuka's zeta Mahler function can be applied to compute the exact value of the arithmetic height of a classical Diophantine variety $\Sigma_{(d)}$, known as the discriminant variety.

1.1. **Main outcomes on explicit descriptions of some zeta Mahler functions.** In this subsection we are going to exhibit the exact values of several zeta Mahler measure functions oriented to understand several properties of zero-dimensional complete intersection projective varieties. In order to achieve this purpose we need to introduce several notations.

Let $\{X_0, \ldots, X_n\}$ be a set of variables and $d \in \mathbb{N}$ be a positive integer. We will denote by $H_d(X_0, \ldots, X_n)$ (or $H_d(\underline{X})$ when the number of variables $n+1$ is fixed) the complex vector space of all homogeneous polynomials in the variables $\{X_0, \ldots, X_n\}$ of degree d with complex coefficients. Namely,

$$H_d(\underline{X}) := \{f \in \mathbb{C}[X_0, \ldots, X_n] : \text{ f is homogeneous, } \deg(f) = d\}.$$

Note that $H_d(\underline{X})$ is a complex vector space of dimension $M(d, n) := \binom{d+n}{n} = N_d + 1$. For a degree list $(d) := (d_1, \ldots, d_n)$ we define $\mathcal{H}_{(d)}$ as the complex space of lists $(f_1, \ldots, f_n)$ of polynomials $f_i \in H_{d_i}$. Namely,

$$\mathcal{H}_{(d)} := \prod_{i=1}^n H_{d_i}(\underline{X}).$$

Again, $\mathcal{H}_{(d)}$ is a complex vector space of dimension $M_{(d)} := \sum_{i=1}^n N_{d_i} + n$. Every list $f := (f_1, \ldots, f_n) \in \mathcal{H}_{(d)}$ defines a projective algebraic variety $V_{\mathbb{P}}(f) \subseteq \mathbb{P}_n(\mathbb{C})$ of their common zeros:

$$V_{\mathbb{P}}(f) := \{\zeta \in \mathbb{P}_n(\mathbb{C}) : f_i(\zeta) = 0, \ 1 \leq i \leq n\} \subseteq \mathbb{P}_n(\mathbb{C}).$$

The affine zeros of the system f are denoted by $V_{\mathbb{A}}(f) \subseteq \mathbb{C}^n$. These affine zeros may be viewed as follows. Let $\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C}) \setminus \{X_0 = 0\}$ be the canonical embedding of the complex affine space into the complex projective space. This embedding φ_0 is given by $\varphi_0(x_1, \ldots, x_n) := (1 : x_1 : \ldots : x_n)$, for all $x := (x_1, \ldots, x_n) \in \mathbb{C}^n$, where $(1 : x_1 : \ldots : x_n)$ are the homogeneous coordinates of the corresponding projective point. Hence, φ_0 identifies some of the projective points in $V_{\mathbb{P}}(f)$ with the affine ones in the following form $V_{\mathbb{A}}(f) = \varphi_0^{-1}(V_{\mathbb{P}}(f))$. Note that $V_{\mathbb{P}}(f)$ is always non-empty. We say that $V_{\mathbb{P}}(f)$ is zero-dimensional if it is a finite set. In this case, $V_{\mathbb{P}}(f)$ is also complete intersection: $V_{\mathbb{P}}(f)$ is a finite set of projective points in $\mathbb{P}_n(\mathbb{C})$, given by n polynomial equations. We say that $V_{\mathbb{P}}(f)$ is smooth if it contains no singular points.

Let $\mathbb{P}(\mathcal{H}_{(d)})$ be the complex projective space defined by $\mathcal{H}_{(d)}$. Then , there is a Diophantine algebraic hyper-surface $\Sigma_{(d)} \subseteq \mathbb{P}(\mathcal{H}_{(d)})$ such that $\forall f \in \mathbb{P}(\mathcal{H}_{(d)}) \setminus \Sigma_{(d)}$, the algebraic variety $V_{\mathbb{P}}(f) \subseteq \mathbb{P}_n(\mathbb{C})$ of its complex projective zeros is zero-dimensional, complete intersection and smooth. This hyper-surface $\Sigma_{(d)} \subseteq \mathbb{P}(\mathcal{H}_{(d)})$ is usually called the *discriminant variety*. The discriminant variety $\Sigma_{(d)}$ is determined as the zero set of a unique multi-homogeneous Diophantine polynomial $\mathrm{Disc}_{(d)}$, which is known as the *discriminant polynomial* (cf. [15] and below for more detailed discussions on the discriminant polynomial $\mathrm{Disc}_{(d)}$).

As the zeta Mahler measure function $\mathcal{Z}_X$ is, in fact, a function defined in terms of expectations, we now discuss the natural probability distributions on the spaces $H_d(\underline{\mathrm{X}})$ and $\mathcal{H}_{(d)}$ (cf. [9] or[14] and references therein for more detailed discussions).

Each complex vector space $H_{d_i}(\underline{X})$ may be endowed with a unique Hermitian product which is invariant under the action of the unitary group $\mathcal{U}(n+1)$ on the elements of $H_{d_i}(\underline{X})$:

$$f \longmapsto f \circ U^* \in H_{d_i}(\underline{X}),$$

where U^* is the conjugate transpose of $U \in \mathcal{U}(n+1)$ and $\circ$ denotes composition. This unique Hermitian product is Bombieri-Weyl Hermitian product (cf. Section 3.2 below for precise definitions).

For every i, $1 \leq i \leq n$, let $\mathbb{S}(H_{d_i}(\underline{X}))$ be the unit sphere in $H_{d_i}(\underline{X})$ with respect to the Bombieri-Weyl Hermitian product. Namely,

$$\mathbb{S}(H_{d_i}(\underline{X})) := \{f \in H_{d_i}(\underline{X}) : \|f\|_{d_i}^2 = 1\},$$

where $\|\cdot\|_{d_i}$ is the norm associated to the Bombieri-Weyl Hermitian product on $H_{d_i}(\underline{X})$.

Bombieri-Weyl Hermitian product is naturally extended to $\mathcal{H}_{(d)}$ by the obvious identity, yielding Bombieri-Weyl norm that may be defined as follows:

$$\|f\|_{\Delta} := \|(f_1, \ldots, f_n)\|_{\Delta} = \big(\sum_{i=1}^{n} \|f_i\|_{d_i}^2\big)^{\frac{1}{2}}, \quad \forall f := (f_1, \ldots, f_n) \in \mathcal{H}_{(d)},$$

we thus consider two natural Riemannian compact manifolds associated to these metrics.

On the one hand, we may consider the unit sphere on $\mathcal{H}_{(d)}$ with respect to Bombieri-Weyl norm

$$\mathbb{S}(\mathcal{H}_{(d)}) := \{f \in \mathcal{H}_{(d)} : \|f\|_{\Delta}^2 = 1\}.$$

This sphere is naturally related to the complex projective space $\mathbb{P}(\mathcal{H}_{(d)})$ and it becomes a very convenient structure when studying homogeneous functions $\phi : \mathcal{H}_{(d)} \longrightarrow \mathbb{C}$. Similarly, in the case ϕ is multi-homogeneous, we may consider the product of spheres $\mathfrak{S}_{(d)}^{(n)} \subseteq \mathcal{H}_{(d)}(\underline{X})$ (we also denote it by $\mathfrak{S}_{(d)}$ when no confusion arises) given by the following identity:

$$\mathfrak{S}_{(d)} := \mathfrak{S}_{(d)}^{(n)} := \prod_{i=1}^{n} \mathbb{S}(H_{d_i}(\underline{X})). \tag{1.2}$$

This product of spheres $\mathfrak{S}_{(d)}$ is naturally related to the product of complex projective spaces $\prod_{i=1}^{n} \mathbb{P}(H_{d_i}(\underline{\mathrm{X}}))$ and it is a natural space to study multi-homogeneous functions $\phi : \mathcal{H}_{(d)} \longrightarrow \mathbb{C}$.

Both $\mathbb{S}(\mathcal{H}_{(d)})$ and $\mathfrak{S}_{(d)}$ are endowed with their respective natural volume forms, $d\nu_{\mathbb{S}}$ and $d\nu_{\mathfrak{S}}$, associated to their respective Riemannian structures. In both cases, the volumes are finite (i.e. $\nu_{\mathbb{S}}[\mathbb{S}(\mathcal{H}_{(d)})] <$

$+\infty$, $\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}] < +\infty$) and they induce natural probability distributions on $\mathbb{S}(\mathcal{H}_{(d)})$ and $\mathfrak{S}_{(d)}$ respectively. These probability distributions are equivalent to standard Gaussian distributions $N_{\Delta}(0, I)$ on the complex space $\mathcal{H}_{(d)}$. Similarly, in the case complex projective spaces (as $\mathbb{P}_n(\mathbb{C})$, $\mathbb{P}(H_d(\underline{X}))$ or $\mathbb{P}(\mathcal{H}_{(d)})$) occur in forthcoming pages, we also consider the natural volume form $d\nu_{\mathbb{P}}$ associated to the corresponding complex Riemannian structure. As these volumes of complex projective spaces are also finite, they also induce probability measures, just dividing by the corresponding volumes. For instance, if $\phi : \mathcal{H}_{(d)} \longrightarrow \mathbb{R}$ is a function, homogeneous of degree k, provided that the real part satisfies $\Re(\frac{tk}{2} + M_{(d)}) > 0$, integrating in polar coordinates, the following equality holds :

$$E_{f\sim N_{\Delta}(0,I)}\left[|\phi|^t\right] = \frac{\Gamma\left(\frac{tk}{2} + M_{(d)}\right)}{\Gamma\left(M_{(d)}\right)} \mathcal{Z}_{\mathbb{S}(\mathcal{H}_{(d)})}(t, \phi), \tag{1.3}$$

where $E_{f\sim N_{\Delta}(0,I)}$ denotes expectation in $\mathcal{H}_{(d)}$ with respect to the standard Normal distribution defined by Bombieri-Weyl's norm. Similarly, if ϕ is multi-homogeneous of degrees $(k_1, \ldots, k_n)$ with respect to each group of variables in the cartesian product $\mathcal{H}_{(d)} := \prod_{i=1}^n H_{d_i}(\underline{X})$, then, provided that the real parts satisfy $\Re(\frac{tk_i}{2} + M(d_i, n)) > 0$, for every i, $1 \leq i \leq n$, the following equality also holds:

$$E_{f\sim N_{\Delta}(0,I)}\left[|\phi|^t\right] = \left(\prod_{i=1}^{n} \frac{\Gamma\left(M(d_i, n) + \frac{tk_i}{2}\right)}{\Gamma(M(d_i, n))}\right) \mathcal{Z}_{\mathfrak{S}_{(d)}}(t, \phi). \tag{1.4}$$

These two identities allow us to work freely either with $\mathcal{Z}_{\mathfrak{S}_{(d)}}$ or $\mathcal{Z}_{\mathbb{S}(\mathcal{H}_{(d)})}$. An immediate consequence of these two equalities (just by differentiating with respect to t), is the following Corollary.

Corollary 1.1 (Relation between Mahler measures in spheres and products of spheres)**.** *With the same notations as above, let $\phi : \mathcal{H}_{(d)} \longrightarrow \mathbb{R}$ be a multi-homogeneous function of multi-degree $(\delta_1, \ldots, \delta_n)$ and total degree δ. Assume that the zeta Mahler measure functions $\mathcal{Z}_{\mathbb{S}(\mathcal{H}_{(d)})}(t, \phi)$ and $\mathcal{Z}_{\mathfrak{S}_{(d)}}(t, \phi)$ exist and are differentiable near $t = 0$. Then, the Mahler measures of ϕ satisfy:*

$$m_{\mathbb{S}(\mathcal{H}_{(d)})}(\phi) + \frac{\delta}{2}\psi(M_{(d)}) = m_{\mathfrak{S}_{(d)}}(\phi) + \sum_{i=1}^{n} \frac{\delta_i}{2}\psi(M(d_i, n)),$$

where ψ is the digamma function and for $X = \mathbb{S}(\mathcal{H}_{(d)})$ or $X = \mathfrak{S}_{(d)}$, $m_X(\phi)$ is the Mahler measure of ϕ, namely

$$m_X(\phi) := \frac{1}{Vol(X)} \int_X \log|\phi(x)| d\nu_x(x).$$

A reader interested in more detailed arguments supporting the use of Bombieri-Weyl Hermitian product, and the probability distributions induced either on $\mathbb{S}(\mathcal{H}_{(d)})$ or $\mathfrak{S}_{(d)}$, may follow [23], [9], [14], [56], [59], [6], [7], [13] and references therein.

Now we introduce several functions defined either in $\mathcal{H}_{(d)}$, $\mathbb{S}(\mathcal{H}_{(d)})$ or $\mathfrak{S}_{(d)}$, through functions that depend on the set of projective zeros $V_{\mathbb{P}}(f)$ associated to f.

We begin our short catalogue of zeta Mahler measure functions with the following one which was already clued in [6]. As above, $\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C})$ is the canonical embedding of the affine space $\mathbb{C}^n$ into the complex projective space $\mathbb{P}_n(\mathbb{C})$. For every system of equations $f \in \mathcal{H}_{(d)}$ and every affine zero $\zeta \in V_{\mathbb{A}}(f)$, we consider the norm of this point in $\mathbb{C}^{n+1}$ with respect to the canonical Hermitian product. Namely,

$$\|\varphi_0(\zeta)\|_{\text{aff}} = (1 + \|\zeta\|^2)^{\frac{1}{2}}.$$

We then consider the following zeta Mahler measure function:

$$\mathcal{Z}(t, {}^1/_{\|\cdot\|_{\text{aff}}}) := \frac{1}{\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]} \int_{f\in\mathfrak{S}_{(d)}} \frac{1}{\mathcal{D}_{(d)}} \left(\int_{\zeta\in V_{\mathbb{A}}(f)} \left(\frac{1}{(1 + \|\zeta\|^2)^{\frac{1}{2}}}\right)^t d\zeta \right) d\nu_{\mathfrak{S}}(f), \tag{1.5}$$

where $\mathcal{D}_{(d)} := \prod_{i=1}^n d_i$ is the Bézout number (see Subsection 5.1 for details). We have considered the multi-homogeneous case (i.e. $\mathfrak{S}_{(d)}$). Generically for $f \in \mathcal{H}_{(d)}$ (and also for $f \in \mathfrak{S}_{(d)}$), the number of affine zeros satisfies $\sharp\,(V_{\mathbb{A}}(f)) = \mathcal{D}_{(d)}$. As the number of finite zeros is generically finite, one could have used $\sum_{\zeta\in V_{\mathbb{A}}(f)}$ instead of $\int_{\zeta\in V_{\mathbb{A}}(f)}$ as they agree up to a zero measure set of f's in $\mathfrak{S}_{(d)}$. We prefer to use the integral sign $\int_{\zeta\in V_{\mathbb{A}}(f)}$ whenever possible. We used the simplified notation $\mathcal{Z}(t,\cdot)$ although, in fact, this function is the zeta Mahler measure function given by

$$\mathcal{Z}(t, {}^{1}/\|\cdot\|_{\rm aff}) := \mathcal{Z}_{\mathcal{V}_{(d)}}(t, {}^{1}/\|\cdot\|_{\rm aff}, NJ\pi_1),$$

where $\mathcal{V}_{(d)}$ is the solution variety described in Identity (5.1) and $NJ\pi_1$ is the Normal Jacobian of the projection $\pi_1 : \mathcal{V}_{(d)} \longrightarrow \mathfrak{S}_{(d)}$.
In [6] a positive answer to Smale 17th Problem (cf. [61]) is exhibited by introducing a Las Vegas algorithm that solves multi-variate polynomial equations in polynomial time, on the average. The algorithm introduced in [6] is shown to compute both affine and projective solutions due to Theorems 1.9 and 4.7 of [6]. These two theorems, use the fact that the expected norms of the affine solutions of systems of polynomial equations are of controlled norm on the average. In our previous notations, Theorem 4.7 of [6] gives an exact description of the value of $\mathcal{Z}(t, {}^{1}/\|\cdot\|_{\rm aff})$ for real values $t\in\mathbb{R}$ satisfying $-2 < t < 2n-1$. The following Proposition extends Theorem 4.7 of [6], by showing the complete description of $\mathcal{Z}(t, {}^{1}/\|\cdot\|_{\rm aff})$ for complex values of t:

Proposition 1.2. *With the same notations as above, let $G\subseteq\mathbb{C}$ be the region given by $G := \{t\in\mathbb{C} : \Re(t) > -2\}$. Then, for every $t\in G$, the zeta Mahler measure function $\mathcal{Z}(t, {}^{1}/\|\cdot\|_{\rm aff})$ satisfies:*

$$\mathcal{Z}(t, {}^{1}/\|\cdot\|_{\rm aff}) = n\mathrm{B}\Big(n, 1+\frac{t}{2}\Big) = \frac{\Gamma(n+1)\Gamma(1+\frac{t}{2})}{\Gamma((n+1)+\frac{t}{2})},$$

where B and Γ are, respectively, beta and gamma functions. In particular, this function admits analytic continuation to $\mathbb{C}\setminus\{z\in\mathbb{Z} : z\le -2\}$, which is the domain of holomorphy of $\mathcal{Z}(t, {}^{1}/\|\cdot\|_{\rm aff})$.

The knowledge of the exact value of this zeta Mahler measure function easily yields almost inmediate relevant properties. As usual, we denote by $H_k := \sum_{i=1}^{k}\frac{1}{i}$ the *k-th harmonic number*. As this function is homogeneous of degree 0, we immediately obtain:

$$E_{f\sim N_\Delta(0,I)}\Big[\frac{1}{\sharp V_{\mathbb{A}}(f)}\sum_{z\in V_{\mathbb{A}}(f)}\log(1+\|z\|^2)^{\frac{1}{2}}\Big] =$$

$$= -\frac{d\mathcal{Z}(t, {}^{1}/\|\cdot\|_{\rm aff})}{dt}\Big|_{t=0} = \frac{1}{2}[\psi(n+1)-\psi(1)] = \frac{1}{2}H_n.$$

Namely, the expectation of the mean of the logarithm of the norms of the affine zeros of a randomly chosen system of polynomial equations equals one half of the n-th harmonic number (i.e. $\frac{1}{2}H_n$). This expected norm of the affine solution of systems of polynomial equations shows that the extreme cases of large zeros of systems of polynomial equations are not very probable. For instance, this expected bound of order $\frac{1}{2}H_n\in O(\log(n))$ is amazingly far away from the Mora-Lazar-Masser-Philippon example of worst case arithmetic solution of Diophantine equations (cf. [38], [39], [21] and references therein on the heights of solutions of systems of polynomial equations). This classical example is given by the homogeneous equations:

$$f_1 := X_1 - 2X_0,\;\; f_2 := X_0^{d_2-1}X_2 - X_1^{d_2},\ldots,\;\; f_n := X_0^{d_n-1}X_n - X_{n-1}^{d_n}.$$

The unique affine solution of this system is $(2, 2^{d_2}, 2^{d_2d_3},\ldots,2^{d_2\cdots d_n})$. Its logarithmic norm is exponential in the number of variables.
A second example of zeta Mahler measure functions we discuss in this short catalogue is the zeta Mahler measure function associated to the inverse of the non-linear condition number $\mu_{\rm norm}$. The normalized condition number $\mu_{\rm norm}$ was first introduced in the manuscript [55] of the series written by M. Shub and S. Smale around the complexity of the Bézout's Theorem (cf. [56, 57, 58]). Since then, the condition number $\mu_{\rm norm}$ has been systematically used in the treatment and analysis of the complexity of the algorithms solving Smale's 17th Problem (cf. [61]). Among the many contributions dealing with this problem we may cite [5], [6], [7], [60], [13] and the books [9], [14] and references therein. The following zeta Mahler measure function was clued in [7].

Let $\zeta \in \mathbb{P}_n(\mathbb{C})$ be a projective point and $p_S : S^{2n+1} \longrightarrow \mathbb{P}_n(\mathbb{C})$ the canonical onto projection from the sphere of radius one $S^{2n+1} \subseteq \mathbb{C}^{n+1}$. The tangent space $T_\zeta\mathbb{P}_n(\mathbb{C})$ is given as the orthogonal complement $< \zeta >^\perp$ of the complex subspace $< \zeta >$ generated by ζ in $\mathbb{C}^{n+1}$, with respect to the canonical Hermitian product in $\mathbb{C}^{n+1}$. We sometimes simplify notations by writing $\zeta^\perp$ instead of $< \zeta >^\perp$. For every homogeneous system $f \in \mathcal{H}_{(d)}$ such that $\zeta \in V_\mathbb{P}(f)$, we may consider the tangent mapping

$$T_\zeta f : T_\zeta\mathbb{P}_n(\mathbb{C}) \longrightarrow T_0\mathbb{C}^n = \mathbb{C}^n.$$

This tangent mapping may be determined as follows. Let $z \in S^{2n+1}$ be a point in the complex sphere such that $p_S(z) = \zeta$. Let $Df(z) \in \mathcal{M}_{n\times(n+1).}(\mathbb{C})$ be the Jacobian matrix of f at z. Then, $T_\zeta f$ is the restriction of $Df(z)$ to the orthogonal complement of ζ in $\mathbb{C}^{n+1}$:

$$T_\zeta f = Df(z)\big|_{\zeta^\perp}.$$

The normalized condition number of f at ζ is given by the following identity:

$$\mu_{\text{norm}}(f, \zeta) := \|f\|_\Delta \|\text{Diag}(d_i^{-\frac{1}{2}})(T_\zeta f)^\dagger\|_{\text{op}},$$

where $\text{Diag}(d_i^{-\frac{1}{2}})$ is the diagonal marix where diagonal entries are $d_1^{-\frac{1}{2}}, \ldots, d_n^{-\frac{1}{2}}$, $\|\cdot\|_\Delta$ is Bombieri-Weyl's norm of f, $(T_\zeta f)^\dagger$ is Moore-Penrose pseudo-inverse and $\|\cdot\|_{\text{op}}$ is the norm as linear operator. In the case $(d) := (1, \ldots, 1)$, systems $f = M$ are matrices with n rows and $n+1$ columns (i.e. in $\mathcal{M}_{n\times(n+1)}(\mathbb{C})$), $\|\cdot\|_\Delta$ is the usual Frobenius norm of the matrix and $\|\text{Diag}(d_i^{-\frac{1}{2}})(T_\zeta f)^\dagger\|_{\text{op}}$ is the norm as linear operator of the Moore-Penrose pseudo-inverse of M. Namely, in the linear case, the condition number becomes the usual Demmel's condition number $\mu(M) = \|M\|_F \|M^\dagger\|_{\text{op}}$. Associated to the normalized condition number μ_{norm} we also have a zeta Mahler measure function which can be defined as follows

$$\mathcal{Z}(t, {}^1/_{\mu_{\text{norm}}}) := \frac{1}{\nu_\mathbb{S}[\mathbb{S}(\mathcal{H}_{(d)})]} \int_{f\in\mathbb{S}(\mathcal{H}_{(d)})} \left(\frac{1}{\mathcal{D}_{(d)}} \int_{\zeta\in V_\mathbb{P}(f)} \mu_{\text{norm}}(f, \zeta)^{-t} \right) d\nu_\mathbb{S}(f).$$

The same comment about the interchangeability of the usage of $\sum_{\zeta\in V_\mathbb{P}(f)}$ or $\int_{\zeta\in V_\mathbb{P}(f)}$ as in Equation (1.5) applies. We do not insist on this aspect again. Note that we have chosen the unit sphere $\mathbb{S}(\mathcal{H}_{(d)})$ here since this normalized condition number is a homogeneous function of degree 0. Once again, we have simplified our notation writing $\mathcal{Z}(t, {}^1/_{\mu_{\text{norm}}})$. In fact, the zeta Mahler measure function of the inverse of the normalized condition number is given as

$$\mathcal{Z}(t, {}^1/_{\mu_{\text{norm}}}) := \mathcal{Z}_{V_{(d)}}(t, {}^1/_{\mu_{\text{norm}}}, NJ\pi_1),$$

where $V_{(d)} := V_{(d)}^{(n)}$ is the solution variety discussed in Subsections 3.3 and $NJ\pi_1$ is the Normal Jacobian of the projection $\pi_1 : V_{(d)} \longrightarrow \mathbb{P}(\mathcal{H}_{(d)})$.

In [7], a second answer to Smale 17th Problem is exhibited. This time the algorithm is much faster, on the average, than the one in [6]. One of the main ingredients of this second algorithm was the knowledge of the exact values of $\mathcal{Z}(t, {}^1/_{\mu_{\text{norm}}})$ for real values of t in the open interval $-4 < t < 0$ (i.e. Theorems 19, 20 and Proposition 22 of [7]). As in the case of the previous Proposition, we now give the complete description of the values of $\mathcal{Z}(t, {}^1/_{\mu_{\text{norm}}})$ for complex values of t, generalizing Theorem 20 and Proposition 22 of [7]:

Proposition 1.3. *With these notations, let let $G \subseteq \mathbb{C}$ be the region given by $G := \{t \in \mathbb{C} : \Re(t) > -4\} \subseteq \mathbb{C}$. Then, the zeta Mahler measure function $\mathcal{Z}(t, {}^1/_{\mu_{\text{norm}}})$ is well-defined and holomorphic in G. Moreover, for every $t \in G$, the following equality holds:*

$$\mathcal{Z}(t, {}^1/_{\mu_{\text{norm}}}) = \frac{\Gamma(M_{(d)})}{\Gamma(M_{(d)} + \frac{t}{2})} \sum_{k=0}^{n-1} \frac{\binom{n+1}{k}\Gamma(n+1-k+\frac{t}{2})}{n^{n-k+1+\frac{t}{2}}\Gamma(n-k)}, \tag{1.6}$$

where $M_{(d)}$ is the dimension of $\mathcal{H}_{(d)}$. In particular, this function admits analytic continuation to $\mathbb{C} \setminus \{z \in \mathbb{Z} \,:\, z \leq -4\}$, which is the domain of holomorphy of $\mathcal{Z}(t, 1/\mu_{\text{norm}})$.

The same expression holds for the case of Demmel's condition number $\mu(A)$, and matrices $A \in \mathcal{M}_{n\times(n+1)}(\mathbb{C})$. Note that, for instance, we may determine the exact value of the expectations of the norm $\|A^{\dagger}\|_{\text{op}}$ for matrices A in the unit sphere $S^{2n(n+1)-1.} \subseteq \mathcal{M}_{n\times(n+1)}(\mathbb{C})$. For instance, in the linear case $(d) = (1, \ldots, 1)$ and $\mu_{\text{norm}} = \mu$, we may compute the expectation of the logarithm of the condition number $E[\log|\mu|]$, noting that $E[\log|\mu|] = -E[\log(1/|\mu|)$, just differentiating the expression above. These results yield:

$$E[\log|\mu|] = -\frac{d\mathcal{Z}(\cdot, 1/\mu)}{dt}\Big|_{t=0} = \frac{1}{2}\sum_{i=0}^{n-1}\binom{n+1}{k}\frac{n-k}{n^{n-k+1}}\left[H_{n^2+n-1} + \log n - H_{n-k}\right].$$

The reader may compare this formula with the bounds obtained in [20], for instance. However, as pointed out by T. Tao in his blog (cf. also [62], Section 2.3) the exact values of the moments of $\|A\|_{\text{op}}$ seem not to be known (only estimates). Similarly, we do not have an exact description of a zeta Mahler function $\mathcal{Z}(t, \|A\|_{\text{op}})$ yet.
The next zeta Mahler measure function we explicitely exhibit in this manuscript is also related to the tangent mapping $T_\zeta f$. We now consider the following zeta Mahler measure function.

Definition 1.1. With the previous notations, we define the zeta Mahler measure function of the Jacobian determinant $\mathcal{Z}(t, \text{JAC})$ by the following identity:

$$\mathcal{Z}(t, \text{JAC}) := \mathcal{Z}_{\mathcal{V}_{(d)}}(t, \text{JAC}, NJ\pi_1) = \frac{1}{\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}\int_{f\in\mathfrak{S}_{(d)}}\left(\frac{1}{\mathcal{D}_{(d)}}\int_{\zeta\in V_{\mathbb{P}}(f)}|\text{JAC}(f,\zeta)|^t dV_{\mathbb{P}}(f)(\zeta)\right)d\nu_{\mathfrak{S}}(f),$$

where the absolute value of the Jacobian determinant $|\text{JAC}(f,\zeta)|$ is defined as $|\text{JAC}(f,\zeta)| = |\det(T_\zeta f)| = |\det(Df(z)Df(z)^*|^{\frac{1}{2}}$, being $z \in S^{2n+1}$ any point such that $p_S(z) = \zeta$, $t \in \mathbb{C}$ is a complex number and the quantities $\mathcal{V}_d$, $NJ\pi_1$ and $\mathcal{D}_{(d)} := \prod_{i=1}^n d_i$ are the same as above.

We prove the following complete description of $\mathcal{Z}(t, \text{JAC})$.

Theorem 1.4. *With these notations, let $G \subseteq \mathbb{C}$ be the complex region given by $G := \{t \in \mathbb{C} \,:\, \Re(t) > -4\} \subseteq \mathbb{C}$. Then, the function $\mathcal{Z}(t, \text{JAC})$ is a well-defined holomorphic function whose domain of holomorphy contains G. Moreover, for every $t \in G \subseteq \mathbb{C}$, the following equality holds:*

$$\mathcal{Z}(t, \text{JAC}) = (\mathcal{D}_{(d)})^{\frac{t}{2}}\prod_{i=1}^{n}\left(\frac{\Gamma(i+\frac{t}{2}+1)}{\Gamma(i+1)}\cdot\frac{\Gamma(M_i)}{\Gamma(M_i+\frac{t}{2})}\right), \tag{1.7}$$

where $M_i := M(d_i, n)$ is the complex dimension of $H_{d_i}(\underline{X})$. In particular, $\mathcal{Z}(t, \text{JAC})$ admits analytic continuation to the complex domain $\mathbb{C} \setminus \{z \in \mathbb{Z} \,:\, z \leq -4\}$, which is its domain of holomorphy.

For some applications to come, we may introduce the following quantity

$$m(\text{JAC}) := \frac{1}{\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}\int_{f\in\mathfrak{S}_{(d)}}\left(\int_{\zeta\in V_{\mathbb{P}}(f)}\log|\text{JAC}(f,\zeta)|dV_{\mathbb{P}}(f)(\zeta)\right)d\nu_{\mathfrak{S}}(f). \tag{1.8}$$

Then, the following equality holds:

$$m(\text{JAC}) = \mathcal{D}_{(d)}\left(\frac{d\mathcal{Z}(t,\text{JAC})}{dt}\Big|_{t=0}\right) = \frac{\mathcal{D}_{(d)}}{2}\Big(\log\mathcal{D}_{(d)} + \sum_{i=1}^{n}\psi(i+1) - \psi(M_i)\Big).$$

Other zeta Mahler functions are discussed in the manuscript as $\mathcal{Z}_{\prod_{i=1}^n S^{2n+1}}(t, \text{DET})$ (the zeta Mahler measure function of complex Wishart matrices) in Corollary 5.8.
As a referee also pointed us out, in the linear case $(d_1, \ldots, d_n) = (1, \ldots, 1)$, the value of $m(JAC)$ can be obtained directly without recourse to Theorem 1.4 or to Corollary 5.8, by an elegant argument combining Binet-Cauchy formula with the knowledge of the height of the projective space.
As an application of zeta Mahler measure functions, we exhibit the exact value of the height of the discriminant variety $\Sigma_{(d)}$ as follows. According to [15], the discriminant variety $\Sigma_{(d)}$ is a hyper-surface

defined by a multi-homogeneous polynomial $\mathrm{Disc}_{(d)}$ known as the *discriminant polynomial* (cf. also [22], [24] or [30]). Here, we compute the exact value of the arithmetic height of this variety (in the sense of [27], [47, 48, 49, 50], [41], [16] and sequels. We have chosen the height with respect to its multi-homogeneous nature, although other heights are comparable in view of equalities (1.3) and (1.4) above.

Theorem 1.5. *With the previous notations, let $(d) := (d_1, \ldots, d_n)$ be a list of degrees and let $\delta_{(d)} := \sum_{i=1}^n d_i$ be the sum of the degrees in the list (d). Assume that $\delta_{(d)} - n = \sum_{i=1}^n (d_i - 1) \geq 1$ holds. Then the arithmetic logarithmic height of the discriminant in the case of generically dense homogeneous polynomials defining a zero-dimensional variety is given by the following equality*

$$ht(\Sigma_{(d)}) := ht(\mathrm{Disc}_{(d)}) = \frac{\mathcal{D}_{(d)}}{2}\left[(\delta_{(d)} - n)\left(\sum_{i=1}^n H_i\right) + \log \mathcal{D}_{(d)}\right].$$

In the simplest case $n = 1$, the discriminant Disc_d of a generic homogeneous polynomial of degree d is the determinant of the Sylvester matrix defined by a generic polynomial f and its derivative f'. Namely, if $f := A_d X_1^d + A_{d-1} X_1^{d-1} + \cdots + A_0$, then

$$\mathrm{Disc}_d(A_0, \ldots A_d) := \det\left(Sylv(f, f')\right),$$

where $Sylv(f, f')$ is the Sylvester matrix defined by f and f'. Theorem 1.5 yields the following exact value of the arithmetic logarithmic height of Disc_d:

$$ht(\mathrm{Disc}_d) = \frac{d}{2}\left((d-1) + \log(d)\right).$$

1.2. **Structure of the manuscript.** The manuscript is structured as follows. Section 2 is devoted to show the most elementary properties of the generalization of zeta Mahler measure function used in this manuscript. Section 3 is devoted to recall some basic facts about the underlying Algebraic Geometry involved in this manuscript. In particular, we include some basic facts about the multivariate discriminant, most of them taken from [15]. Section 4 is devoted to introduce some basic facts from Integral Geometry. For instance, we recall Federer's Co-area Formula. Section 5 is devoted to compute some of the zeta Mahler measure functions stated at the Introduction. In Subsubsection 5.1.1 we prove Proposition 1.2. In Subsubsection 5.2.1 we Prove Proposition 1.3. In this Section we also recall the main outcomes of the work of N.R. Goodman [31], [32] and we rewrite them as a zeta Mahler measure function $\mathcal{Z}_{\prod_{i=1}^n S^{2n+1}}(t, \mathrm{DET})$, where DET is the determinant of a complex Wishart matrix (see Subsection 5.3). Finally, Section 6 is devoted to prove Theorem 1.4 by computing the exact value of $\mathcal{Z}(t, \mathrm{JAC})$. Last but not least, Section 7 is devoted to derive the arithmetic height of $\Sigma_{(d)}$ (and its defining polynomial $\mathrm{Disc}_{(d)}$) using the knowledge of computation of $\mathcal{Z}(t, \mathrm{JAC})$, proving Theorem 1.5. Although some of the zeta Mahler functions treated in these pages have immediate translation to numerical analysis algorithms (as those associated to condition numbers), it may be less clear in the case of the main outcome of Theorem 1.5. At the end of the manuscript we have included Subsection 7.2 which sketches how the main outcome of Theorem 1.5 can be applied to show upper bounds for the error probability of some modular arithmetics algorithms. As the goal of these pages was not to show these applications to algorithms, we just sketch the ideas leaving the details to other discussions.

2. On the existence of holomorphic generalized zeta Mahler measure functions

We begin by the following statement which generalizes and quantifies Proposition 2.1 of [1], provided that the main outcome of [43] is taken into account. It also generalizes the usage done in [16] of [8] to further classes of functions (other than polynomials) and other probability spaces. The proof of Theorem 8 in [1] is based on a result by [26] and both are based on a confuse hypothesis which make somehow difficult to follow the arguments. Both publications ([1] and [26]) use an inductive argument, based on the existence of a particular factorization of multi-variate polynomials that does not apply to many of them as, for instance, $f := X_2(X_1^q - 1) - 1$, among others easy to construct(see Remark 2.2 below for details). Additionally, our generalization introduces quantitative bounds for the domain of holomorphy of a zeta Mahler measure function $\mathcal{Z}_X(t,f)$, which are not present in any former statement of similar results.

Lemma 2.1 (Complex differentiation under the integral sign)**.** *Let X be a compact topological space endowed with a measure $(X,\mathcal{A},\mu)$ such that $\mu(X)<\infty$ and the Borel measurable subsets of X are in the $\sigma-$algebra $\mathcal{A}$. Let $f,g: X\longrightarrow\mathbb{C}$ be two continuous real-valued functions defined on X. Assume that the following properties hold:*

- *f and g are not equal to 0 almost everywhere,*
- *there is some positive real number $p\in\mathbb{R}$, $p>0$ such that for all positive real number $\varepsilon>0$, the following equality holds*

$$\frac{1}{\mu(X)}\int_X \chi_{X(f,\varepsilon)}(x)|g(x)|d\mu(x)\le C(f,g)\varepsilon^p, \tag{2.1}$$

where $\chi_{X(f,\varepsilon)}$ is the characteristic function of the set

$$X(f,\varepsilon):=\{x\in X\ :\ |f(x)|\le\varepsilon\},$$

and $C(f,g)$ is a constant which only depends on f, g and X.

Then, the following function

$$\mathcal{Z}_X(t,f,g):=\frac{1}{I_X[|g|]}\int_X |f(x)|^t|g(x)|d\mu(x),$$

is well-defined and holomorphic in the complex domain $\{t\in\mathbb{C}\ :\ \Re(t)>-p\}$, where $\Re(t)$ denotes the real part of t and $I_X[|g|]$ denotes the integral $\int_{x\in X}|g(x)|d\mu(x)$. Moreover

$$\frac{d^k\mathcal{Z}_X(t,f,g)}{dt^k}=\frac{1}{I_X[|g|]}\int_X |f(x)|^t(\log|f(x)|)^k|g(x)|d\mu(x)$$

Proof. Let $G\subseteq\mathbb{C}$ the complex domain given by the following identity:

$$G:=\{t\in\mathbb{C}\ :\ \Re(t)>-p\}.$$

We first prove the following two inequalities for every $t\in G$. Let us denote by $I(t,f,g)$ the following integral:

$$I(t,f,g):=\int_X ||f(x)|^t|g(x)||d\mu(x)=\int_X |f(x)|^{\Re(t)}|g(x)|d\mu(x).$$

Then, we have

$$\begin{cases} I(t,f,g)\le\mu(X)M^{\Re(t)}N & \text{if } \Re(t)\ge 0,\\ I(t,f,g)<\mu(X)\left(N+\frac{C(f,g)}{2^{\Re(t)}-2^{-p}}\right) & \text{if } -p<\Re(t)<0,\end{cases} \tag{2.2}$$

where $M:=\max\{1,\max\{|f(x)|\ :\ x\in X\}\}$ and $N:=\max\{|g(x)|\ :\ x\in X\}$. The case $\Re(t)\ge 0$ being obvious, we concentrate our arguments in the case $-p<\Re(t)<0$. We prove the inequality of this case by following the same arguments as in the proof of Proposition 2.1 of [1].

First of all, we introduce the following two measurable subsets of X:

$$X_+(f):=\{x\in X\ :\ |f(x)|>1\},\ \text{ and }\ X_-(f):=\{x\in X\ :\ |f(x)|\le 1\}.$$

Then,

$$I(t,f,g)=I_+(t,f,g)+I_-(t,f,g),$$

where

$$I_+(t,f,g):=\int_{X_+(f)}|f(x)|^{\Re(t)}|g(x)|d\mu(x),$$

and

$$I_-(t,f,g):=\int_{X_-(f)}|f(x)|^{\Re(t)}|g(x)|d\mu(x).$$

We obviously conclude that

$$I_+(t,f,g)\leq \mu(X)N. \tag{2.3}$$

As for $I_-(t,f,g)$ we decompose $X_-(f)$ as the disjoint union of the following measurable subsets:

$$X_-(f)=f^{-1}(0)\cup\bigcup_{k=0}^{\infty}X_-^{(k)}(f),$$

where

$$X_-^{(k)}(f):=\{x\in X\;:\;\frac{1}{2^{k+1}}<|f(x)|\leq \frac{1}{2^k}\}.$$

Then, as $\mu(f^{-1}(0))=0$, we have

$$I_-(t,f,g)=\sum_{k=0}^{\infty}I_-^{(k)}(t,f,g),$$

where

$$I_-^{(k)}(t,f,g):=\int_{X_-^{(k)}(f)}|f(x)|^{\Re(t)}|g(x)|d\mu(x).$$

Now, for $x\in X_-^{(k)}(f)$, we have $\frac{1}{2^{k+1}}<|f(x)|\leq\frac{1}{2^k}$ and, as $\Re(t)<0$, we conclude that for all $x\in X_-^{(k)}(f)$ we have:

$$|f(x)|^{\Re(t)}<2^{-\Re(t)(k+1)}.$$

Then, we have:

$$I_-^{(k)}(t,f,g)<2^{-\Re(t)(k+1)}\int_{X_-^{(k)}(f)}|g(x)|d\mu(x). \tag{2.4}$$

Then, applying the hypothesis described in Equation (2.1), we conclude:

$$I_-^{(k)}(t,f,g)<2^{-\Re(t)}\frac{\mu(X)C(f,g)(\frac{1}{2^k})^p}{2^{\Re(t)k}}<\frac{2^{-\Re(t)}C(f,g)\mu(X)}{2^{(\Re(t)+p)k}}. \tag{2.5}$$

Hence, if $\Re(t)>-p$, we have

$$I_-(t,f,g)<2^{-\Re(t)}C(f,g)\mu(X)\left(\sum_{k=0}^{\infty}\frac{1}{2^{(\Re(t)+p)k}}\right)=\frac{2^{-\Re(t)}C(f,g)\mu(X)}{1-2^{-(\Re(t)+p)}}=\frac{C(f,g)\mu(X)}{2^{\Re(t)}-2^{-p}}<\infty. \tag{2.6}$$

Putting together inequalities in Equations (2.3) and (2.6) we obviously conclude the second inequality in Equation (2.2) and both inequalities are then proved.

Now, we have proved that the function $|f(x)|^t|g(x)|$ is in $L^1(X,\mathcal{A},\mu)$ (i.e. it is absolutely integrable) for every $t\in G$. Moreover, under our hypothesis,

$$I_X[|g|]\mathcal{Z}_X(t,f,g)=\int_{X'}|f(x)|^t|g(x)|d\mu(x), \tag{2.7}$$

where $X'\subseteq X$ is the set of points in X where f does not vanish. Namely,

$$X':=\{x\in X\;:\;f(x)\neq 0\}.$$

Now, we are in conditions to apply the main outcome of [43] to the function $h:G\times X'\longrightarrow\mathbb{C}$ given by the following identity:

$$h(t,x):=|f(x)|^t|g(x)|.$$

Namely, we observe that we have:

- The function $h(t,\cdot)$ is measurable for every $t\in G$ (in fact, we have seen that $h(t,\cdot)$ is in $L^1(X')$ for every $t\in G$),
- the function $h(\cdot,x)$ is holomorphic for every $x\in X'$ and,

- the integral $\int |h(\cdot,x)|d\mu(x)$ is locally bounded, namely for all $t_0 \in G$, there is some positive real number $\delta > 0$ such that
$$\sup_{t\in G, |t-t_0|<\delta} \int_{X'} |h(t,x)|d\mu(x) < \infty.$$

The third claim follows from the inequalities given in Equation (2.2). From these two inequalities it is obvious how to find for every $t \in G$ a neighborhood such that $I(t,f,g)$ is absolutely and uniformly bounded in this neighborhood.
Thus, applying the main outcome from [43], we conclude that $\mathcal{Z}_X(t,f,g)$ is a holomorphic function in G and we can differentiate under the integral sign. This yields the statement and the proof is finished. □

The following Lemma exhibits sufficient conditions for Lemma 2.1 to hold.

Lemma 2.2. *Let X be a compact topological space and let $f, g : X \longrightarrow \mathbb{C}$ be two continuous real-valued functions defined on X. Assume that X is endowed with a measure $(X, \mathcal{A}, \mu)$ such that $\mu(X) < \infty$ and the Borel measurable subsets of X are in the $\sigma-$algebra $\mathcal{A}$. Let $q < 0$ be a negative real number and assume that the following two properties hold:*

- *the functions f and g are not equal to 0 almost everywhere,*
- *the moment of order q of f is finite (i.e. $f \in L^q(X)$ for the measure μ).*

Then, with the same notations as in Equation (2.1) above, the following inequality holds

$$\frac{1}{\mu(X)} \int_X \chi_{X(f,\varepsilon)}(x)|g(x)|d\mu(x) \leq C(f,g)\varepsilon^{-q}. \tag{2.8}$$

In particular, the function $\mathcal{Z}_X(t,f,g)$ is well-defined and holomorphic in the complex domain $G := \{t \in \mathbb{C} \ : \ \Re(t) > q\}$ and we may differentiate inside the integral sign.

Proof. We just use the Markov inequality in the following form. Let us introduce the function $h(x) := (f(x))^{-1}$ defined almost everywhere in X (since $\mu[\{x \in X \ : \ f(x) = 0\}] = 0$). Then, observe that for every $\delta > 0$ the following inequalities hold:

$$L(f,q) := \int_X |f(x)|^q d\mu(x) = \int_X |h(x)|^{-q} d\mu(x) \geq \int_{T(h,\delta)} |h(x)|^{-q} d\mu(x),$$

where

$$T(h,\delta) := \{x \in X \ : \ |h(x)| \geq \delta\}.$$

Then,

$$L(f,q) \geq \int_{T(h,\delta)} \delta^{-q} d\mu(x) = \delta^{-q}\mu[T(h,\delta)].$$

Hence,

$$\mu[T(h,\delta)] \leq L(f,q)\delta^q.$$

Replacing $\delta = \varepsilon^{-1}$ and noting that

$$T(h,\varepsilon^{-1}) = X(f,\varepsilon),$$

we immediately conclude that Inequality (2.1) holds with $p = -q$ and $C(f,g) := \frac{NL(f,q)}{\mu(X)}$ where $N := \max\{|g(x)| \ : \ x \in X\}$. The rest of the claims of Lemma 2.1 also hold. □

From now own, we shall denote by $\mathcal{Z}_X(t,f) := \mathcal{Z}(t,f,1)$ the zeta Mahler measure function in the case $g = 1$. The following Proposition replaces Theorem 8 of [1] and gives quantitative bounds for the neighborhood of $t = 0$ where the zeta Mahler measure function is defined according to [16].

Proposition 2.3. *Let $f \in \mathbb{C}[X_0, \ldots, X_n] \setminus \mathbb{C}$ be a non-constant complex homogeneous polynomial of degree d. Let $G \subseteq \mathbb{C}$ be the complex domain $G := \{t \in \mathbb{C} \ : \ \Re(t) > -2/d\}$. Then, the zeta Mahler measure function of f in S^{2n+1}:*

$$\mathcal{Z}_{S^{2n+1}}(t,f) := \frac{1}{\nu_{S^{2n+1}}[S^{2n+1}]} \int_{S^{2n+1}} |f(z)|^t d\nu_{S^{2n+1}}(z),$$

is well-defined and holomorphic in G.

Proof. According to Łojasiewicz Inequality as in [37], the following property holds for every non-constant homogeneous polynomial $f \in \mathbb{C}[X_0, \ldots, X_n]$ and for every $z \in S^{2n+1} \subseteq \mathbb{C}^{n+1}$:

$$\mathrm{dist}(z, V)^d \leq C|f(z)|,$$

where C is a constant, $V := V_{\mathbb{A}}(f)$ is the complex hyper-surface defined by f on $\mathbb{C}^{n+1}$, and

$$\mathrm{dist}(z, V) := \inf\{\|z - x\| \ : \ x \in V_{\mathbb{A}}(f)\}.$$

Let us denote by $V_S(f) := V_{\mathbb{A}}(f) \cap S^{2n+1}$. Let $p_S : S^{2n+1} \longrightarrow \mathbb{P}_n(\mathbb{C})$ be the canonical projection onto $\mathbb{P}_n(\mathbb{C})$, let $d_{\mathbb{P}} : \mathbb{P}_n(\mathbb{C})^2 \longrightarrow \mathbb{R}$ be the sine of the Fubini–Study distance in $\mathbb{P}_n(\mathbb{C})$ and let $V_{\mathbb{P}}(f)$ be the projective hyper-surface defined by f. As V is a cone (f is homogeneous), the following property holds:

$$\mathrm{dist}(z, V) = d_{\mathbb{P}}(p_S(z)), V_{\mathbb{P}}(f)).$$

Then, we return to Lemma 2.1 above (with $g \equiv 1$) and we want to study the probability of the set $X(f, \varepsilon) := \{z \in S^{2n+1} \ : \ |f(z)| \leq \varepsilon\}$. From Łojasiewicz Inequality, we conclude that:

$$X(f, \varepsilon) \subseteq \{z \in S^{2n+1} \ : \ d_{\mathbb{P}}(p_S(z), V_{\mathbb{P}}(f))^d \leq C\varepsilon\} = (V_S(f))_{C^{1/d}\varepsilon^{1/d}},$$

where $(V_S(f))_{C^{1/d}\varepsilon^{1/d}}$ is the inverse image under the canonical projection p_S of the tube (with respect to the projective distance $d_{\mathbb{P}}$) around $V_{\mathbb{P}}(f)$ of radius $C^{1/d}\varepsilon^{1/d}$. According to [4], Theorem 1, as $V_{\mathbb{P}}(f)$ is a projective hyper-surface of degree at most d, we have

$$\frac{\nu_{\mathbb{P}}[(V_{\mathbb{P}}(f))_{C^{1/d}\varepsilon^{1/d}}]}{\nu_{\mathbb{P}}[\mathbb{P}_n(\mathbb{C})]} \leq 2d\left(enC^{1/d}\varepsilon^{1/d}\right)^2 = C(f)\varepsilon^{2/d},$$

where e is the basis of the natural logarithm and $C(f)$ is a constant that depends on f.
As the covering $p_S : S^{2n+1} \longrightarrow \mathbb{P}_n(\mathbb{C})$ has constant Jacobian determinant, just by integrating by change of variables, it is easy to conclude that

$$\frac{\nu_S[(V_S(f))_{\varepsilon(f)}]}{\nu_S[S^{2n+1}]} = \frac{\nu_{\mathbb{P}}[(V_{\mathbb{P}}(f))_{\varepsilon(f)}]}{\nu_{\mathbb{P}}[\mathbb{P}_n(\mathbb{C})]} \leq C(f)\varepsilon^{2/d}.$$

Finally, as $V_S(f)$ is a hyper-surface, it is of measure 0. Then, according to Lemma 2.1 and Equation (2.1) we conclude that the following function is well-defined and holomorphic in the complex domain $\{t \in \mathbb{C} \ : \ \Re(t) > -2/d\}$:

$$\mathcal{Z}_{S^{2n+1}}(t, f) := \frac{1}{\nu_S[S^{2n+1}]} \int_{S^{2n+1}} |f(z)|^t d\nu_S(z),$$

and the Proposition is proved.

□

Remark 2.1. We believe that this bound $-2/d$ is not satisfactory. As a consequence of Theorem 1.2 of [29], the domain of holomorphy of the zeta Mahler measure function $\mathcal{Z}_{S^{2n+1}}(t, p)$ of a random homogeneous polynomial $p \in H_d(\underline{X})$ should contain $G := \{t \ : \ \Re(t) > -2\}$, on the average.

Remark 2.2. We had some troubles when verifying the correctness of the proof of Theorem 8 of [1] that does not change the essentials of its claim (as the reader may see in Proposition 2.3 above). As required by some referee, we give some more details on these troubles. In the proof of his Theorem 8, the author of [1] used his Proposition 2.1 and a Lemma (Lemma 2.5) which is, in his own words, *"essentially due to Everest and Ward"*. The author means Lemma 3.8 of [26].
However, both the proof of Lemma 2.5 in [1] and the proof of Lemma 3.8 of [26] are based on an inductive argument based on a hypothesis which is not completely true as stated in both manuscripts. This hypothesis is the existence of a "factorization" of multi-variate polynomials that we reproduce here:
"Let f be r-variable nonzero polynomials. For $(z_1, ..., z_{r-1}) \in \mathbb{C}^{r-1}$ we factorize f as

$$f(z_1, ...z_{r-1}, X_r) = a(z_1, ..., z_{r-1}) \prod_{j=1}^{m} (X_r - g_j(z_1, ..., z_{r-1})), \tag{2.9}$$

where $m = \deg_{X_r}(f)$, $a(X_1, ..., X_{r-1}) \in \mathbb{C}[X_1, ..., X_{r-1}] \setminus \{0\}$ is the coefficient of X_r^m for f and g_j are suitable branches of algebraic functions."

The meaning of the words "suitable branches of algebraic functions" is somehow confuse. And it is even more confuse if the "functions" g_i are defined "for $(z_1, \ldots, z_r) \in \mathbb{C}^{r-1}$". It is rather easy to find bi-variate polynomials as $f \in \mathbb{C}[X_1, X_2]$, given by

$$f := X_2(X_1^q - 1) - 1,$$

where $q \in \mathbb{N}$, and such that f does not admit any "factorization" of the kind

$$f = a(z_1)(X_2 - g(z_1)), \quad z_1 \in \mathbb{C}. \tag{2.10}$$

No $g_1 : \mathbb{C} \longrightarrow \mathbb{C}$ exists satisfying this equality. Even if we restrict ourselves to values z_1 in the complex torus $z_1 \in S^1 = \mathbb{T}^1 \subseteq \mathbb{C}$, no function $g_1 : \mathbb{T}^1 \longrightarrow \mathbb{C}$ exists such the Equality (2.10) holds for all $z_1 \in \mathbb{T}^1$. This happens because not every hyper-surface is in Noether position with respect to some of the variables. This may be arranged just by a generic linear change of variables. However, a linear change of variables changes the Mahler measure of a polynomial when defined on the product of spheres $(\mathbb{T}^1)^n = (S^1)^n$.

Probably, the authors of [1] or [26] mean something different to what they claimed in their respective (and similar) proofs. But, in their way to prove the statement, they are facing the problem of gluing multi-valued functions or some Riemann surface theory (which none of them do in their manuscript). We do not now how to arrange their argument. There may be additional arguments to replace this hypothesis and fix the difficulty in the proof. Anyway, our goal was not to correct their proof but to give a new one. In this sense, Proposition 2.3 above imply Theorem 8 of [1] and there is no risk of serious error in his statement.

3. The underlying Geometry

3.1. **Notations for Lists of polynomials.** We follow most of the notations in [9] and those used at the Introduction. Let $n, d \in \mathbb{N}$ be two positive integers. We denote by $P_d(\underline{X})$ the complex vector space of all polynomials of degree at most d in $\mathbb{C}[X_1, \ldots, X_n]$. Complex vector spaces $H_d(\underline{X})$ and $P_d(\underline{X})$ are obviously isomorphic of dimension $M(d, n) := \binom{d+n}{n}$ and the isomorphism is given by the mapping ${}^a : H_d(\underline{X}) \longrightarrow P_d(\underline{X})$, which associates to every homogeneous polynomial $f \in H_d(\underline{X})$ its affine trace ${}^a f := f(1, X_1, \ldots, X_n) \in P_d(\underline{X})$.

As for lists of polynomial equations, let $(d) := (d_1, \ldots, d_n) \in \mathbb{N}^n$ be a list of degrees. As in the Introduction, we introduce the complex vector space of affine polynomials $\mathcal{P}_{(d)} := \prod_{i=1}^n P_{d_i}(\underline{X})$, given as lists $f := (f_1, \ldots, f_n)$ of polynomials $f_i \in \mathbb{C}[X_1, \ldots, X_n]$ of respective degrees bounded by the list $(d) := (d_1, \ldots, d_n)$. The affine trace obviously defines an isomorphism between $\mathcal{P}_{(d)}$ and $\mathcal{H}_{(d)}$. The complex dimension of both vector spaces satisfies:

$$M_{(d)} := \dim_{\mathbb{C}}(\mathcal{H}_{(d)}) = \dim_{\mathbb{C}}(\mathcal{P}_{(d)}) = \sum_{i=1}^n \binom{d_i + n}{n} = \sum_{i=1}^n M(d_i, n).$$

We sometimes consider the complex projective space defined by any of these spaces. We denote by $\mathbb{P}(\mathcal{H}_{(d)})$ this complex vector space and we denote by $N_{(d)} := M_{(d)} - 1$ (or simply by N) its complex dimension. Analogously we denote by $N(d_i, n)$ (or N_{d_i}) the complex dimension of $\mathbb{P}(H_{d_i}(\underline{X}))$.

For every list $f := (f_1, \ldots, f_n) \in \mathcal{H}_{(d)}$, let $V_{\mathbb{P}}(f) \subseteq \mathbb{P}_n(\mathbb{C})$ be the complex projective variety of their common zeros as stated at the Introduction. Similarly, for every list $g := (g_1, \ldots, g_n) \in \mathcal{P}_{(d)}$, we define the affine algebraic variety $V_{\mathbb{A}}(g) \subseteq \mathbb{C}^n$ of their common affine zeros:

$$V_{\mathbb{A}}(g) := \{x \in \mathbb{C}^n \ : \ g_i(x) = 0, \ 1 \leq i \leq n\} \subseteq \mathbb{C}^n.$$

As in the Introduction, let φ_0 be the standard embedding of $\mathbb{C}^n$ into $\mathbb{P}_n(\mathbb{C})$. Observe that φ_0 identifies $V_{\mathbb{A}}({}^a f)$ with $V_{\mathbb{P}}(f) \cap (\mathbb{P}_n(\mathbb{C}) \setminus \{X_0 = 0\})$. Namely, $V_{\mathbb{A}}({}^a f) = \varphi_0^{-1}(V_{\mathbb{P}}(f))$, for every $f \in \mathcal{H}_{(d)}$. Because of this obvious identification between $\mathcal{P}_{(d)}$ and $\mathcal{H}_{(d)}$, we usually omit the super-script ^{a} in forthcoming pages. So, we will simply write $V_{\mathbb{A}}(f)$, for $f \in \mathcal{H}_{(d)}$, to denote the affine zeros in $V_{\mathbb{A}}({}^a f)$.

In what follows, we denote by $\delta_{(d)} := \sum_{i=1}^n d_i$ the sum of the degrees in the list and by $\mathcal{D}_{(d)} := \prod_{i=1}^n d_i$ we denote the *Bézout number* associated to the degree list $(d) := (d_1, \ldots, d_n)$.

3.2. **A more precise description of Bombieri–Weyl Hermitian product.** As in [55] or [9] (Sec. 12.1) we may equip $\mathcal{H}_{(d)}$ with the unitarily invariant Bombieri-Weyl Hermitian product,

$$\langle \cdot, \cdot \rangle_\Delta : \mathcal{H}_{(d)} \times \mathcal{H}_{(d)} \longrightarrow \mathbb{C}.$$

This Hermitian product may be introduced as follows. Let $f, g \in H_d(\underline{X})$ be two homogeneous complex polynomials of degree d in $n+1$ variables and assume that the following are their respective monomial expansions:

$$f := \sum_{\substack{\mu \in \mathbb{N}^{n+1} \\ |\mu| = d}} a_\mu X_0^{\mu_0} \cdots X_n^{\mu_n}, \quad g := \sum_{\substack{\mu \in \mathbb{N}^{n+1} \\ |\mu| = d}} b_\mu X_0^{\mu_0} \cdots X_n^{\mu_n},$$

where $\mu := (\mu_0, \ldots, \mu_n) \in \mathbb{N}^{n+1}$ and $|\mu| := \mu_0 + \ldots + \mu_n$, $\forall \mu \in \mathbb{N}^{n+1}$. We define the Bombieri–Weyl Hermitian product $\langle f, g \rangle_d$ by the following identity:

$$\langle f, g \rangle_d := \sum_{\substack{\mu \in \mathbb{N}^{n+1} \\ |\mu| = d}} \binom{d}{\mu}^{-1} a_\mu \overline{b}_\mu, \quad \text{where} \quad \binom{d}{\mu} := \frac{d!}{\mu_0! \cdots \mu_n!}, \tag{3.1}$$

is the multi-nomial coefficient and $\overline{\cdot}$ denotes complex conjugation. For every polynomial $f \in H_d(\underline{X})$ we denote by $\|f\|_d := \sqrt{\langle f, f \rangle_d}$ the *Bombieri-Weyl norm of* f. For every degree list (d) we extend this Hermitian product in the obvious way. Namely, if $f := (f_1, \ldots, f_n) \in \mathcal{H}_{(d)}$ and $g := (g_1, \ldots, g_m) \in \mathcal{H}_{(d)}$, then we define $\langle f, g \rangle_\Delta := \sum_{i=1}^n \langle f_i, g_i \rangle_{d_i}$. We denote by $\| \cdot \|_\Delta := \sqrt{\langle \cdot, \cdot \rangle_\Delta}$ the corresponding norm. As in the Introduction, we denote by $\mathbb{S}(\mathcal{H}_{(d)})$ the sphere of radius one in $\mathcal{H}_{(d)}$ with respect to the norme $\| \cdot \|_\Delta$. Similarly, for every degree list (d) we denote by $\mathfrak{S}_{(d)}$ the product of spheres (with Bombieri-Weyl metric) given by the following identity $\mathfrak{S}_{(d)} := \prod_{i=1}^m \mathbb{S}(H_{d_i}(\underline{X}))$, where $\mathbb{S}(H_{d_i}(\underline{X})) := \{f \in H_{d_i}(\underline{X}) : \|f\|_{d_i}^2 = 1\}$.
The following well–known statement shows that Bombieri–Weyl norm is simply an expectation.

Proposition 3.1 (Bombieri-Weyl's norm as L^2 norm (cf. [23], for instance))**.** *For every homogeneous polynomial $f \in H_d(\underline{X})$, its Bombieri-Weyl norm satisfies:*

$$\|f\|_d^2 = \binom{d+n}{n} \frac{1}{\nu_S[S^{2n+1}]} \int_{S^{2n+1}} |f(z)|^2 d\nu_S(z) = \binom{d+n}{n} \mathcal{Z}_{S^{2n+1}}(2, f),$$

where $d\nu_S$ is the canonical volume form in S^{2n+1} associated to its Riemannian structure, $\nu_S[S^{2n+1}] := \frac{2\pi^{n+1}}{\Gamma(n+1)}$, is the volume of the complex sphere $S^{2n+1} := \{z \in \mathbb{C}^{n+1} : \|z\| = 1\}$ and $\mathcal{Z}_{S^{2n+1}}(2, f)$ is the value of the zeta Mahler measure function of f over S^{2n+1} at $t = 2$.

This Proposition yields a direct proof of the unitary invariance of Bombieri–Weyl Hermitian product (cf. [9], for instance). Namely, let $\mathcal{U}(n+1)$ be the unitary group of $(n+1) \times (n+1)$ complex matrices. Let us consider the following action of $\mathcal{U}(n+1)$ on $\mathcal{H}_{(d)}^{(m)}(\underline{X})$:

$$\begin{array}{ccc} \mathcal{U}(n+1) \times \mathcal{H}_{(d)}^{(m)}(\underline{X}) & \longrightarrow & \mathcal{H}_{(d)}^{(m)}(\underline{X}) \\ (U, (f_1, \ldots, f_m)) & \longmapsto & (f_1 \circ U^*, \ldots, f_m \circ U^*), \end{array} \tag{3.2}$$

where $f \circ U^*$ denotes composition. Then, this action is isometric with respect to Bombieri-Weyl Hermitian product: For every $f, g \in \mathcal{H}_{(d)}^{(m)}(\underline{X})$ and for every $U \in \mathcal{U}(n+1)$ the following equality holds:

$$\langle f, g \rangle_\Delta = \langle f \circ U^*, g \circ U^* \rangle_\Delta.$$

3.3. **The solution variety.** Some of the main advances in [55, 56, 58, 59] are due to the smart exploration of a geometric structure related to the polynomial system solving: the *solution variety*. Same can be said about [6, 7]. We follow the notations used in these manuscripts.
We begin by introducing some notations. We consider the canonical Hermitian form $\langle \cdot, \cdot \rangle : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ and the bilinear mapping:

$$\begin{array}{cccc} \cdot : & \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} & \longrightarrow & \mathbb{C} \\ & (u, v) & \longmapsto & u \cdot v := \sum_{i=0}^n u_i v_i, \end{array}$$

where $u := (u_0, \ldots, u_n) \in \mathbb{C}^{n+1}$, $v := (v_0, \ldots, v_n) \in \mathbb{C}^{n+1}$. We denote by $\|u\| \in \mathbb{R}$ the norm of a vector $u \in \mathbb{C}^{n+1}$ with respect to the canonical Hermitian form. The bilinear mapping $\cdot$ has isotropic vectors given as the zero set $V_{\mathbb{P}}(\mathcal{Q}) \subseteq \mathbb{P}_n(\mathbb{C})$ of the quadratic polynomial

$$\mathcal{Q}(X) := \frac{1}{2} X \cdot X = \frac{1}{2} \sum_{i=0}^{n} X_i^2 \in \mathbb{C}[X_0, \ldots, X_n]. \tag{3.3}$$

Let us fix the set $\{X_0, \ldots, X_n\}$ of homogeneous variables and let $(d) := (d_1, \ldots, d_n)$ be a list of positive degrees. We define the projective solution variety $V_{(d)} \subseteq \mathbb{P}(\mathcal{H}_{(d)}) \times \mathbb{P}_n(\mathbb{C})$ by the following equality:

$$V_{(d)} := \{(f, \zeta) \in \mathbb{P}(\mathcal{H}_{(d)}) \times \mathbb{P}_n(\mathbb{C}) \ : \ f_i(\zeta) = 0, \ 1 \leq i \leq n\}.$$

This algebraic variety $V_{(d)}$ is a complex smooth multi–homogeneous algebraic variety of co-dimension n. Thus, its complex dimension is $N_{(d)}$.
The tangent space $T_\zeta \mathbb{P}_n(\mathbb{C})$ of the complex Riemannian manifold $\mathbb{P}_n(\mathbb{C})$ at $\zeta \in \mathbb{P}_n(\mathbb{C})$ is identified with the orthogonal complement of the complex vector space spanned by ζ in $\mathbb{C}^{n+1}$ which we simply denote by $\zeta^\perp$. For a zero $\zeta \in V_{\mathbb{P}}(f)$ we may consider the tangent mapping

$$T_\zeta f : T_\zeta \mathbb{P}_n(\mathbb{C}) \longrightarrow T_0 \mathbb{C}^n = \mathbb{C}^n.$$

As in the Introduction, for every $f := (f_1, \ldots, f_n) \in \mathcal{H}_{(d)}$ and every $\zeta \in V_{\mathbb{P}}(f)$, we denote by $Df(\zeta) := Df(z) \in \mathcal{M}_{n \times (n+1)}(\mathbb{C})$ the Jacobian matrix of f at some representant z of $\zeta \in \mathbb{P}_n(\mathbb{C})$ in S^{2n+1}. The rows of this matrix $Df(\zeta)$ are the gradients $\nabla_z f_1, \ldots, \nabla_z f_n$. Leibnitz rule implies that $Df(z) z^t = 0$ or, equivalently, $\nabla_z f_i \cdot z = 0$ for every i, $1 \leq i \leq n$.
At every point $(f, \zeta) \in V_{(d)}$ the tangent space $T_{(f,\zeta)} V_{(d)}$ is given by the following equality:

$$T_{(f,\zeta)} V_{(d)} := \{(\dot{f}, \dot{\zeta}) \in T_f \mathbb{P}(\mathcal{H}_{(d)}) \times T_\zeta \mathbb{P}_n(\mathbb{C}) : \dot{f}(\zeta) + T_\zeta f(\dot{\zeta}) = 0\},$$

where $T_\zeta f : T_\zeta \mathbb{P}_n(\mathbb{C}) \longrightarrow \mathbb{C}^n$ is the restriction of the Jacobian matrix $Df(\zeta)$ to $T_\zeta \mathbb{P}_n(\mathbb{C}) = \zeta^\perp$, which is the orthogonal complement of ζ in $\mathbb{C}^{n+1}$ with respect to the canonical Hermitian product in $\mathbb{C}^{n+1}$. Namely, we have

$$T_\zeta f = Df(\zeta)|_{\zeta^\perp}.$$

We have two canonical projections defined in the solution variety:

$$\begin{array}{llll} \pi_1 : & V_{(d)} & \longrightarrow & \mathbb{P}(\mathcal{H}_{(d)}), \\ & (f, \zeta) & \longmapsto & f \end{array} \quad \begin{array}{llll} \pi_2 : & V_{(d)} & \longrightarrow & \mathbb{P}_n(\mathbb{C}) \\ & (f, \zeta) & \longmapsto & \zeta. \end{array}$$

The following proposition resumes the main properties of these two canonical projections.

Proposition 3.2 (cf. [9] and [5]). *With these notations, the following properties hold:*

i) *Both mappings π_1 and π_2 are onto.*
ii) *The mapping $\pi_2 : V_{(d)} \longrightarrow \mathbb{P}_n(\mathbb{C})$ is a submersion at every point $(f, \zeta) \in V_{(d)}$ and for every $\zeta \in \mathbb{P}_n(\mathbb{C})$ the fiber $\pi_2^{-1}(\{\zeta\})$ can be identified with a complex projective linear submanifold of $\mathbb{P}(\mathcal{H}_{(d)})$ of co-dimension n given by the following equality:*

$$V_\zeta := \pi_2^{-1}(\{\zeta\}) = \{f \in \mathbb{P}(\mathcal{H}_{(d)}) \ : \ f_i(\zeta) = 0, \ 1 \leq i \leq m\}.$$

iii) *For every $f \in \mathbb{P}(\mathcal{H}_{(d)})$ we have $V_{\mathbb{P}}(f) = \pi_1^{-1}(\{f\})$.*
iv) *The set of critical values of π_1 is a projective hyper–surface $\Sigma_{(d)} \subseteq \mathbb{P}(\mathcal{H}_{(d)})$, known as the discriminant variety.*
v) *For every system $f \in \mathbb{P}(\mathcal{H}_{(d)}) \setminus \Sigma_{(d)}$, outside the discriminant variety, the fiber $V_{\mathbb{P}}(f) = \pi_1^{-1}(\{f\})$ is a smooth complete intersection complex projective subvariety of dimension zero.*
vi) *A system $f \in \mathbb{P}(\mathcal{H}_{(d)})$ is in the discriminant variety $\Sigma_{(d)}$ if and only if there is some zero $\zeta \in V_{\mathbb{P}}(f)$ such that the tangent mapping $T_\zeta f$ is singular.*

3.4. **The discriminant of a family of polynomials.** In this section we discuss some basic properties of the discriminant variety $\Sigma_{(d)}$ and its defining equation: the discriminant polynomial. We shall follow the notations of Section 3.3 and [15]. As in Subsection 3.3 above, we consider the "solution variety" $V_{(d)} \subseteq \mathbb{P}(\mathcal{H}_{(d)}) \times \mathbb{P}_n(\mathbb{C})$ and the two canonical projections $\pi_1 : V_{(d)} \longrightarrow \mathbb{P}(\mathcal{H}_{(d)})$ and $\pi_2 : V_{(d)} \longrightarrow \mathbb{P}_n(\mathbb{C})$. The discriminant variety $\Sigma_{(d)}$ is the hyper-surface of the critical values of π_1. It is determined by the set of zeros of a polynomial $\mathrm{Disc}_{(d)}$. Moreover, this variety $\Sigma_{(d)}$ is definable over the rationals and, hence, we may assume $\mathrm{Disc}_{(d)}$ is a primitive polynomial with integer coefficients. Here we discuss a few properties of $\mathrm{Disc}_{(d)}$, most of them extracted from [15], which are going to be used in Section 7 below.

In forthcoming pages we shall use the notation Res to denote the *multi-variate resultant polynomial*. Given a degree list $(d) := (d_1, \ldots, d_{n+1})$, with $n+1$ terms, there is a unique multi-variate, irreducible and primitive Diophantine polynomial $\mathrm{Res}_{(d)}$, whose variables are the coefficients of a list of $n+1$ homogeneous polynomials in $n+1$ variables $f := (f_1, \ldots, f_{n+1}) \in \prod_{i=1}^{n+1} H_{d_i}(X_0, \ldots, X_n)$ such that the following property holds:

$$\mathrm{Res}_{(d)}(f_1, \ldots, f_{n+1}) = 0 \iff \exists \zeta \in \mathbb{P}_n(\mathbb{C}),\ f_1(\zeta) = \cdots = f_{n+1}(\zeta) = 0.$$

Observe that in $\mathrm{Res}_{(d)}$ the number of polynomials involved in the list $f = (f_1, \ldots, f_{n+1})$ equals the number of homogeneous variables $\{X_0, \ldots, X_n\}$. Sometimes, the multi-variate resultant is called an elimination polynomial. As the resultant is not a primary object of our study, we refer to [36], [19], [24], [18], [15], [52] or [29] and references there in for more detailed expositions on the properties on the Resultant polynomial $\mathrm{Res}_{(d)}$.

Much literature has been devoted for years to understand the resultant polynomial $\mathrm{Res}_{(d)}$. Much less has been written about the discriminant $\mathrm{Disc}_{(d)}$. We may cite [15] which strongly inspired us. The discriminant polynomial $\mathrm{Disc}_{(d)}$ and the variety of its zeros $\Sigma_{(d)}$ are central mathematical (arithmetic) objects with applications in many different fields. The discriminant variety $\Sigma_{(d)}$ plays a central role in many different fields as Foundations of Numerical Analysis (cf [55], [59], [6], [9], [14] and references therein) or Algebraic Geometry and Singularity Theory (cf. [15], [22], [24] and references therein). Another central text which discusses resultants and discriminants is [30].

An essential feature distinguishes the nature of resultants and discriminants. In the multi-variate resultant, the polynomial "to eliminate" is the polynomial f_{n+1} whose coefficients are algebraically independent from the coefficients of the "given" polynomials in the list $f := (f_1, \ldots, f_n)$. This facilitates the determination and formalization of the resultant. However, in the case of the discriminant $\mathrm{Disc}_{(d)}$, "the polynomial to eliminate" is the "function" $\det(T_\zeta f)$, whose "coefficients" are also the coefficients of the given list $f := (f_1, \ldots, f_n)$. This dependence makes a little bit harder to operate with the discriminant and, in particular, it makes more complicated any calculation of integrals and expectations of functions of $\mathrm{Disc}_{(d)}$ when trying to determine the height of $\Sigma_{(d)}$ (see Section 7).

In [15], the discriminant is characterized in the following terms. For every homogeneous polynomial $F \in \mathbb{C}[X_0, \ldots, X_n]$ of degree d we consider the extended $(n+1) \times (n+1)$ matrix:

$$\Delta(Df(X), \nabla_X F) := \begin{pmatrix} \frac{\partial F}{\partial X_0} & \frac{\partial F}{\partial X_1} & \cdots & \frac{\partial F}{\partial X_n} \\ \frac{\partial f_1}{\partial X_0} & \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \cdots & \vdots \\ \frac{\partial f_n}{\partial X_0} & \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{pmatrix} = \begin{pmatrix} \nabla_X F \\ \\ Df(X) \end{pmatrix}.$$

Namely, $\Delta(Df(X), \nabla_X F)$ is the matrix obtained by extending the Jacobian matrix $Df(X)$ with a new row given as the gradient $\nabla_X F$ of F. Let us denote by $J(f_1, \ldots, f_n, F)$ the determinant of this matrix. This is an homogeneous polynomial in $\mathbb{C}[X_0, \ldots, X_n]$ of total degree $\delta\big((d), d\big) := d - 1 + \sum_{i=1}^n (d_i - 1)$. According to Definition 3.5, and Identity (3.1.5) of [15], we may define the discriminant as follows:

For every given a degree list $(d) = (d_1, \ldots, d_n)$, for every $i,\ 1 \leq i \leq n$, let $\mathcal{A}_{d_i}^{(i)}$ be the following set of algebraically independent variables

$$\mathcal{A}_{d_i}^{(i)} := \{A_\mu^{(i)} : \mu \in \mathbb{N}^{n+1}, |\mu| = d_i\}, \tag{3.4}$$

where $\mu := (\mu_0, \ldots, \mu_n) \in \mathbb{N}^{n+1}$, and $|\mu| := \mu_0 + \ldots + \mu_n$. The variables in the list $\mathcal{A}^{(i)}_{d_i)}$ are exactly the list of coefficients of a generic (dense) polynomial F_i in $H_{d_i}(\underline{X})$ given by

$$F_i(\mathcal{A}^{(i)}_{d_i}, \underline{X}) := \sum_{\substack{\mu \in \mathbb{N}^{n+1} \\ |\mu|=d}} A^{(i)}_\mu X_0^{\mu_0} \cdots X_n^{\mu_n}.$$

In particular, every $f_i \in H_{d_i}(\underline{X})$ is obtained by specializing the variables in $\mathcal{A}^{(i)}_{(d_i)}$ into constants $\underline{a}^{(i)} \in \mathbb{C}^{N_{d_i}+1}$ such that $f_i = F_i(\underline{a}^{(i)}, \underline{X})$. We also consider a complete set of variables, algebraically independent over $\mathbb{C}$, formed by all these sets of variables:

$$\mathcal{A}_{(d)} := \bigcup_{i=1}^{n} \mathcal{A}^{(i)}_{d_i},$$

where $(d) := (d_1, \ldots, d_n)$. As usual, for every $P \in \mathbb{Z}[\mathcal{A}_{(d)}]$, we denote by $P(f_1, \ldots, f_n)$ the value of P at the coefficients of the polynomials in the list $f := (f_1, \ldots, f_n)$.

Definition 3.1 (cf. [15])**.** Let $\mathcal{Q}(X) := \frac{1}{2}\sum_{i=0}^n X_i^2 \in \mathbb{C}[X_0, \ldots, X_n]$ be the quadratic polynomial introduced above. For every degree list $(d) := (d_1, \ldots, d_n)$ such that $\delta_{(d)} - n = \sum_{i=1}^n (d_i - 1) \geq 1$, the discriminant is the unique non–zero Diophantine polynomial $\mathrm{Disc}_{(d)} \in \mathbb{Z}[\mathcal{A}_{(d)}]$, such that the following property holds:

$$\mathrm{Res}_{\widetilde{(d)}}(f_1, \ldots, f_n, J(f_1, \ldots, f_n, \mathcal{Q})) = 2^{\mathcal{D}_{(d)}} \mathrm{Disc}_{(d)}(f_1, \ldots, f_n) \,\mathrm{Res}_{(\bar{d})}(f_1, \ldots, f_n, \mathcal{Q}), \tag{3.5}$$

where $\widetilde{(d)} := (d_1, \ldots, d_n, \delta\big((d), 2\big))$ and $(\bar{d}) := (d_1, \ldots, d_n, 2)$ are degree lists and $\mathrm{Res}_{\widetilde{(d)}}$ and $\mathrm{Res}_{(\bar{d})}$ are the respective multi-variate resultants corresponding to these degree lists.

The following statement resumes the main properties of this discriminant polynomial as shown in [15].

Theorem 3.3. *Let $(d) := (d_1, \ldots, d_n)$ be a list of degrees such that $\delta_{(d)} - n \geq 1$. Then, the discriminant polynomial $\mathrm{Disc}_{(d)} \in \mathbb{Z}[\mathcal{A}_{(d)}]$ is a polynomial with integer coefficients that satisfies the following properties:*

i) $\mathrm{Disc}_{(d)}$ *is irreducible and, in particular, the integer coefficients are set-wise co-prime (i.e. their greatest common divisor is 1).*

ii) $\mathrm{Disc}_{(d)}$ *is a multi-homogeneous polynomial with respect to each group of variables $\mathcal{A}^{(i)}_{d_i} := \{A^{(i)}_\mu : \mu \in \mathbb{N}^{n+1}, |\mu| = d_i\}$ determined as the generic coefficients of f_i. Moreover, the degree of $\mathrm{Disc}_{(d)}$ with respect to the coefficients of f_i is given by the following identity:*

$$\deg_{f_i} \mathrm{Disc}_{(d)} = \Big(\prod_{j\neq i} d_j\Big)\big(\delta_{(d)} + d_i - (n+1)\big).$$

iii) *The total degree of $\mathrm{Disc}_{(d)}$ is given by the following identity:*

$$\deg \mathrm{Disc}_{(d)} = \sum_{i=1}^n \deg_{f_i} \mathrm{Disc}_{(d)} = \sum_{i=1}^n \Big(\prod_{j\neq i} d_j\Big)\big(\delta_{(d)} - (n+1)\big) + n\mathcal{D}_{(d)}.$$

iv) *The zero set of $\mathrm{Disc}_{(d)}$ in $\mathbb{P}(\mathcal{H}_{(d)})$ is the discriminant variety $\Sigma_{(d)}$, i.e.*

$$V_{\mathbb{P}(\mathcal{H}_{(d)})}(\mathrm{Disc}_{(d)}) = \Sigma_{(d)}.$$

For every system $f \in \mathcal{H}_{(d)}$ and for every $\zeta := (\zeta_0, \ldots, \zeta_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, we may consider the following two square matrices in $\mathcal{M}_{n+1}(\mathbb{C})$:

$$\Delta(Df(\zeta), \zeta) := \begin{pmatrix} \zeta \\ \\ Df(\zeta) \end{pmatrix}, \quad \overline{\Delta}(Df(\zeta), \zeta) := \begin{pmatrix} \overline{\zeta} \\ \\ Df(\zeta) \end{pmatrix},$$

where $Df(\zeta)$ has been extended by adding ζ as row (in the case of Δ) or adding the conjugate $\overline{\zeta} := (\overline{\zeta_0}, \ldots, \overline{\zeta_n})$ (in the case of $\overline{\Delta}$). Note that

$$\Delta(Df(\zeta), \zeta) = \Delta(Df(\zeta), \nabla_\zeta \mathcal{Q}),$$

where $\mathcal{Q} := \frac{1}{2}\sum_{i=0}^n X_i^2$.
The following is an elementary exercise of Linear Algebra.

Lemma 3.4. *With the same notations. Let $f \in \mathcal{H}_{(d)}$ be a system of polynomial equations and let $\zeta \in \mathbb{C}^{n+1}\setminus\{0\}$ be any non–zero point in the complex affine cone over $V_{\mathbb{P}}(f)$. Assume that the Jacobian matrix $Df(\zeta)$ is of maximal rank. Then, if $\zeta \notin V_{\mathbb{P}}(\mathcal{Q})$, the matrix $\Delta(Df(\zeta), \zeta)$ is non-singular and satisfies:*

$$|\det(\Delta(Df(\zeta),\zeta))| = \frac{|2\mathcal{Q}(\zeta)|}{\|\zeta\|^2}|\det(\overline{\Delta}(Df(\zeta),\zeta))| = \frac{|2\mathcal{Q}(\zeta)|}{\|\zeta\|}\sqrt{\det(Df(\zeta)Df(\zeta)^*)}.$$

Otherwise, if $\zeta \in V_{\mathbb{P}}(\mathcal{Q})$, $\det(\Delta(Df(\zeta),\zeta)) = 0$ and the equality obviously holds.

Proposition 3.5. *Let $(d) := (d_1, \ldots, d_n)$ be a degree list such that $\sum_{i=1}^n (d_i - 1) \geq 1$. Let $f := (f_1, \ldots, f_n) \in \mathcal{H}_{(d)}$ be a list of polynomials and let $\mathcal{Q} \in \mathbb{C}[X_0, \ldots, X_n]$ be the quadratic form introduced in Equation (3.3) above. Assume that the following properties hold:*

- i) *No point in $V_{\mathbb{P}}(f)$ is isotropic with respect to $\cdot$. Namely, $V_{\mathbb{P}}(f_1, \ldots, f_n) \cap V_{\mathbb{P}}(\mathcal{Q}) = \emptyset$.*
- ii) *The list f is outside the discriminant variety, i.e. $f \notin \Sigma_{(d)}$.*
- iii) *The list $f := (f_1, \ldots, f_n)$ is a generalized Pham system, i.e. no point in $V_{\mathbb{P}}(f_1, \ldots, f_n)$ lies in the infinity hyper-plane $\{X_0 = 0\} \subseteq \mathbb{P}_n(\mathbb{C})$.*
- iv) *The multiplicity of all affine zeros of $V_{\mathbb{A}}(f)$ is one.*

Then, the absolute value $|\operatorname{Disc}_{(d)}(f)| := |\operatorname{Disc}_{(d)}(f_1, \ldots, f_n)|$satisfies:

$$\frac{|\operatorname{Disc}_{(d)}(f)|}{2^{\sharp(V_{\mathbb{A}}(f))-\mathcal{D}_{(d)}}} = |\operatorname{Res}_{(d)}(f_{1,0}, \ldots, f_{n,0})|^{\delta_{(d)}-(n+1)} \prod_{\zeta\in V_{\mathbb{A}}(f)} \frac{|\det(Df(1,\zeta)Df(1,\zeta)^*)|^{1/2}}{(1+\|\zeta\|^2)^{1/2}}, \tag{3.6}$$

where $f_{i,0} := f_i(0, X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]$ is the restriction of the polynomials f_i to the infinity hyper-plane $\{X_0 = 0\} \subseteq \mathbb{P}_n(\mathbb{C})$, $\operatorname{Res}_{(d)}$ is the multivariate resultant in n homogeneous variables $\{X_1, \ldots, X_n\}$, determined by the degree list $(d) = (d_1, \ldots, d_n)$ and $\delta_{(d)} := \sum_{i=1}^n d_i$.

Proof. Let us denote by $\widetilde{R} := \operatorname{Res}_{\widetilde{(d)}}\left(f_1, \ldots, f_n, J(f_1, \ldots, f_n, \mathcal{Q})\right)$ the value of the resultant $\operatorname{Res}_{(d)}$ at the coefficients of the polynomials in the list $\left(f_1, \ldots, f_n, J(f_1, \ldots, f_n, \mathcal{Q})\right)$. Then, according to Poisson Formula (see [19] or [36]), noting that the degree of $J(f_1, \ldots, f_n, \mathcal{Q})$ is $1 + \sum_{i=1}^n (d_i - 1)$, we conclude

$$\widetilde{R} = \operatorname{Res}_{(d)}(f_{1,0}, \ldots, f_{n,0})^{1+\left(\sum_{i=1}^n (d_i-1)\right)} \prod_{\zeta\in V_{\mathbb{A}}(f)} J(f_1, \ldots, f_n, \mathcal{Q})(1,\zeta),$$

and

$$\operatorname{Res}_{\overline{(d)}}(f_1, \ldots, f_n, \mathcal{Q}) = \operatorname{Res}_{(d)}(f_{1,0}, \ldots, f_{n,0})^2 \prod_{\zeta\in V_{\mathbb{A}}(f)} \mathcal{Q}(1,\zeta),$$

where $(f_{i,0} := f_i(0, X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]$ is the restriction of the polynomials f_i to the infinity hyper-plane $\{X_0 = 0\} \subseteq \mathbb{P}_n(\mathbb{C})$, and

$$\widetilde{(d)} := (d_1, \ldots, d_n, 1 + \sum_{i=1}^n (d_i - 1)), \quad \overline{(d)} := (d_1, \ldots, d_n, 2).$$

From our assumption (iii), $V_{\mathbb{P}}(f_{1,0}, \ldots, f_{n,0}) = \emptyset$ and, hence,

$$\operatorname{Res}_{(d)}(f_{1,0}, \ldots, f_{n,0}) \neq 0.$$

Accordingly, assumption (i) implies that $\mathcal{Q}(1,\zeta) \neq 0$, for all $\zeta \in V_{\mathbb{A}}(f)$. Next, from Equation (3.5) of Definition 3.1 and the previous identities, we conclude:

$$2^{\mathcal{D}_{(d)}} \operatorname{Disc}_{(d)}(f) = \operatorname{Res}_{(d)}(f_{1,0}, \ldots, f_{n,0})^{\left(\sum_{i=1}^n (d_i-1)\right)-1} \prod_{\zeta\in V_{\mathbb{A}}(f)} \frac{J(f_1, \ldots, f_n, \mathcal{Q})(1,\zeta)}{\mathcal{Q}(1,\zeta)}.$$

Now, recall that $J(f_1, \ldots, f_n, \mathcal{Q}) := \det\left(\Delta(Df(X), X)\right)$. Hence, from Lemma 3.4 we conclude:

$$\frac{|J(f_1, \ldots, f_n, \mathcal{Q})(1,\zeta)|}{|\mathcal{Q}(1,\zeta)|} = \frac{|\det\left(\Delta(Df(1,\zeta),(1,\zeta))\right)|}{|\mathcal{Q}(1,\zeta)|} = \frac{2|\det(Df(1,\zeta)Df(1,\zeta)^*|^{1/2}}{||(1,\zeta)||}.$$

In other words, we have:

$$\frac{\mathrm{Disc}_{(d)}(f)}{2^{\sharp(V_{\mathbb{A}}(f))-\mathcal{D}_{(d)}}} = \mathrm{Res}_{(d)}(f_{1,0},\ldots,f_{n,0})^{(\sum_{i=1}^{n}(d_i-1))-1}\prod_{\zeta\in V_{\mathbb{A}}(f)}\frac{|\det(Df(1,\zeta)Df(1,\zeta)^*|^{1/2}}{(1+\|\zeta\|^2)^{1/2}},$$

as wanted. □

4. Some basic Integral Geometry

4.1. **Normal Jacobians and the Co-area Formula.** The Co-area Formula is a classic integral formula which generalizes Fubini's Theorem. The most general version we know is Federer's Co-area Formula (cf. [28]), but for our purposes a smooth version as used in [9] and references therein, or [34] suffices.

Definition 4.1. Let X and Y be Riemannian manifolds, and let $F : X \longrightarrow Y$ be a C^1 surjective map. Let $k = \dim(Y)$ be the real dimension of Y. For every point $x \in X$ such that the differential $DF(x)$ is surjective, let $v_1^x,\ldots,v_k^x$ be an orthonormal basis of $Ker(DF(x))^{\perp}$. Then, we define the Normal Jacobian of F at x, NJ_xF, as the volume in the tangent space $T_{F(x)}Y$ of the parallelepiped spanned by $DF(x)(v_1^x),\ldots,DF(x)(v_k^x)$. In the case that $DF(x)$ is not surjective, we define $NJ_xF = 0$.

The following Proposition is easy to prove from this Definition.

Proposition 4.1. *Let X, Y be two Riemannian manifolds, and let $F : X \longrightarrow Y$ be a C^1 map. Let $x_1, x_2 \in X$ be two points. Assume that there exist isometries $\varphi_X : X \longrightarrow X$ and $\varphi_Y : Y \longrightarrow Y$ such that $\varphi_X(x_1) = x_2$, and*

$$F \circ \varphi_X = \varphi_Y \circ F.$$

Then, the following equality holds:

$$NJ_{x_1}F = NJ_{x_2}F.$$

Moreover, if there exists an inverse $G : Y \longrightarrow X$, then

$$NJ_xF = \frac{1}{NJ_{F(x)}G}.$$

A relevant tool to be used in forthcoming pages is the following classical statement of Integral Geometry:

Theorem 4.2 (Co–area Formula)**.** *Consider a surjective C^1 differentiable map $F : X \longrightarrow Y$, where X, Y are Riemannian manifolds of real dimensions $n_1 \geq n_2$. Assume that F is a submersion almost everywhere on X. Consider a measurable function $f : X \longrightarrow \mathbb{R}$, such that f is integrable. Then, for every $y \in Y$ except a zero–measure set, $F^{-1}(y)$ is empty or a real submanifold of X of real dimension $n_1 - n_2$. Moreover, the following equality holds (and the integrals appearing on it are well-defined):*

$$\int_X f NJ_xF \; dX = \int_{y\in Y}\int_{x\in F^{-1}(y)} f(x)\; dF^{-1}(y)dY.$$

The following statement is Lemma 21 of [4].

Lemma 4.3. *Let $\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C})$ be the canonical embedding .Then, the following equality holds for every $z \in \mathbb{C}^n$:*

$$NJ_z\varphi_0 := \frac{1}{(1+\|z\|^2)^{n+1}}.$$

In particular, for every $f \in \mathbb{C}[X_1,\ldots,X_n]$, the following inequality holds:

$$\int_{z\in\mathbb{C}^n}\frac{\log|f(z_1,\ldots,z_n)|}{(1+\|z\|^2)^{n+1}}dz = \int_{x\in\mathbb{P}_n(\mathbb{C})}\log|f(\varphi_0^{-1}(x))|d\nu_{\mathbb{P}}(x),$$

where $d\nu_{\mathbb{P}}$ is the canonical form associated with the Fubini–Study metric in $\mathbb{C}^{n+1}$.

4.2. **An integral equality from** [7]**.** Prior to recall this integral equality, we need to introduce some of the notations used in [7]. For every system of polynomial equations $f \in \mathcal{H}_{(d)}$, we consider the set of solutions of f along the complex sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$. Namely, we introduce:

$$V_S(f) := \{\zeta \in S^{2n+1} \ : \ f(\zeta) = 0\}.$$

Using this spherical zero set, we also consider the cone over the solution variety $V_{(d)}$, defined as follows:

$$\widetilde{V}_{(d)} := \{(f, \zeta) \in \mathcal{H}_{(d)} \setminus \{0\} \times S^{2n+1} \ : \ f(\zeta) = 0\} \subseteq \mathcal{H}_{(d)} \times S^{2n+1}.$$

For $\zeta \in \mathbb{P}_n(\mathbb{C})$ we consider the vector subspaces of $\mathcal{H}_{(d)}$,

$$R_\zeta := \{h \in \mathcal{H}_{(d)} \ : \ h(\zeta) = 0, \ Dh(\zeta) = 0\}, \ \ L_\zeta := (R_\zeta)^\perp,$$

where $^\perp$ here denotes orthogonal complement with respect to Bombieri-Weyl Hermitian product.
The structures of R_ζ and L_ζ are better understood if we first fix $\zeta := e_0 := (1, 0, \ldots, 0)^t$. Indeed, R_{e_0} is the set of polynomial systems $h := (h_1, \ldots, h_n) \in \mathcal{H}_{(d)}$ such that $h(e_0) = 0$ and $Dh(e_0) = 0$, namely

$$h_i(X) = X_0^{d_i-2} p_{d_i-2}(X_1, \ldots, X_n) + \cdots + X_0 p_1(X_1, \ldots, X_n) + p_0(X_1, \ldots, X_n),$$

for some homogeneous polynomials $p_j, 0 \leq j \leq d_i - 2$. Thus, a polynomial system h is in R_{e_0} if all the coefficients of the monomials containing $X_0^{d_i}$ and $X_0^{d_i-1}$ are zero. Reciprocally, a polynomial system h is in L_{e_0} if all the non zero monomials contain $X_0^{d_i}$ or $X_0^{d_i-1}$. Note that for such a $h \in L_{e_0}$ we have that $h(1, X_1, \ldots, X_n)$ defines a linear function of $X_1, \ldots, X_n$. Thus, for any $h \in \mathcal{H}_{(d)}$ we can think on the orthogonal projection of h onto L_ζ as the "linear part" of h with respect to e_0.
Now, let $\zeta \in S^{2n+1}$ and consider a $(n+1) \times (n+1)$ unitary matrix U such that $Ue_0 = \zeta$. Then, by the unitary invariance of the Bombieri–Weyl product in $\mathcal{H}_{(d)}$ we have

$$R_\zeta = \{h \circ U^* : h \in R_{e_0}\}, \quad L_\zeta = \{h \circ U^* : h \in L_{e_0}\}.$$

For every matrix $M \in \mathcal{H}_{(1)}$, we denote by $V_S(M) \subseteq S^{2n+1}$ the intersection of its kernel with the complex unit sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$. Namely, $V_S(M) = \{\zeta \in S^{2n+1} \ : \ M\zeta = 0\}$. Let $\varphi(M, \zeta) \in L_\zeta$ be the system of equations defined by

$$\varphi(M, \zeta)(z) := \mathrm{Diag}(\langle z, \zeta \rangle^{d_i - 1} d_i^{1/2}) Mz \in \mathcal{H}_{(d)}. \tag{4.1}$$

The following equalities are claimed in [7]:

$$\|\varphi(M, \zeta)\|_\Delta = \|M\|_F, \tag{4.2}$$

where $\| \cdot \|_F$ is the usual Frobenius norm on $\mathcal{H}_{(1)}$, and

$$D(\varphi(M, \zeta))(\zeta) = Diag(d_i^{1/2})M. \tag{4.3}$$

Again, these formulas become clearer if we first fix $\zeta := e_0$. Then, $M = (0 \mid A)$ where A is a square matrix of size n. Let a_{ij}, $1 \leq i, j \leq n$ be the entries of A, and let $\varphi(M, \zeta) := (f_1, \ldots, f_n)$. Then,

$$f_i(X_0, \ldots, X_n) = d_i^{1/2} X_0^{d_i - 1} \sum_{1 \leq j \leq n} a_{ij} X_j.$$

Theorem 4.4 ([7])**.** *Let $\widetilde{\Theta} : \widetilde{V}_{(d)} \longrightarrow [0, \infty]$ be a measurable mapping. Then, the following equality holds:*

$$\int_{f \in \mathcal{H}_{(d)}} \int_{\zeta \in V_S(f)} \widetilde{\Theta}(f, \zeta) dV_S(f) d\mathcal{H}_{(d)} =$$

$$= \mathcal{D}_{(d)} \int_{M \in \mathcal{H}_{(1)}} \int_{\zeta \in V_S(M)} \int_{h \in R_\zeta} \widetilde{\Theta}(h + \varphi(M, \zeta), \zeta) dR_\zeta dV_S(M) d\mathcal{H}_{(1)}.$$

5. Computing some zeta Mahler function

In this Section we compute some of the zeta Mahler functions defined at the Introduction.

5.1. **Zeta Mahler measure function of the norm of the affine solutions.** Let $(d) := (d_1, \ldots, d_n)$ be a degree list. We may introduce a variation of the solution variety $V_{(d)}$ introduced in previous pages. We define the incidence variety $\mathcal{V}_{(d)} \subseteq \mathfrak{S}_{(d)} \times \mathbb{P}_n(\mathbb{C})$ in the following terms:

$$\mathcal{V}_{(d)} := \{(f, \zeta) \in \mathfrak{S}_{(d)} \times \mathbb{P}_n(\mathbb{C}) \ : \ f_i(\zeta) = 0, \ 1 \leq i \leq n, \ f := (f_1, \ldots, f_n)\}. \tag{5.1}$$

As in previous pages we may also consider two canonical projections:

- $\pi_1 : \mathcal{V}_{(d)} \longrightarrow \mathfrak{S}_{(d)}, \ \pi_1(f, \zeta) := f, \ \forall (f, \zeta) \in \mathcal{V}_{(d)}$.
- $\pi_2 : \mathcal{V}_{(d)} \longrightarrow \mathbb{P}_n(\mathbb{C}), \ \pi_2(f, \zeta) := \zeta, \ \forall (f, \zeta) \in \mathcal{V}_{(d)}$.

From the unitary invariance of the Bombieri-Weyl metric, there is an isometric action of the unitary group $\mathcal{U}(n+1)$ on $\mathcal{V}_{(d)}$ given in the following terms:

$$\begin{array}{ccc} \mathcal{U}(n+1) \times \mathcal{V}_{(d)} & \longrightarrow & \mathcal{V}_{(d)} \\ (U, (f, \zeta)) & \longmapsto & (f \circ U^*, U\zeta). \end{array} \tag{5.2}$$

The following *double fibration argument* is a well-known statement with several formulations. We include here the version of [29].

Proposition 5.1. *Let $g : \mathbb{P}_n(\mathbb{C}) \longrightarrow \mathbb{R}_+$ be an integrable function. Then, the following equality holds:*

$$\int_{f \in \mathfrak{S}_{(d)}} \left(\sum_{\zeta \in V_{\mathbb{P}}(f)} g(\zeta) \right) d\nu_{\mathfrak{S}}(f) = \frac{\mathcal{D}_{(d)} \nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}{\nu_{\mathbb{P}}[\mathbb{P}_n(\mathbb{C})]} \int_{z \in \mathbb{P}_n(\mathbb{C})} g(z) d\nu_{\mathbb{P}}(z),$$

where $d\nu_{\mathbb{P}}$ is the differential form associated to the canonical Fubini-Study metric in $\mathbb{P}_n(\mathbb{C})$.

Corollary 5.2 (cf. [29])**.** *With the previous notations and assumptions, this Proposition may be rephrased:*

$$E_{\mathfrak{S}_{(d)}} \left[\sum_{\zeta \in V_{\mathbb{P}}(f)} g(\zeta) \right] = \mathcal{D}_{(d)} E_{\mathbb{P}_n(\mathbb{C})} \left[g \right].$$

As in Identity (1.1) of the Introduction, we denote by $I_X[|g|]$ the following integral:

$$I_X[|g|] = \int_X |g(x)| d\mu(x) s.$$

Corollary 5.3. *With the previous notations, the following equalities hold:*

- *Let $X := V_{(d)} \subseteq \mathbb{P}(\mathcal{H}_{(d)}) \times \mathbb{P}_n(\mathbb{C})$ be the solution variety introduced in Subsection 3.4 and $\pi_1 : V_{(d)} \longrightarrow \mathbb{P}(\mathcal{H}_{(d)})$ the canonical projection associated to this case. Then, $I_X[|NJ\pi_1|]$ satisfies:*
$$I_{V_{(d)}}[|NJ\pi_1|] = \mathcal{D}_{(d)} \nu_{\mathbb{P}}[\mathbb{P}(\mathcal{H}_{(d)})].$$
- *Similarly, let $X := \mathcal{V}_{(d)} \subseteq \mathfrak{S}_{(d)} \times \mathbb{P}_n(\mathbb{C})$ be the solution variety introduced above in this Subsection and $\pi_1 : \mathcal{V}_{(d)} \longrightarrow \mathfrak{S}_{(d)}$ the corresponding canonical projection. Then, we have*
$$I_{\mathcal{V}_{(d)}}[|NJ\pi_1|] = \mathcal{D}_{(d)} \nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}].$$

Proof. We just prove the first one. The second one having the same proof. We recall that

$$I_{V_d}[|NJ\pi_1|] := \int_{(f,\zeta) \in V_{(d)}} |NJ_{(f,\zeta)} \pi_1| \ dV_{(d)}(f, \zeta).$$

Using the Co-area Formula, the following equality holds for every function $\varphi : V_{(d)} \longrightarrow \mathbb{R}$:

$$\int_{f \in \mathbb{P}(\mathcal{H}_{(d)})} \int_{\zeta \in V_{\mathbb{P}}(f)} \varphi(f, \zeta) \ \ dV_{\mathbb{P}}(f)(\zeta) d\nu_{\mathbb{P}}(f) = \int_{(f,\zeta) \in V_{(d)}} \varphi(f, \zeta) NJ_{(f,\zeta)} \pi_1 \ dV_{(d)}(f, \zeta).$$

Hence, taking $\varphi \equiv 1$ we conclude that

$$\int_{(f,\zeta) \in V_{(d)}} 1 NJ_{(f,\zeta)} \pi_1 \ dV_{(d)}(f, \zeta) = I_{V_{(d)}}[NJ\pi_1],$$

and generically for $f \in \mathbb{P}(\mathcal{H}_{(d)})$ the following equality holds:

$$\int_{\zeta \in V_{\mathbb{P}}(f)} 1 \, dV_{\mathbb{P}}(f)(\zeta) = \sharp V_{\mathbb{P}}(f) = \mathcal{D}_{(d)},$$

where $\sharp V_{\mathbb{P}}(f)$ is the number of points in $V_{\mathbb{P}}(f)$. Finally, we have

$$I_{V_{(d)}}[NJ\pi_1] = \int_{\mathbb{P}(\mathcal{H}_{(d)})} \mathcal{D}_{(d)} \, d\nu_{\mathbb{P}}(f) = \mathcal{D}_{(d)} \nu_{\mathbb{P}}[\mathbb{P}(\mathcal{H}_{(d)})]$$

□

Now we are in conditions to prove Proposition 1.2 as stated in the Introduction.

5.1.1. *Proof of Proposition 1.2.*

Proof. We proceed as in the proof of Lemma 4.6 of [6]. From the results stated in Subsection 2 it is enough to prove that the integral involved in $\mathcal{Z}(t, 1/\|\cdot\|_{\rm aff})$ is finite for every real number t, such that $t > -2$. Moreover, as both sides of the main equality are holomorphic functions in the complex region G, it is enough to prove that the equality holds for every real number $t > -2$. Now, observe that $\mathcal{Z}(t, 1/\|\cdot\|_{\rm aff})$ is the following expectation:

$$\mathcal{Z}(t, 1/\|\cdot\|_{\rm aff}) = E_{\mathfrak{S}_{(d)}}\left[\frac{1}{\mathcal{D}_{(d)}} \sum_{\zeta \in V_{\mathbb{A}}(f)} (1 + \|\zeta\|^2)^{-\frac{t}{2}}\right].$$

Now, from Proposition 5.1 we conclude:

$$\mathcal{Z}(t, 1/\|\cdot\|_{\rm aff}) = E_{x \in \mathbb{P}_n(\mathbb{C})}[(1 + \|\varphi_0^{-1}(x)\|^2)^{-\frac{t}{2}}],$$

where $\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C})$ is the canonical embedding of the affine space $\mathbb{C}^n$ into the complex projective space.

Now, as in the proof of Lemma 4.6 of [6], we use Lemma 21 of [4] (cf. Lemma 4.3 above). Then, we have

$$\mathcal{Z}(t, 1/\|\cdot\|_{\rm aff}) = E_{\mathbb{P}_n(\mathbb{C})}[(1 + \|\varphi_0^{-1}(x)\|^2)^{-\frac{t}{2}}] = \frac{1}{\nu_{\mathbb{P}}[\mathbb{P}_n(\mathbb{C})]} \int_{\mathbb{C}^n} \frac{(1 + \|x\|^2)^{-\frac{t}{2}}}{(1 + \|x\|^2)^{n+1}} \, dx.$$

Using spherical coordinates, we conclude:

$$\mathcal{Z}(t, 1/\|\cdot\|_{\rm aff}) = \frac{1}{\nu_{\mathbb{P}}[\mathbb{P}_n(\mathbb{C})]} \int_{S^{2n-1}} \int_0^\infty \frac{(1 + r^2)^{-\frac{t}{2}} r^{2n-1}}{(1 + r^2)^{n+1}} \, dr.$$

Hence,

$$\mathcal{Z}(t, 1/\|\cdot\|_{\rm aff}) = \frac{\nu_S[S^{2n-1}]}{\nu_{\mathbb{P}}[\mathbb{P}_n(\mathbb{C})]} \int_0^\infty \frac{(1 + r^2)^{-\frac{t}{2}} r^{2n-1}}{(1 + r^2)^{n+1}} \, dr.$$

Replacing $s := r^2$, we conclude:

$$\mathcal{Z}(t, 1/\|\cdot\|_{\rm aff}) = \frac{\nu_S[S^{2n-1}]}{2\nu_{\mathbb{P}}[\mathbb{P}_n(\mathbb{C})]} \int_0^\infty \frac{s^{n-1}}{(1 + s)^{(n+1) + \frac{t}{2}}} \, ds = \frac{\nu_S[S^{2n-1}]}{2\nu_{\mathbb{P}}[\mathbb{P}_n(\mathbb{C})]} \mathrm{B}(n, 1 + \tfrac{t}{2}),$$

Now, replacing $\nu_S[S^{2n-1}]$ and $\nu_{\mathbb{P}}(\mathbb{P}_n(\mathbb{C})]$ by their values, we get:

$$\mathcal{Z}(t, 1/\|\cdot\|_{\rm aff}) = n\mathrm{B}(n, 1 + \tfrac{t}{2}) = \frac{n\Gamma(n)\Gamma(1 + \frac{t}{2})}{\Gamma\big((n+1) + \frac{t}{2}\big)} = \frac{\Gamma(n+1)\Gamma(1 + \frac{t}{2})}{\Gamma\big((n+1) + \frac{t}{2}\big)}.$$

□

Corollary 5.4. *With the same notations as above, the following holds:*

$$E_{\mathfrak{S}_{(d)}}\left[\frac{1}{\mathcal{D}_{(d)}} \sum_{\zeta \in V_{\mathbb{A}}({}^a f)} \log(1 + \|\zeta\|^2)\right] = \psi(n+1) - \psi(1) = H_n,$$

where H_n is the $n-$th harmonic number and ψ is the digamma function.

Proof. Now, observe that

$$-\left.\frac{d\mathcal{Z}(t, 1/\|\cdot\|_{\text{aff}})}{dt}\right|_{t=0} = E_{\mathfrak{S}_{(d)}}\left[\frac{1}{\mathcal{D}_{(d)}}\sum_{\zeta\in V_{\mathbb{A}}(f)}\frac{1}{2}\log(1+\|\zeta\|^2)\right]. \tag{5.3}$$

The derivative of $\mathcal{Z}(t, 1/\|\cdot\|_{\text{aff}})$ at $t=0$ satisfies

$$\left.\frac{d\mathcal{Z}(t, 1/\|\cdot\|_{\text{aff}})}{dt}\right|_{t=0} = \frac{1}{\Gamma\big((n+1)+\frac{t}{2}\big)^2}[C_1(t)-C_2(t)]\Big|_{t=0},$$

where

$$C_1(t) := \Gamma(n+1)\Gamma'(1+\tfrac{t}{2})(\tfrac{1}{2})\Gamma\big((n+1)+\tfrac{t}{2}\big),$$
$$C_2(t) := \Gamma(n+1)\Gamma(1+\tfrac{t}{2})\Gamma'\big((n+1)+\tfrac{t}{2}\big)(\tfrac{1}{2}).$$

Thus,

$$-\left.\frac{d\mathcal{Z}(t, 1/\|\cdot\|_{\text{aff}})}{dt}\right|_{t=0} = \frac{1}{2}\left[\Gamma(1)\cdot\frac{\Gamma'(n+1)}{\Gamma(n+1)} - \frac{\Gamma'(1)}{\Gamma(1)}\cdot\frac{\Gamma(n+1)}{\Gamma(n+1)}\right] = \frac{1}{2}[\psi(n+1)-\psi(1)],$$

where ψ is the digamma function that satisfies $\psi(n) = H_{n-1}-\gamma$, and γ is Euler-Mascheroni constant. The Corollary obviously follows from Equality (5.3). □

5.2. **An excursus on zeta Mahler measure functions and condition numbers.** This is just to explain how zeta Mahler measure functions may be viewed as potential instruments in the knowledge of condition numbers. As it is not the goal of these pages, we just keep it as a short excursus into the subject.

For a $n\times(n+1)$ complex matrix $A\in\mathcal{M}_{n\times(n+1)}(\mathbb{C}) = \mathcal{H}_{(1)}$, Demmel's condition number can be defined by

$$\mu(A) := \|A\|_F\|A^\dagger\|_2,$$

where $\|\cdot\|_F$ denotes Frobenius norm and $\|A^\dagger\|_2$ is the norm as operator of the Moore-Penrose pseudo-inverse $A^\dagger$ of A. One may define the zeta Mahler measure function of the inverse of this condition number by the following identity:

$$\mathcal{Z}(t, 1/\mu) := \mathcal{Z}_{\mathbb{S}(\mathcal{M}_{n\times(n+1)}(\mathbb{C}))}(t, 1/\mu) := \frac{1}{\nu_{\mathbb{S}}[\mathbb{S}(\mathcal{M}_{n\times(n+1)}(\mathbb{C}))]}\int_{\mathbb{S}(\mathcal{M}_{n\times(n+1)}(\mathbb{C}))}\mu^{-t}(A)d\nu_{\mathbb{S}}(A),$$

where $\mathbb{S}(\mathcal{M}_{n\times(n+1)}(\mathbb{C}))$ is the sphere of radius one centered at the origin in $\mathcal{M}_{n\times(n+1)}(\mathbb{C})$ with respect to the Frobenius norm.

Corollary 5.5. *With these notations, let $G := \{t\in\mathbb{C} : \Re(t) > -4\}$. Then, the zeta Mahler measure function $\mathcal{Z}(t, 1/\mu)$ is well-defined and homolorphic in G. Moreover, for every $t\in G$ the following equality holds:*

$$\mathcal{Z}(t, 1/\mu) = \frac{\Gamma(n^2+n)}{\Gamma(n^2+n+\frac{t}{2})}\sum_{k=0}^{n-1}\frac{\binom{n+1}{k}\Gamma(n-k+1+\frac{t}{2})}{n^{n-k+1+\frac{t}{2}}\Gamma(n-k)}. \tag{5.4}$$

In particular, $\mathcal{Z}(t, 1/\mu)$ admits analytic continuation to the complex domain $\mathbb{C}\setminus\{z\in\mathbb{Z} : z\le -4\}$.

Proof. As $\mathcal{M}_{n\times(n+1)}(\mathbb{C}) = \mathcal{H}_{(1)}$, from the main outcome of [56], we know that there is a constant $C(n)$ such that

$$\frac{1}{\nu_{\mathbb{S}}[\mathbb{S}(\mathcal{M}_{n\times(n+1)}(\mathbb{C}))]}\nu_{\mathbb{S}}[A\in\mathbb{S}(\mathcal{M}_{n\times(n+1)}(\mathbb{C})) : \mu(A) > 1/\varepsilon] \le C(n)\varepsilon^4.$$

Then, as it is well-known $\nu_{\mathbb{S}}[\{A\in M_{n\times n+1}(\mathbb{C}) : 1/\mu(A) = 0\}] = 0$ we apply Lemma 2.1 above and we immediately conclude that $\mathcal{Z}(t, 1/\mu)$ is a well-defined holomorphic function over G. Next, as both functions on the two sides of the Equality (5.4) are holomorphic, the Identity Principle for univariate holomorphic functions implies that if these two functions agree on a subset $A\subseteq G$ having accumulation points, then they are equal as functions defined on the connected set G. But, from Theorem 19 of [7], we know that these two functions agree for real values of t in the open real interval $t\in(-4,0)$. Then, they agree in G and the main claim of this Corollary follows. Additionally, the holomorphic function on the right hand side of Equation (5.4) admits analytic continuation to the complex domain $\mathbb{C} \setminus \{z \in \mathbb{Z} \, : \, z \leq -4\}$ and the last claim also follows. □

We do the same kind of study of the non-linear condition number μ_{norm} introduced in [55]. For a degree list (d) and for every list of polynomials $f \in \mathcal{H}_{(d)}$ and for every point $\zeta \in V_{\mathbb{P}}(f)$, we define the normalized condition number $\mu_{\text{norm}}(f, \zeta)$ by the following identity:

$$\mu_{\text{norm}}(f, \zeta) := \|f\|_\Delta \|\text{Diag}(d_i^{-1/2}) Df(z)^\dagger\|,$$

where $z \in S^{2n+1} \subseteq \mathbb{C}^{n+1}$ is any representant of ζ in the unit sphere S^{2n+1}, $|| \cdot ||_\Delta$ is Bombieri-Weyl's norm and $\text{Diag}(d_i^{-1/2})$ is the diagonal matrix whose diagonal entries are $d_1^{-1/2}, \ldots, d_n^{-1/2}$. We then define the zeta Mahler measure function associated to the inverse of the normalized condition number $1/\mu_{\text{norm}}$ as follows:

$$\mathcal{Z}(t, {}^1/\mu_{\text{norm}}) := \mathcal{Z}_{\mathbb{S}(\mathcal{H}_{(d)})}(t, {}^1/\mu_{\text{norm}}) := \frac{1}{\nu_{\mathbb{S}}[\mathbb{S}(\mathcal{H}_{(d)})]} \int_{f \in \mathbb{S}(\mathcal{H}_{(d)})} \left(\frac{1}{\mathcal{D}_{(d)}} \sum_{\zeta \in V_{\mathbb{P}}(f)} \mu_{\text{norm}}(f, \zeta)^{-t} \right) d\nu_{\mathbb{S}}(f).$$

We prove Proposition 1.3 as stated at the Introduction.

5.2.1. *Proof of Proposition 1.3.*

Proof. It's well-known that $\mu[\{f \in \mathcal{H}_{(d)} \, : \, {}^1/\mu_{\text{norm}}(f) = 0\}] = 0$ so Lemma 2.1 above combined with the main outcome of [56] imply that $\mathcal{Z}(t, {}^1/\mu_{\text{norm}})$ is well-defined and holomorphic in G. As for the equality, Theorem 23 of [7] claims that these two holomorphic functions agree on the real interval $(-4, 0) \subseteq G$. Hence, applying again the Identity Principle for univariate holomorphic functions, we obtain the equality for all $t \in G$. The last claim of the Corollary is, again, a consequence of the impossibility of the existence of analytic continuations of $\Gamma(2 + t/2)$ (i.e. $k = n - 1$ in Equation (1.6)). □

5.3. **Zeta Mahler function of the determinant of a complex Wishart matrix.** This manuscript owes much of its inspiration to a seminal idea developed in a series of three manuscripts of 1963, written by N.R. Goodman (mostly to [31], [32]). In these manuscripts Goodman computes, among other things, the characteristic function of the logarithm of a complex Wishart Matrix. Since then, revisions and analysis of the results in [32] have been spread along the literature and its academic trace may be followed through many authors, references and applications as [54], [44], [33], [2], [45] and many others. It is not our purpose here to make a survey of the many applications of those ideas from N.R. Goodman. His results obviously inspired our notions of zeta Mahler measure functions. Here, we just want to rewrite his main outcomes in the same language as the one we used in the previous subsection.
First of all, let us denote by (1) the list of degrees $(1) := (1, \ldots, 1) \in \mathbb{N}^n$ and let us denote by $\mathcal{H}_{(1)} := \mathcal{H}_{(1)}(X_0, \ldots, X_n)$. This is the space of $n \times (n + 1)$ complex matrices. Namely,

$$\mathcal{H}_{(1)} = \mathcal{M}_{n \times (n+1)}(\mathbb{C}).$$

In this case Bombieri-Weyl Hermitian product agrees with the usual Frobenius Hermitian product. Namely, for $A, B \in \mathcal{H}_{(1)}$, Frobenius Hermitian product is given by:

$$\langle A, B \rangle_F := \text{Tr}\left(AB^*\right),$$

where Tr denotes trace and * denotes conjugate transpose. Accordingly, Frobenius norm is denoted by $\|A\|_F := \left(\text{Tr}\left(AA^*\right)\right)^{1/2}$. Next, let us consider the product of complex spheres $\prod_{i=1}^n S^{2n+1} \subseteq \mathcal{H}_{(1)}$ as the matrices $X \in \mathcal{H}_{(1)}$ whose rows are vectors in the sphere $S^{2n+1} := \{z \in \mathbb{C}^{n+1} \, : \, \|z\| = 1\}$, i.e.

$$X := \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \quad \theta_i \in S^{2n+1}.$$

We then consider the following zeta Mahler measure function:

$$\mathcal{Z}(t, \mathrm{DET}) := \mathcal{Z}_{\prod_{i=1}^n S^{2n+1}}(t, \mathrm{DET}) := \frac{1}{\nu_S[\prod_{i=1}^n S^{2n+1}]} \int_{X \in \prod_{i=1}^n S^{2n+1}} |\det(XX^*)|^t dX, \tag{5.5}$$

where $t \in \mathbb{C}$ is a complex number.
Let us denote by $\Omega_n \subseteq \mathcal{M}_n(\mathbb{C})$ the set of all Hermitian semi-definite positive complex matrices, endowed with the Borel-Lebesgue measure. Let $W : \mathcal{H}_{(1)} \longrightarrow \Omega_n$ be the Wishart matrix transformation:

$$W(M) := MM^* \in \Omega_n, \forall M \in \mathcal{H}_{(1)}.$$

Assume that $\mathcal{H}_{(1)}$ is endowed with the Gaussian distribution $N(0, I_{n(n+1)})$ that we denote by γ. Let $W_*\gamma$ be the pushforward probability distribution on Ω_n induced by W and the Gaussian distribution in $\mathcal{H}_{(1)}$. Then, [31] (cf. also [33] and other references) proves the following statement:

Theorem 5.6 ([31])**.** *With these notations, the probability distribution $W_*\gamma$ on Ω_n has a probability density function f_W given by the following equality:*

$$f_W(P) := \frac{1}{\pi^{\frac{1}{2}n(n-1)} \prod_{j=2}^{n+1} \Gamma(j)} \det(P) e^{-Tr(P)},$$

where $\det(\cdot)$ is the determinant and $\mathrm{Tr}(\cdot)$ is the trace of the matrix.

In terms of integral identities, denoting by μ the usual Lebesgue measure of any space $\mathbb{C}^n$, the following is an equivalent form of presenting the previous Theorem:

$$\frac{1}{\pi^{n(n+1)}} \int_{\mathcal{H}_{(1)}} f(W(M)) e^{-||M||_F^2} d\mu = \int_{\Omega_n} \frac{f(P)}{\pi^{\frac{1}{2}n(n-1)} \prod_{j=2}^{n+1} \Gamma(j)} \det(P) e^{-Tr(P)} d\mu,$$

where $f : \Omega_n \longrightarrow \mathbb{R}$ is any integrable function. The following is an immediate consequence of the main outcomes of N. R. Goodman and it has been observed in [33], Identity (1.2):

Theorem 5.7 (cf. [32], [33], [54])**.** *Let $t \in \mathbb{R}$ be a real number and assume that $t > -2$. Then, the following equality holds:*

$$\int_{\mathcal{H}_{(1)}} \frac{|\det(XX^*)|^t e^{-||X||_F^2}}{\pi^{n(n+1)}} d\mu(X) = \int_{\Omega_n} \frac{\det(P)^{t+1} e^{-Tr(P)}}{\pi^{\frac{1}{2}n(n-1)} \prod_{j=2}^{n+1} \Gamma(j)} d\mu = \prod_{j=2}^{n+1} \frac{\Gamma(j+t)}{\Gamma(j)}.$$

where μ is the Lebesgue measure in $\mathcal{H}_{(1)}$.

From this equality we easily conclude the following one, just using integration in spherical coordinates:

Corollary 5.8. *With the same notations as above, let G be the complex domain given by $G := \{t \in \mathbb{C} : \Re(t) > -2\}$. Then, the $\mathcal{Z}(t, \mathrm{DET})$ is a well-defined and holomorphic function for $t \in G$. Moreover, the following equality holds for every $t \in G$:*

$$\mathcal{Z}(t, \mathrm{DET}) = \left(\frac{\Gamma(n+1)}{\Gamma(t+n+1)} \right)^n \left(\prod_{j=2}^{n+1} \frac{\Gamma(j+t)}{\Gamma(j)} \right), \tag{5.6}$$

In particular, $\mathcal{Z}(t, \mathrm{DET})$ admits analytic continuation to the complex domain $\mathbb{C} \setminus \{z \in \mathbb{Z} : z \leq -2\}$. Additionally, the derivatives satisfy:

$$\frac{d^k \mathcal{Z}(t, \mathrm{DET})}{dt^k} = \frac{1}{\nu_S[\prod_{i=1}^n S^{2n+1}]} \int_{X \in \prod_{i=1}^n S^{2n+1}} |\det(XX^*)|^t \log^k (\det(XX^*))\, d\nu_S(X).$$

Proof. Although it is an almost obvious consequence of the works by N.R. Goodman, we include a proof for completeness. We first prove that the Equality (5.6) holds for every real number $t \in \mathbb{R}$, with $t > -2$. Then, applying Lemma 2.2 we know that the hypothesis of Lemma 2.1 holds and, then, the first claim of the Corollary holds. Then, $\mathcal{Z}(t, \mathrm{DET})$ is holomorphic in G. Since G is connected, if Equality (5.6) holds for every real number $t \in G$, then the Identity Principle of holomorphic functions imply that this equality also holds for every complex point $t \in G$ and the remaining claims also hold.

According to Theorem 5.7, we have for $t \in \mathbb{R}$, $t > -2$:

$$M(t) := \int_{\mathcal{H}_{(1)}} |\det(XX^*)|^t e^{-||X||_F^2} d\mu(X) = \pi^{n(n+1)} \prod_{j=2}^{n+1} \frac{\Gamma(j+t)}{\Gamma(j)}.$$

We proceed by combining Tonelli-Fubini's Theorem with integration in spherical coordinates (cf. [9], Ex. 8, p. 206, for instance) to conclude:

$$M(t) = \int_0^\infty \cdots \int_0^\infty \int_{\underline{\Theta} \in \prod_{i=1}^n S^{2n+1}} |\det\left(\underline{r\Theta} \cdot \underline{r\Theta}^*\right)|^t e^{-\sum_{i=1}^n r_i^2} \prod_{i=1}^n r_i^{2n+1} d\underline{\Theta} d\underline{r},$$

where

$$\underline{r} := (r_1, \ldots, r_n) \in [0,\infty)^n, \quad \underline{\Theta} := \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} \in \prod_{i=1}^n S^{2n+1},$$

$$d\underline{r} := dr_1 \cdots dr_n, \quad d\underline{\Theta} := d\theta_1 \cdots d\theta_n,$$

$$\underline{r\Theta} := \begin{pmatrix} r_1\theta_1 \\ \vdots \\ r_n\theta_n \end{pmatrix}.$$

As

$$\det\left(\underline{r\Theta r\Theta}^*\right) = \left(\prod_{i=1}^n r_i^2\right) \det(\underline{\Theta\Theta}^*), \tag{5.7}$$

we conclude:

$$M(t) = \prod_{i=1}^n \left(\int_0^\infty r_i^{2t+2n+1} e^{-r_i^2} dr_i\right) \left(\int_{X \in \prod_{i=1}^n S^{2n+1}} |\det(XX^*)|^t dX\right).$$

We thus conclude:

$$\int_{X \in \prod_{i=1}^n S^{2n+1}} |\det(XX^*)|^t dX = \frac{2^n M(t)}{\Gamma(t+n+1)^n},$$

and, hence,

$$\int_{X \in \prod_{i=1}^n S^{2n+1}} |\det(XX^*)|^t dX = 2^n \pi^{n(n+1)} \prod_{j=2}^{n+1} \frac{\Gamma(j+t)}{\Gamma(j)\Gamma(t+n+1)}.$$

Then, as $\nu_S[S^{2n+1}] = \frac{2\pi^{n+1}}{\Gamma(n+1)}$, dividing by $\prod_{i=1}^n \nu_S[S^{2n+1}]$ the equality follows. □

6. The zeta Mahler measure function of the Jacobian determinant: Proof of Theorem 1.4

In this Section we prove Theorem 1.4 of the Introduction. As in the proof of Corollary 5.8, we first prove that Identity (1.7) of Theorem 1.4 holds for every real number $t \in \mathbb{R}$, $t > -4$. Then, Lemmata 2.2 and 2.1 imply that $\mathcal{Z}(t, \mathrm{JAC})$ is a well-defined and holomorphic function in G. Finally, the Identity Principle for holomorphic functions would imply that Identity (1.7) holds for every complex number $t \in G$. The dependence on the gamma function will imply that $\mathcal{Z}(t, \mathrm{JAC})$ admits analytic continuation to the complex domain $\mathbb{C} \setminus \{z \in \mathbb{Z} \ : \ z \leq -4\}$ and the proof of the Theorem would be finished. We follow the same strategy as the one used in Corollary 18 of [7] to prove Identity (1.7) for real numbers $t \in \mathbb{R}$, such that $t > -4$. As in the Introduction, we consider the Jacobian determinant

$$\mathrm{JAC}(f, \zeta) := \det(T_\zeta f),$$

for every $f \in \mathfrak{S}_{(d)}$ and $\zeta \in V_S(f)$. Under the same hypothesis the following equality holds:

$$|\mathrm{JAC}(f,\zeta)| = |\det(T_\zeta f)| = |\det(Df(\zeta)Df(\zeta)^*)|^{\frac{1}{2}} = |\mathrm{WJAC}(f,\zeta)|^{\frac{1}{2}},$$

where $\mathrm{WJAC}(f,\zeta) := |\det(Df(\zeta)Df(\zeta)^*)|$ is the Wishart matrix of the Jacobian for every $f \in \mathcal{H}_{(d)}$ and for every $\zeta \in V_S(f)$. Let us introduce an auxiliary function $\mathcal{Z}(t, \mathrm{WJAC})$ as follows:

$$\mathcal{Z}(t,\mathrm{WJAC}) := \frac{1}{\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}\int_{f\in\mathfrak{S}_{(d)}}\left(\frac{1}{\mathcal{D}_{(d)}}\int_{\zeta\in V_{\mathbb{P}}(f)}|\mathrm{WJAC}(f,\zeta)|^t\, dV_{\mathbb{P}}(f)(\zeta)\right)d\nu_{\mathfrak{S}}(f).$$

Next, observe that for every real number $t > -4$, we have

$$\mathcal{Z}(t,\mathrm{JAC}) = \mathcal{Z}(t/2,\mathrm{WJAC}). \tag{6.1}$$

We thus proceed by computing the value of $\mathcal{Z}(t, \mathrm{WJAC})$ for every real number $t \in \mathbb{R}$, $t > -2$. Using Corollary 5.3, we have that $I_{\mathcal{V}_{(d)}}[|NJ\pi_1|] = \mathcal{D}_{(d)}\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]$ and, hence, we conclude that

(6.2)
$$\mathcal{Z}(t,\mathrm{WJAC}) = \frac{1}{I_{\mathcal{V}_{(d)}}[|NJ\pi_1|]}\int_{(f,\zeta)\in\mathcal{V}_{(d)}}|\mathrm{WJAC}(f,\zeta)|^t NJ\pi_1(f,\zeta)d\mathcal{V}_{(d)}(f,\zeta) = \mathcal{Z}_{\mathcal{V}_{(d)}}(t,\mathrm{WJAC},NJ\pi_1),$$

and both $\mathcal{Z}(t, \mathrm{JAC})$ and $\mathcal{Z}(t, \mathrm{WJAC})$ fall into the scope of Lemma 2.1.
Note that for every $f := (f_1, \ldots, f_n) \in \mathcal{H}_{(d)}$ such that $f_i \neq 0$, for all i, $1 \leq i \leq n$, and for every $\zeta \in V_S(f)$ the following equality holds :

$$|\mathrm{WJAC}(f,\zeta)| := |\det\left(Df(\zeta)Df(\zeta)^*\right)| = \left(\prod_{i=1}^n \|f_i\|_{d_i}^2\right)|\det(D\widetilde{f}(\zeta)D\widetilde{f}(\zeta)^*)|, \tag{6.3}$$

where

$$\widetilde{f} := \left(\frac{f_1}{\|f_1\|_{d_1}},\ldots,\frac{f_n}{\|f_n\|_{d_n}}\right)\in\mathfrak{S}_{(d)}.$$

Next, let us observe that

$$\mathcal{Z}(t,\mathrm{WJAC}) = \frac{1}{2\pi}\frac{1}{\mathcal{D}_{(d)}\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}\int_{f\in\mathfrak{S}_{(d)}}\int_{\zeta\in V_S(f)}|WJAC(f,\zeta)|^t\, dV_S(f)(\zeta)d\nu_{\mathfrak{S}}(f).$$

Now, let $\mathbb{B}(\mathcal{H}_{(d)})$ be the product of unit balls given by the following identity:

$$\mathbb{B}(\mathcal{H}_{(d)}) := \prod_{i=1}^n B_{H_{d_i}}(0,1),$$

where $B_{H_{d_i}}(0, 1)$ is the closed ball of radius one centered at the origin in H_{d_i}.
Let us introduce the integral:

$$\mathrm{WJ}(t) := \int_{f\in\mathbb{B}(\mathcal{H}_{(d)})}\int_{\zeta\in V_S(f)}|\mathrm{WJAC}(f,\zeta)|^t\, dV_S(f)(\zeta)d\mathcal{H}_{(d)}(f).$$

Combining Fubini's Theorem and integration in spherical coordinates, we obtain:

$$\mathrm{WJ}(t) = \int_{f\in\mathfrak{S}_{(d)}}\int_{\zeta\in V_S(f)}\int_0^1\cdots\int_0^1|\mathrm{WJAC}(f_{\underline{r}},\zeta)|^t\left(\prod_{i=1}^n r_i^{2M_i-1}\right)dV_S(f)(\zeta)d\nu_{\mathfrak{S}}(f)d\underline{r},$$

where M_i is the complex dimension of H_{d_i} and $\underline{r} := (r_1, \ldots, r_n) \in [0, 1]^n$, $d\underline{r} := dr_1 \cdots dr_n$ and $f_{\underline{r}} := (r_1f_1, \ldots, r_nf_n)$, $f_i \in \mathbb{S}(H_{d_i})$, $1 \leq i \leq n$.
Using the equality of Equation (6.3) above, we conclude:

$$\mathrm{WJ}(t) = \int_{f\in\mathfrak{S}_{(d)}}\int_{\zeta\in V_S(f)}\int_0^1\cdots\int_0^1|\mathrm{WJAC}(f,\zeta)|^t\left(\prod_{i=1}^n r_i^{2M_i-1+2t}\right)dV_S(f)(\zeta)d\nu_{\mathfrak{S}}(f)d\underline{r}.$$

Namely, we have

$$\mathrm{WJ}(t) = \left(\prod_{i=1}^n\left(\int_0^1 r_i^{2(M_i+t)-1}dr_i\right)\right)\left(\int_{f\in\mathfrak{S}_{(d)}}\int_{\zeta\in V_S(f)}|\mathrm{WJAC}(f,\zeta)|^t\, dV_S(f)(\zeta)d\nu_{\mathfrak{S}}(f)\right).$$

As $\int_0^1 r^k dr = \frac{1}{k+1}$, we obviously conclude:

$$\mathrm{WJ}(t) = \left(\frac{1}{\prod_{i=1}^n 2(M_i+t)}\right)\left(\int_{f\in\mathfrak{S}_{(d)}}\int_{\zeta\in V_S(f)} |\mathrm{WJAC}(f,\zeta)|^t \, dV_S(f)(\zeta) d\nu_{\mathfrak{S}}(f)\right).$$

We then have

$$\mathscr{Z}(t,\mathrm{WJAC}) = \frac{2^{n-1}\prod_{i=1}^n (M_i+t)}{\pi \mathcal{D}_{(d)} \nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]} \mathrm{WJ}(t). \tag{6.4}$$

Now, we proceed by computing $\mathrm{WJ}(t)$. Now, observe that, in the definition of $WJ(t)$ above we have included the index function $\chi_{\mathbb{B}(\mathcal{H}_{(d)})}$ of the product of balls $\mathbb{B}(\mathcal{H}_{(d)})$, although it has not been explicitly given in the formula, but in the integration space. Thus, we may have written:

$$\mathrm{WJ}(t) := \int_{f\in\mathcal{H}_{(d)}}\int_{\zeta\in V_S(f)} \chi_{\mathbb{B}(\mathcal{H}_{(d)})}(f) |\mathrm{WJAC}(f,\zeta)|^t \, dV_S(f)(\zeta) d\mathcal{H}_{(d)}(f).$$

Then, we may decompose the index function $\chi_{\mathbb{B}(\mathcal{H}_{(d)})}$ as a product of two index functions $\chi_{\mathbb{B}_F(\mathcal{H}_{(1)})}$ and $\chi_{\mathbb{B}_{R_\zeta}(M,\zeta)}$ defined as follows.

First of all, let $\mathbb{B}_F(\mathcal{H}_{(1)})$ be the product of the closed unit balls $\mathbb{C}^{n+1}$ with respect to the canonical Hermitian norm. Namely,

$$\mathbb{B}_F(\mathcal{H}_{(1)}) := \{M := \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \; : \; m_i \in \mathbb{C}^{n+1}, \|m_i\|_2^2 \le 1\} = \prod_{i=1}^n B_{\mathbb{C}^{n+1}}(0,1).$$

For every $M \in \mathbb{B}_F(\mathcal{H}_{(1)})$ and $\zeta \in S^{2n+1}$ such that $\zeta \in \ker(M)$, we may also define a product of balls in R_ζ given by the following identity:

$$\mathbb{B}_{R_\zeta}(M,\zeta) := \{h := (h_1,\ldots,h_n) \in R_\zeta \; : \; \|h_i\|_{d_i}^2 \le 1 - \|\varphi_i(M,\zeta)\|_{d_i}^2\},$$

where $\varphi_i(M,\zeta)$ are the rows of $\varphi(M,\zeta)(z) := \begin{pmatrix} \varphi_1(M,\zeta)(z) \\ \vdots \\ \varphi_n(M,\zeta)(z) \end{pmatrix}$.

According with the same properties of Equation (4.2) (cf. also [7]), we have

$$\|\varphi_i(M,\zeta)\|_{d_i}^2 = \|m_i\|_2^2,$$

where $m_i \in \mathbb{C}^{n+1}$ is the $i-$th row of M viewed as vector in $\mathbb{C}^{n+1}$ and $\|m_i\|_2$ is the norm of m_i with respect to the canonical Hermitian product in $\mathbb{C}^{n+1}$. Thus, the product of balls $\mathbb{B}_{R_\zeta}(M,\zeta)$ can also be given by:

$$\mathbb{B}_{R_\zeta}(M,\zeta) := \{h := (h_1,\ldots,h_n) \in R_\zeta \; : \; \|h_i\|_{d_i}^2 \le 1 - \|m_i\|_2^2, \;\; M := \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}, m_i \in \mathbb{C}^{n+1}\}. \tag{6.5}$$

In particular, for every $M \in \mathcal{H}_{(1)}$ of maximal rank with kernel generated by $\zeta \in S^{2n+1}$ and for every $h \in R_\zeta$, and given $h + \varphi(M,\zeta) \in L_\zeta \perp R_\zeta$, the coordinates h_i and $\varphi_i(M,\zeta)$ are orthogonal in every $H_{d_i}(\underline{X})$, with respect to the Bombieri-Weyl norm. Hence, the following equivalence holds:

$$h + \varphi(M,\zeta) \in \mathbb{B}(\mathcal{H}_{(d)}) \iff M \in \mathbb{B}_F(\mathcal{H}_{(1)}) \wedge h \in \mathbb{B}_{R_\zeta}(M,\zeta),$$

which corresponds to the equality of index functions

$$\chi_{\mathbb{B}(\mathcal{H}_{(d)})}(h + \varphi(M,\zeta)) = \chi_{\mathbb{B}_F(\mathcal{H}_{(1)})}(M) \cdot \chi_{\mathbb{B}_{R_\zeta}(M,\zeta)}(h).$$

Having this in mind and applying Corollary 17 of [7] (cf. also Theorem 4.4 above), the following equality also holds:

$$\frac{\mathrm{WJ(t)}}{\mathcal{D}_{(d)}} = \int_{M\in\mathbb{B}_F(\mathcal{H}_{(1)})}\int_{\zeta\in V_S(M)}\int_{h\in\mathbb{B}_{R_\zeta}(M,\zeta)} |\mathrm{WJAC}(h+\varphi(M,\zeta),\zeta)|^t \, dR_\zeta(h) dV_S(M)(\zeta) d\mathcal{H}_{(1)}(M). \tag{6.6}$$

Noting that for $h \in R_\zeta$, $D(h)(\zeta) = 0$, and recalling Equation (4.3) above (cf. also [7]) we have:

$$D(h + \varphi(M,\zeta))(\zeta) = D(\varphi(M,\zeta)(\zeta) = \text{Diag}(d_i^{1/2})M,$$

where $\text{Diag}(d_i^{1/2})$ is the diagonal matrix whose $i-$diagonal term is $d_i^{1/2}$. Hence,

$$|\text{WJAC}(h+\varphi(M,\zeta),\zeta)| := |\det\left(D(h+\varphi(M,\zeta))(\zeta)D(h+\varphi(M,\zeta))(\zeta)^*\right)| = |\prod_{i=1}^n d_i \det(MM^*)|.$$

Thus, Equation (6.6) becomes:

$$\frac{\text{WJ(t)}}{\mathcal{D}_{(d)}} = \int_{M\in\mathbb{B}_F(\mathcal{H}_{(1)})}\int_{\zeta\in V_S(M)}\int_{h\in\mathbb{B}_{R_\zeta}(M,\zeta)} |\prod_{i=1}^n d_i \det(MM^*)|^t \, dR_\zeta(h)dV_S(M)(\zeta)d\mathcal{H}_{(1)}(M). \tag{6.7}$$

Next, extracting the Bézout numbers, Equation (6.7) becomes:

$$\frac{\text{WJ(t)}}{\mathcal{D}_{(d)}{}^{t+1}} = \int_{M\in\mathbb{B}_F(\mathcal{H}_{(1)})} |\det(MM^*)|^t \left(\int_{\zeta\in V_S(M)}\int_{h\in\mathbb{B}_{R_\zeta}(M,\zeta)} 1 \, dR_\zeta(h)dV_S(M)(\zeta)\right) d\mathcal{H}_{(1)}(M). \tag{6.8}$$

Generically on $\mathcal{H}_{(1)}$, the kernel of a matrix $M \in \mathcal{H}_{(1)}$ is a vector subspace of $\mathbb{C}^{n+1}$ of dimension 1 and $V_S(M) = S^1\zeta$ is the orbit, under the action of the unit sphere $S^1 \subseteq \mathbb{C}$, on any fixed $\zeta \in V_S(M)$. In particular, the volume

$$\text{vol}[\mathbb{B}_{R_\zeta}(M,\zeta)] := \int_{h\in\mathbb{B}_{R_\zeta}(M,\zeta)} 1 \, dR_\zeta(h),$$

does not depend on the chosen orbit generator $\zeta \in V_S(M)$. Namely, let us assume $M, M' \in \mathcal{H}_{(1)}$ be any two matrices with kernel of dimension one and assume

$$M := \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}, \quad M' := \begin{pmatrix} m_1' \\ \vdots \\ m_n' \end{pmatrix},$$

where $m_i, m_i' \in \mathbb{C}^{n+1}$ and $||m_i\|_2^2 = \|m_i'\|_2^2$, for every i, $1 \le i \le n$. Then, we have:

$$\text{vol}[\mathbb{B}_{R_\zeta}(M,\zeta)] := \text{vol}[\mathbb{B}_{R_{\zeta'}}(M',\zeta')],$$

where $\zeta \in V_S(M)$ and $\zeta' \in V_S(M')$. Namely, we may assume $\zeta = e_0 := (1,0,\ldots,0) \in \mathbb{C}^{n+1}$ and that $M \in \mathcal{H}_{(1)}$ is any matrix that vanishes on e_0. Then, $\mathbb{B}_{R_{e_0}}(M,e_0)$ is a product of closed balls:

$$\mathbb{B}_{R_{e_0}}(M,e_0) = \prod_{i=1}^n B_{R_{i,e_0}}(0, \sqrt{1-\|m_i\|_2^2}),$$

where

$$B_{R_{i,e_0}}(0,\sqrt{1-\|m_i\|_2^2}) := \{f \in R_{i,e_0} \ : \ \|f\|_{d_i} \le \sqrt{1-\|m_i\|_2^2}\},$$

and

$$R_{i,e_0} := \{f \in H_{d_i} \ : \ f(e_0) = 0, \ \nabla_{e_0} f = 0\}.$$

Next, observe that R_{i,e_0} is a vector subspace of H_{d_i} of co-dimension $n+1$. Namely,

$$\dim_{\mathbb{C}}(R_{i,e_0}) = M_i - (n+1) = \binom{d_i+n}{n} - (n+1).$$

As the volume of the unit ball in the complex space $\mathbb{C}^k$ satisfies $\text{vol}[B_{\mathbb{C}^k}(0,1)] = \frac{\pi^k}{\Gamma(k+1)}$, we then conclude:

$$\text{vol}[\mathbb{B}_{R_\zeta}(M,\zeta)] = \prod_{i=1}^n \left(\left(1-\|m_i\|_2^2\right)^{\frac{2(M_i-(n+1))}{2}} \text{vol}[B_{R_{i,e_0}}(0,1)]\right) = \prod_{i=1}^n \frac{\left(\pi\left(1-\|m_i\|_2^2\right)\right)^{M_i-(n+1)}}{\Gamma(M_i-n)}.$$

Then, Equation (6.8) becomes:

$$\frac{\mathrm{WJ(t)}}{2\pi\mathcal{D}_{(d)}^{\;t+1}} = \left(\prod_{i=1}^{n} \frac{\pi^{M_i-(n+1)}}{\Gamma(M_i-n)}\right)\mathrm{K(t)}. \tag{6.9}$$

where

$$\mathrm{K(t)} := \int_{M\in\mathbb{B}_F(\mathcal{H}_{(1)})} |\det(MM^*)|^t \prod_{i=1}^{n}\left(1-\|m_i\|_2^2\right)^{M_i-(n+1)} d\mathcal{H}_{(1)}(M). \tag{6.10}$$

We now focus our interest on the determination of K(t). We integrate in spherical coordinates, noting $\mathbb{B}_F(\mathcal{H}_{(1)}) = \prod_{i=1}^n B_{\mathbb{C}^{n+1}}(0,1)$ is a product of closed unit balls. We proceed as in the proof of Corollary 5.8.

$$\mathbb{B}_F(\mathcal{H}_{(1)}) := \{M := \begin{pmatrix} m_1\\ \vdots \\ m_n\end{pmatrix} \;:\; m_i\in\mathbb{C}^{n+1},\; \|m_i\|_2^2\leq 1\} = \prod_{i=1}^n B_{\mathbb{C}^{n+1}}(0,1).$$

Using Equation (5.7), and integrating in spherical coordinates, we obtain:

$$\mathrm{K(t)} = \int_{M\in\prod_{i=1}^n S^{2n+1}}\int_0^1\cdots\int_0^1\left(\prod_{i=1}^n\left(1-r_i^2\right)^{M_i-(n+1)} r_i^{2t} r_i^{2n+1}\right)|\det(MM^*)|^t\, dM d\underline{r}.$$

Then,

$$\mathrm{K(t)} = \left(\prod_{i=1}^n\int_0^1 r_i^{2(n+t)+1}(1-r_i^2)^{M_i-(n+1)}dr_i\right)\left(\int_{M\in\prod_{i=1}^n S^{2n+1}} |\det(MM^*)|^t\, dM\right).$$

Namely,

$$\mathrm{K(t)} = \left(\prod_{i=1}^n \frac{\mathrm{B}(n+t+1, M_i-n)}{2}\right)\left(\int_{M\in\prod_{i=1}^n S^{2n+1}} |\det(MM^*)|^t\, dM\right).$$

We finally use Corollary 5.8 to conclude:

$$\mathrm{K(t)} = \left(\prod_{i=1}^n \frac{\mathrm{B}(n+t+1, M_i-n)}{2}\right)\left(2^n\pi^{n(n+1)}\prod_{j=2}^{n+1}\frac{\Gamma(j+t)}{\Gamma(j)\Gamma(t+n+1)}\right). \tag{6.11}$$

Then, we go back to Equation (6.9) to conclude:

(6.12)

$$\frac{\mathrm{WJ(t)}}{2\pi\mathcal{D}_{(d)}^{\;t+1}} = \left(\prod_{i=1}^{n} \frac{\pi^{M_i-(n+1)}}{\Gamma(M_i-n)}\right)\left(\prod_{i=1}^n \frac{\mathrm{B}(n+t+1, M_i-n)}{2}\right)\left(2^n\pi^{n(n+1)}\prod_{j=2}^{n+1}\frac{\Gamma(j+t)}{\Gamma(j)\Gamma(t+n+1)}\right).$$

Replacing the Beta function by its value, we get

$$\frac{\mathrm{WJ(t)}}{2\pi\mathcal{D}_{(d)}^{\;t+1}} = \pi^{M_{(d)}}\prod_{i=1}^n\frac{1}{\Gamma(M_i+t+1)}\prod_{j=2}^{n+1}\frac{\Gamma(j+t)}{\Gamma(j)}, \tag{6.13}$$

where $M_{(d)}$ is the dimension of $\mathcal{H}_{(d)}$, $M_{(d)} = \sum_{i=1}^n M_i$.
Now, as the volume of the product of spheres satisfies:

$$\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}] = \prod_{i=1}^n\frac{2\pi^{M_i}}{\Gamma(M_i)} = 2^n\pi^{M_{(d)}}\prod_{i=1}^n\frac{1}{\Gamma(M_i)},$$

combining this last equality with Equation (6.4) and Equation (6.13), we conclude

$$\mathcal{Z}(t,\mathrm{WJAC}) = \frac{2^{n-1}\prod_{i=1}^n(M_i+t)}{\pi\mathcal{D}_{(d)}\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}\mathrm{WJ}(t) = \mathcal{D}_{(d)}^{\;t}\prod_{i=1}^n\frac{\Gamma(i+t+1)\Gamma(M_i)(M_i+t)}{\Gamma(i+1)\Gamma(M_i+t+1)}. \tag{6.14}$$

Namely, as $\Gamma(z+1) = z\Gamma(z)$ we conclude

$$\mathcal{Z}(t, \mathrm{WJAC}) = \mathcal{D}_{(d)}{}^{t} \prod_{i=1}^{n} \frac{\Gamma(i+t+1)\Gamma(M_i)}{\Gamma(i+1)\Gamma(M_i+t)}. \tag{6.15}$$

Now, the main result follows from Equality (6.1) and hence the following holds for every real number $t \in \mathbb{R}$, $t > -4$:

$$\mathcal{Z}(t, \mathrm{JAC}) = (\mathcal{D}_{(d)})^{\frac{t}{2}} \prod_{i=1}^{n} \left(\frac{\Gamma(i+\frac{t}{2}+1)}{\Gamma(i+1)} \cdot \frac{\Gamma(M_i)}{\Gamma(M_i+\frac{t}{2})} \right).$$

And this finishes the computation of the value of $\mathcal{Z}(t, \mathrm{JAC})$. Observe that the holomorphic function on the right hand side of Equality (1.7) contains the term $\Gamma(2+\frac{t}{2})$. Thus, $\mathcal{Z}(t, \mathrm{JAC})$ admits analytic continuation to the complex domain $\mathbb{C} \setminus \{z \in \mathbb{Z} \,:\, z \leq -4\}$.

Now, we finish the proof of this Theorem by computing $m(\mathrm{JAC})$ as defined in the Introduction. According to Lemma 2.1, we observe that $m(\mathrm{JAC})$ is related to the derivative of $\mathcal{Z}(t, \mathrm{JAC})$ at $t = 0$, i.e.:

$$m(\mathrm{JAC}) = \mathcal{D}_{(d)} \frac{d\mathcal{Z}(t, JAC)}{dt}\Big|_{t=0}.$$

Then, after some elementary calculations from the previous expression, we obtain:

$$\frac{d\mathcal{Z}(t, \mathrm{JAC})}{dt}\Big|_{t=0} = \frac{1}{2}\big(\log \mathcal{D}_{(d)} + \sum_{i=1}^{n} \psi(i+1) - \psi(M_i) \big),$$

where ψ is the digamma function, i.e. $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Hence, we conclude

$$m(\mathrm{JAC}) = \frac{\mathcal{D}_{(d)}}{2} \Big(\log \mathcal{D}_{(d)} + \sum_{i=1}^{n} \psi(i+1) - \psi(M_i) \Big),$$

and the last statement of the Theorem follows.

7. Height of the Discriminant polynomial: Proof of Theorem 1.5

Elimination Theory is a term originated in the nineteenth century whose nowadays translation are several fields called Computational Algebraic Geometry or Effective Methods in Algebraic Geometry, among others. The main task in Elimination Theory is the design of efficients algorithms to "eliminate" quantifiers. The central problem in Elimination Theory is, then, the elimination of a block of existencial quantifiers in a formula involving polynomial equations. The basic elimination polynomial is the multivariate resultant $\mathrm{Res}_{(d)}$ which determines a quantifier free formula equivalent to another one containing a block of existential quantifiers (i.e. a Nullstellensatz) in the projective case.

Together with the resultant polynomial $\mathrm{Res}_{\overline{(d)}}$, there is another central Diophantine polynomial in Elimination Theory: the *discriminant polynomial* $\mathrm{Disc}_{(d)}$ which has been discussed in Subsection 3.4 above. Since both of them are Diophantine, both of them must be subject of study in terms of arithmetic height in the sense of [27], [16], [12], [50], [21], [52] and references therein. In the path to prove the Arithmetic Bézout Inequality (cf. [12], [47, 48, 49, 50], [41] and continuators) Philippon's school computed the exact value of the arithmetic height fof the resultant polynomial $\mathrm{Res}_{(d)}$ (cf. [51], [52] and references therein). The strong techniques developed by these authors do not seem to produce the exact value of the arithmetic height of the discriminant polynomial $\mathrm{Disc}_{(d)}$. Only upper bounds are available from the techniques related to the Arithmetic Bézout inequality.

A measure of the Diophantine properties of $\mathrm{Disc}_{(d)}$ is its Mahler measure (cf. [47, 48, 49, 50], [41] and sequels). On the other hand, Mahler measure becomes entropy in Dynamical Systems of Algebraic Origin ([53], [25], [26], [42] and references therein). As $\mathrm{Disc}_{(d)}$ is multi-homogeneous, its natural Mahler measure will be the one given by the product of spheres $\mathfrak{S}_{(d)}$. Our exact knowledge of the zeta Mahler measure functions $\mathcal{Z}(t, 1/\|\cdot\|_{\mathrm{aff}})$ and $\mathcal{Z}(t, JAC)$ and its derivatives will provide the exact

value of the arithmetic height of the discriminant polynomial $\mathrm{Disc}_{(d)}$, this shows the strength of zeta Mahler measure functions and the information they provide.
We follow [50], to define the arithmetic height of a multi-homogeneous polynomial using its Mahler measure and its partial degrees.

Definition 7.1 (Unitarily invariant height of $\Sigma_{(d)}$ and of $\mathrm{Disc}_{(d)}$)**.** With these notations, the invariant logarithmic height of the discriminant variety $\Sigma_{(d)}$ is given by the following identity:

$$ht(\Sigma_{(d)}) = ht(\mathrm{Disc}_{(d)}) = m_{\mathfrak{S}_{(d)}}(\mathrm{Disc}_{(d)}) + \sum_{i=1}^{n} \frac{\deg_{f_i} \mathrm{Disc}_{(d)}}{2} H_{M_i - 1},$$

where:

- $m_{\mathfrak{S}_{(d)}}(\mathrm{Disc}_{(d)})$ is the logarithmic Mahler measure of the discriminant $\mathrm{Disc}_{(d)}$ in the product of spheres $\mathfrak{S}_{(d)}$:
$$m_{\mathfrak{S}_{(d)}}(\mathrm{Disc}_{(d)}) := \frac{1}{\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]} \int_{\mathfrak{S}_{(d)}} \log |\mathrm{Disc}_{(d)}(f_1, \ldots, f_n)| d\nu_{\mathfrak{S}}(f_1, \ldots, f_n).$$
- $\deg_{f_i} \mathrm{Disc}_{(d)}$ is the partial degree of $\mathrm{Disc}_{(d)}$ with respect to the variables representing the coefficients of f_i,
- $M_i := M(d_i, n)$ is the dimension of $H_{d_i}(\underline{X})$,

First of all, observe that we have chosen the product of spheres $\mathfrak{S}_{(d)}$ instead of the sphere $\mathbb{S}(\mathcal{H}_{(d)})$ because $\mathrm{Disc}_{(d)}$ is multi-homogeneous. Equations (1.3) and (1.4) of the introduction show that one may easily deduce the logarithmic Mahler measure of $\mathrm{Disc}_{(d)}$ with respect to $\mathbb{S}(\mathcal{H}_{(d)})$ from our results below.
According to [15] (cf. Theorem 3.3 above, where these properties are resumed) $\mathrm{Disc}_{(d)}$ is multi-homogeneous polynomial such that the degrees with respect to each group of variables (represented by a polynomial $f_i \in H_{d_i}(\underline{X})$ is given by:

$$\deg_{f_i} \mathrm{Disc}_{(d)} = \Big(\prod_{j \neq i} d_j\Big)\big(\delta_{(d)} + d_i - (n+1)\big).$$

7.1. **Proof of Theorem 1.5.** Hence, the following Proposition immediately yields the Theorem 1.5 of the Introduction:

Proposition 7.1. *With the previous notations, let $(d) := (d_1, \ldots, d_n)$ be a list of degrees and let $\delta_{(d)} := \sum_{i=1}^n d_i$ be the sum of the degrees in the list (d). Assume that $\delta_{(d)} - n = \sum_{i=1}^n (d_i - 1) \geq 1$ holds. Then, the logarithmic Mahler measure of the discriminant polynomial in the case of dense homogeneous polynomials defining a zero-dimensional variety is given by the following equality:*

$$m_{\mathfrak{S}_{(d)}}(\mathrm{Disc}_{(d)}) = A_{(d)} - B_{(d)},$$

where

$$A_{(d)} := \frac{\mathcal{D}_{(d)}}{2}\left[(\delta_{(d)} - n)\left(\sum_{i=1}^{n} H_i\right) + \log \mathcal{D}_{(d)}\right],$$

and

$$B_{(d)} := \sum_{i=1}^{n} \frac{\deg_{f_i} \mathrm{Disc}_{(d)}}{2} H_{M_i-1} = \sum_{i=1}^{n} \frac{\prod_{j\neq i} d_j}{2}\big(\delta_{(d)} + d_i - (n+1)\big) H_{M_i - 1},$$

Proof. First of all, for every list of polynomials $f \in \mathcal{H}_{(d)}$ that satisfies the hypothesis of Proposition 3.5, the following equality (Equation (3.6)) holds:

$$\frac{|\mathrm{Disc}_{(d)}(f)|}{2^{\sharp(V_{\mathbb{A}}(f)) - \mathcal{D}_{(d)}}} = |\mathrm{Res}_{(d)}(f_{1,0}, \ldots, f_{n,0})|^{(\sum_{i=1}^n (d_i - 1)) - 1} \prod_{\zeta \in V_{\mathbb{A}}(f)} \frac{|\det(Df(1,\zeta)Df(1,\zeta)^*)|^{1/2}}{(1 + \|\zeta\|^2)^{1/2}}. \tag{7.1}$$

As in the Proof of Theorem 1.4, for every $\zeta \in \mathbb{P}_n(\mathbb{C})$, let $\mathrm{WJAC}(f, \zeta)$ the determinant $\det(Df(z)Df(z)^*)$, where $z \in V_S(f)$ and $p_S(z) = \zeta$.

Next, for every affine point $(1,\zeta)\in\mathbb{C}^{n+1}\setminus\{0\}$, let $\xi:=\pi(1,\zeta)\in\mathbb{P}_n(\mathbb{C})$ be the associated projective point, and the following equality holds:

$$|\det(Df(1,\zeta)Df(1,\zeta)^*)|=(1+\|\zeta\|)^2)^{\sum_{i=1}^n(d_i-1)}|\det\left(Df\left(\frac{(1,\zeta)}{(1+\|\zeta\|^2)^{1/2}}\right)Df\left(\frac{(1,\zeta)}{(1+\|\zeta\|^2)^{1/2}}\right)^*\right)|.$$

Namely,

$$|\det(Df(1,\zeta)Df(1,\zeta)^*)|=(1+\|\zeta\|)^2)^{\sum_{i=1}^n(d_i-1)}|\mathrm{WJAC}(f,\xi)|,$$

where the notations are those recently introduced. Hence, for every system $f\in\mathcal{H}_{(d)}$ whose projective zeros are not in the infinity hyper-plane $H_\infty:=V_{\mathbb{P}}(X_0)$, we conclude:

$$\prod_{\zeta\in V_{\mathbb{A}}(f)}\frac{|\det(Df(1,\zeta)Df(1,\zeta)^*)|^{1/2}}{(1+\|\zeta\|^2)^{1/2}}=\prod_{\zeta\in V_{\mathbb{A}}(f)}(1+\|\zeta\|^2)^{\frac{(\sum_{i=1}^n d_i)-(n+1)}{2}}\prod_{\xi\in V_{\mathbb{P}}(f)}|\mathrm{WJAC}(f,\xi)|^{1/2},$$

The hypothesis of Proposition 3.5 are satisfied for generic choices of systems of polynomial equations $f:=(f_1,\ldots,f_n)\in\mathcal{H}_{(d)}$. Hence, up to a set of measure zero in $\mathcal{H}_{(d)}$, the logarithm of the absolute value of the discriminant satisfies:

$$\log|\mathrm{Disc}_{(d)}(f)|=J_1(f)+J_2(f)+J_3(f)+J_4(f),$$

where

- $J_1(f):=(\sharp(V_{\mathbb{A}}(f))-\mathcal{D}_{(d)})\log 2=0$, up to a set of measure zero of $\mathcal{H}_{(d)}$.
- $J_2(f):=\big(\big(\delta_{(d)}-(n+1)\big)\log|\mathrm{Res}_{(d)}(f_{1,0},\ldots,f_{n,0})|$.
- $J_3(f):=\frac{1}{2}\sum_{\zeta\in V_{\mathbb{P}}(f)}\log|\mathrm{WJAC}(f,\zeta)|=\sum_{\zeta\in V_{\mathbb{P}}(f)}\log|\mathrm{JAC}(f,\zeta)|$.
- $J_4(f):=\frac{(\delta_{(d)}-(n+1))}{2}\sum_{\zeta\in V_{\mathbb{A}}(f)}\log\left|1+\|\zeta\|^2\right|$.

Hence, taking integrals, the logarithmic Mahler measure of the discriminant satisfies:

$$m_{\mathfrak{S}_{(d)}}(\mathrm{Disc}_{(d)})=\frac{1}{\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}\int_{\mathfrak{S}_{(d)}}\log|\mathrm{Disc}_{(d)}(f)|=I_2+I_3+I_4,$$

where, for $2\le i\le 4$,

$$I_i:=\frac{1}{\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}\int_{\mathfrak{S}_{(d)}}J_i(f)d\nu_{\mathfrak{S}}(f).$$

The value I_2 is given by the following identity,

$$I_2:=\Big(\delta_{(d)}-(n+1)\Big)\frac{1}{\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}\int_{\mathfrak{S}_{(d)}}\log|\mathrm{Res}_{(d)}(f_{1,0},\ldots,f_{n,0})|d\nu_{\mathfrak{S}}(f),$$

where the polynomials $f_{i,0}\in\mathbb{C}[X_1,\ldots,X_n]$ and, in particular, they belong to $H_{d_i}(X_1,\ldots,X_n)$. We thus introduce the following two cartesian products, depending on the number of variables:

$$\mathfrak{S}_{(d)}(\underline{X}^{(n+1)}):=\prod_{i=1}^n\mathbb{S}\big(H_{d_i}(X_0,\ldots,X_n)\big),$$

$$\mathfrak{S}_{(d)}(\underline{X}^{(n)}):=\prod_{i=1}^n\mathbb{S}\big(H_{d_i}(X_1,\ldots,X_n)\big),$$

where $\mathbb{S}$ stands for the Bombiery-Weyl unit sphere. According to Lemma 3.11 in [29] the following equality holds:

$$I_2=\big(\delta_{(d)}-(n+1)\big)\left(m_{\mathfrak{S}_{(d)}(\underline{X}^{(n)})}\big(\mathrm{Res}_{(d)}(f_{1,0},\ldots,f_{n,0})\big)-\mathcal{J}\big((d),n\big)\right),\tag{7.2}$$

where:

- $$\mathcal{J}\big((d), n\big) := \sum_{i=1}^{n} \frac{\prod_{j\neq i} d_j}{2} (H_{M_i-1} - H_{L_i-1}),$$
  where H_k denotes de k-th harmonic number and
  $$M_i := \binom{d_i+n}{n}, \quad L_i := \binom{d_i+n-1}{n-1}$$
- $m_{\mathfrak{S}_{(d)}(\underline{X}^{(n)})}\big(\mathrm{Res}_{(d)}(f_{1,0},\ldots,f_{n,0})\big)$ is the Mahler measure of the multi-variate resultant in n variables.

The Mahler measure of $\mathrm{Res}_{(d)}$ satisfies (cf. [51], [52] or [29], for instance):

$$m_{\mathfrak{S}_{(d)}(\underline{X}^{(n)})}(\mathrm{Res}_{(d)}) = \frac{\mathcal{D}_{(d)}}{2}\Big(\sum_{j=1}^{n-1} H_j\Big) - \Big(\sum_{i=1}^{n} \frac{\prod_{j\neq i} d_j}{2} H_{L_i-1}\Big).$$

Thus, Equation (7.2) becomes:

$$I_2 = \big(\delta_{(d)} - (n+1)\big)\frac{\mathcal{D}_{(d)}}{2}\Big[\sum_{j=1}^{n-1} H_j - \sum_{i=1}^{n} \frac{H_{M_i-1}}{d_i}\Big]$$

Now, observe that I_3 is the quantity $m(JAC)$ computed in Equation (1.8) stated as a consequence of Theorem 1.4. Thus, we have

$$I_3 := m(\mathrm{JAC}) = \frac{\mathcal{D}_{(d)}}{2}\left(\log(\mathcal{D}_{(d)}) + \sum_{i=1}^{n}\Big(\psi(i+1) - \psi(M_i)\Big)\right) = \frac{\mathcal{D}_{(d)}}{2}\left(\log(\mathcal{D}_{(d)}) + \sum_{i=1}^{n}\Big(H_i - H_{M_i-1}\Big)\right).$$

We finally compute I_4,

$$I_4 := \frac{1}{\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}\int_{\mathfrak{S}_{(d)}} J_4(f)\,d\nu_{\mathfrak{S}}(f) = \frac{\big(\delta_{(d)} - (n+1)\big)}{2\nu_{\mathfrak{S}}[\mathfrak{S}_{(d)}]}\int_{\mathfrak{S}_{(d)}} \sum_{\zeta\in V_{\mathbb{A}}(f)} \log(1+\|\zeta\|^2)\,d\nu_{\mathfrak{S}}(f).$$

Thus, according to Corollary 5.4, we conclude:

$$I_4 := (\delta_{(d)} - (n+1))\frac{\mathcal{D}_{(d)}}{2} H_n.$$

We put all these quantities in a single identity to conclude the following value of the logarithmic Mahler measure of the multi-variate discriminant.

$$m_{\mathfrak{S}_{(d)}}(\mathrm{Disc}_{(d)}) = A_{(d)} - B_{(d)},$$

where:

$$A_{(d)} := \frac{\mathcal{D}_{(d)}}{2}\left[(\delta_{(d)} - n)\left(\sum_{i=1}^{n} H_i\right) + \log \mathcal{D}_{(d)}\right],$$

$$B_{(d)} := \sum_{i=1}^{n} \frac{\prod_{j\neq i} d_j}{2}\big(\delta_{(d)} + d_i - (n+1)\big) H_{M_i-1},$$

where $\mathcal{D}_{(d)} := \prod_{i=1}^{n} d_i$, $\delta_{(d)} := \sum_{i=1}^{n} d_i$ and $H_r := \sum_{j=1}^{r} \frac{1}{j}$ are the quantities used along the text.

□

7.2. **Some comments on Algorithmic Applications of Theorem 1.5.** The main outcome of Theorem 1.5 may be applied directly to the design of deterministic or Monte Carlo algorithms, based on modular arithmetic that decide the following problem:

Problem 7.2. *Given a list of homogeneous polynomial equations with Gaussian integer coefficients $f := (f_1, \ldots, f_n) \in \mathbb{Z}[i][X_1, \ldots, X_n]^n$, satisfying $\deg(f_j) = d_j$, $1 \le j \le n$, and whose coefficients have a total bit length bounded by h. Decide whether the projective variety $V_{\mathbb{P}}(f) \subseteq \mathbb{P}_n(\mathbb{C})$ has a singular zero.*

The technique to treat this Problem is based on the use of the discriminant, since the variety $V_{\mathbb{P}}(f)$ contains a singular zero if and only if $\mathrm{Disc}_{(d)}(f_1, \ldots, f_n) = 0$. Since $\mathrm{Disc}_{(d)}(f_1, \ldots, f_n) \in \mathbb{Z}$, this reduces to apply either Monte Carlo Zero Tests for Integers as in [35] or deterministic algorithms based on the Chinese Remainder Theorem.

We do not discuss it in full detail, since it is not the main goal of these pages. However, we give some indications on how to proceed. Roughly speaking, these methods are based on determining an upper bound of $|\mathrm{Disc}_{(d)}(f_1, \ldots, f_n)|$. In the deterministic case, for instance, it is enough to have a finite set $\{p_1, \ldots, p_s\}$ of pairwise co-prime positive integers. Then, the Chinese Remainder Theorem implies that the following are equivalent conditions:

- $\mathrm{Disc}_{(d)}(f_1, \ldots, f_n) = 0 \mod p_i,\ 1 \leq i \leq s$,
- $\mathrm{Disc}_{(d)}(f_1, \ldots, f_n) = 0 \mod T$, where $T := \prod_{i=1}^{s} p_i$,

If the numbers $p_1, \ldots, p_s$ are prime numbers, the first condition means that the projective varieties defined over the finite fields $\mathbb{F}_{p_i}$ by the list of polynomials $f_1, \ldots, f_n$ contains a singular zero.

Then, the answer to a zero test for the discriminant based on modular arithmetics that satisfies any of these two equivalent conditions would be: *Either* $\mathrm{Disc}_{(d)}(f_1, \ldots, f_n) = 0$ *or* $|\mathrm{Disc}_{(d)}(f_1, \ldots, f_n)| > \frac{1}{2}\prod_{i=1}^{s} p_i - 1 = \frac{1}{2}T - 1$. The error in this answer is determined and controlled by the inequality $|\mathrm{Disc}_{(d)}(f_1, \ldots, f_n)| > \frac{1}{2}\prod_{i=1}^{s} p_i - 1$. In the case we prefer to use randomized Monte Carlo tests (as the one in [35]) the lower bound $\frac{1}{2}T - 1$ is replaced by a much bigger function of T.

As the Discriminant $\mathrm{Disc}_{(d)}$ is multi-homogeneous, this inequality controlling the error is equivalent to

$$|\mathrm{Disc}_{(d)}(\overline{f_1}, \ldots, \overline{f_n})| > M(\underline{p}, f) := \frac{\frac{1}{2}\prod_{i=1}^{s} p_i - 1}{\left(\prod_{i=1}^{n} \|f_i\|_{d_i}^{\deg_{f_i}(\mathrm{Disc}_{(d)})}\right)}, \tag{7.3}$$

where $\overline{f_i} := f_i/\|f_i\|_{d_i}$ is the trace in the complex sphere $\mathbb{S}(H_{d_i}(\underline{X}))$ of the polynomial f_i. The distribution of the traces in $\prod_{i=1}^{n} \mathbb{S}(\mathcal{H}_{(d)})$ of polynomials with Gaussian integer coefficients of bounded bit length h are very close to the continuous distribution. The difference between both probability distributions is bounded by discrepancy bounds as those introduced in [3] or [17] and references there in. Up to these discrepancy bounds, the probability that $|\mathrm{Disc}_{(d)}(\overline{f_1}, \ldots, \overline{f_n})| > M(\underline{p}, f)$ for a system f of height bounded by h, is bounded by the probability that a random point $g := (g_1, \ldots, g_n) \in \mathfrak{S}_{(d)} := \prod_{i=1}^{n} \mathbb{S}(H_{d_i}(\underline{X}))$ satisfies the same inequality, i.e.

$$P(\mathrm{Disc}_{(d)}, (d), h) := Prob_{\mathfrak{S}_{(d)}}[(g_1, \ldots, g_n) \in \mathfrak{S}_{(d)} \ : \ |\mathrm{Disc}_{(d)}(g_1, \ldots, g_n)| > \mathcal{M}(\underline{p}, (d), h)\ ],$$

where $\mathcal{M}(\underline{p}, (d), h)$ is a quantity which depends on $\prod_{i=1}^{s} p_i$, the degree list (d) and h. Note that the quantity $P(\mathrm{Disc}_{(d)}, (d), h)$ will be *the probability that if* $\mathrm{Disc}_{(d)}(f_1, \ldots, f_n) = 0 \mod p_i,\ 1 \leq i \leq s,$ *and we answer* $\mathrm{Disc}_{(d)}(f_1, \ldots, f_n) = 0$*, our answer is wrong.* Namely, $P(\mathrm{Disc}_{(d)}, (d), h)$ bounds the *error probability.* In order to minimize the error probability we just have to increase either $\prod_{i=1}^{s} p_i$ or h or maybe both. These calculations are omitted here.

The role of the height (or the Mahler measure) here is to control this continuous probability. Note that this continuous probability can also be written in logarithmic terms as

$$P(\mathrm{Disc}_{(d)}, (d), h) = Prob_{\mathfrak{S}_{(d)}}[(g_1, \ldots, g_n) \in \mathfrak{S}_{(d)} \ : \ \log\left(|\mathrm{Disc}_{(d)}(g_1, \ldots, g_n)|\right) > \log\left(\mathcal{M}(\underline{p}, (d), h)\right)\ ].$$

Using either Chebyshef or Markov's inequality, we may bound $P(\mathrm{Disc}_{(d)}, (d), h)$. For instance, using Markov's inequality we have:

$$P(\mathrm{Disc}_{(d)}, (d), h) \leq \frac{E_{\mathfrak{S}_{(d)}}[\log|Disc_{(d)}|]}{\log\left(\mathcal{M}(\underline{p}, (d), h)\right)}.$$

But the expectation $E_{\mathfrak{S}_{(d)}}[\log|Disc_{(d)}|]$ is the logarithmic Mahler measure of $\mathrm{Disc}_{(d)}$ as in Definition 7.1. Namely, we have shown that the error probability of answering $\mathrm{Disc}_{(d)}(f_1, \ldots, f_n) = 0$ when this is not the case is bounded by:

$$P(\mathrm{Disc}_{(d)}, (d), h) \leq \frac{m_{\mathfrak{S}_{(d)}}(Disc_{(d)})}{\log\left(\mathcal{M}(\underline{p}, (d), h)\right)},$$

where $m_{\mathfrak{S}_{(d)}}(Disc_{(d)})$ is the logarithmic Mahler measure of the discriminant. Thus, using the logarithmic Mahler measure of the discriminant (see Proposition 7.1) we already have the wanted estimate of the error probability.

Some authors would perhaps prefer the use of the arithmetic height because of their use in Arithmetic Intersection Theory (see the references cited at the beginning of Section 7). Because it satisfies an Arithmetic Bézout Inequality, height may be more "canonical" as quantity determining the arithmetic properties of a variety.

Thus, according with Definition 7.1, the logarithmic Mahler measure of the discriminant and the height are related by:

$$ht(\mathrm{Disc}_{(d)}) - \sum_{i=1}^{n} \frac{\deg_{f_i} \mathrm{Disc}_{(d)}}{2} H_{M_i-1} = m_{\mathfrak{S}_{(d)}}(\mathrm{Disc}_{(d)}),$$

where the notations are those of Theorem 1.5 and Definition 7.1. Hence, the error probability of our modular algorithm will be bounded (up to discrepancy bounds) in terms of arithmetic height by:

$$P(\mathrm{Disc}_{(d)}, (d), h) \leq \frac{ht(\mathrm{Disc}_{(d)}) - \sum_{i=1}^{n} \frac{\deg_{f_i} \mathrm{Disc}_{(d)}}{2} H_{M_i-1}}{\log\left(\mathcal{M}(\underline{p}, (d), h)\right)}.$$

And, hence, Theorem 1.5 or Proposition 7.1 (depending on the personal taste of the reader) applies to yield upper bounds for the probability of error of modular algorithms for testing whether a system of homogeneous polynomial equations have a singular zero. Details and precise calculations are omitted since they are not the main stream of this manuscript.

References

[1] H. Akatsuka, *Zeta Mahler measures*, Journal of Number Theory **129** (2009), 2713–34.

[2] S. N. Anfinsen, A. P. Doulgeris, T. Eltoft, *Estimation of the Equivalent Number of Looks in Polarimetric Synthetic Aperture Radar Imagery.* IEEE Transactions on Geoscience and Remote Sensing **47** (2009), 3795–3809.

[3] C. Beltrán, Luis M. Pardo, *On the Probability Distribution of singular varieties of given co-rank.* J. of Symb. Comput. **42** (2007), 4-29.

[4] C. Beltrán, L.M. Pardo, *Estimates on the Distribution of the Condition Number of Singular Matrices* Foundations of Computational Mathematics **7** (2007), 87–134.

[5] C. Beltrán, L.M. Pardo, *On Smale's 17th Problem: A Probabilistic Positive Solution.* Foundations of Computational Mathematics **8** (2008), 1–43.

[6] C. Beltrán, L.M. Pardo, *Smale's 17th Problem: Average Polynomial Time to Compute Affine and Projective Solutions.* J. of the Amer. Math Soc. **22** (2009), 363–385.

[7] C. Beltrán, L.M. Pardo, *Fast linear homotopy to find approximate zeros of polynomial systems*, Found. of Comput. Math. **11** (2011), 95–129.

[8] I.N. Bernšteĭn, S.I. Gel'fand, *Meormorphy of the function P^{λ}*, Funktional. Anal. i Priložen **3** (1969), 84–85.

[9] L. Blum, F. Cucker, M. Shub, S. Smale, *Complexity and real computation*, Springer-Verlag, New York, 1998.

[10] J.M. Borwein, A. Straub, *Log-sine evaluations of Mahler measures.* J. Aust. Math. Soc. **92** (2012), 15–36.

[11] J.M. Borwein, A. Straub, J. Wan, W. Zudilin, *Densities of short uniform random walks. With an appendix by Don Zagier.* Canad. J. Math. **64** (2012), 961–990.

[12] J.B. Bost, H. Gillet, C. Soulé, *Heights of projective varieties and positive Green forms.* J. Amer. Math. Soc. **7** (1994), 903–1027.

[13] P. Bürgisser, F. Cucker, *On a Problem Posed by Steve Smale.* Annals of Math. **174** (2011), 1785–1836.

[14] P. Bürgisser, F. Cucker, *Condition: The Geometry of Numerical Algorithms*, Die Grundlehren der mathematischen Wissenschaften, Band **349**, Springer, 2013.

[15] L. Busé, J.P. Jouanolou, *On the discriminant scheme of homogeneous polynomials.* Math. in Comput. Sci. **8** (2014), 175-234.

[16] J. Cassaigne, V. Maillot, *Hauteur des hypersurfaces et fonctions Zêta d'Igusa.* J. of Numb. Theor. **83** (2000), 226-255.

[17] D. Castro, J.L. Montaña, J. San Martín, Luis M. Pardo, *The distribution of condition numbers of rational data of bounded bit length.* Foundations of Computational Mathematics **2** (2002), 1-52.

[18] D. Castro, M. Giusti, J. Heintz, G. Matera, L.M. Pardo, *The hardness of polynomial equation solving.* Found. Comput. Math. **3** (2003), 347–420.

[19] E. Cattani, A. Dickenstein, *Introduction to residues and resultants*, in A.Dickenstein, I.Z.Emiris (Eds.), Solving Polynomial Equations: Foundations, Algorithms, and Applications, Algorithms and Comput. Math. **14**, Springer-Verlag, 2005, 1–61.

[20] J.A. Cuesta-Albertos, M. Wschebor, *Some remarks on the condition number of a real random square matrix*, J. of Complex. **19** (2004), 548–554.
[21] C. D'Andrea, T. Krick, M. Sombra, *Heights of varieties in multiprojective spaces and arithmetic Nullstellensätze*, Annales Sci. de l'École Norm. Sup. **46** (2013), 549–627.
[22] A. Dickenstein, I. Z. Emiris (Eds), *Solving Polynomial Equations: Foundations, Algorithms, and Applications*, Algorithms and Computation in Mathematics 14, Springer, 2005.
[23] J.P. Dedieu, *Points fixes, zéros et la méthode de Newton*, Springer, 2006.
[24] M. Demazure, *Résultant, discriminant.* L' Enseignement Mathématique **58** (2012), 333–373.
[25] M. Einsiedler, *A generalization of Mahler measure and its application in algebraic dynamical systems* Acta Arithmetica LXXXVIII.1 (1999), 15–29.
[26] G. Everest, T. Ward, *Height of Polynomials and Entropy in Algebraic Dynamics*, Springer Verlag, 1999.
[27] G. Faltings, *Diophatine approximation on Abelian varieties*, Ann. of Math. **133** (1991), 549-576.
[28] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band **153**, Springer-Verlag New York Inc., New York, 1969.
[29] M. Fernández, L. M. Pardo, *An Arithmetic Poisson Formula for the Multi-variate Resultant.* J. of Complexity **29** (2013), 323–350.
[30] I. M. Gelfand, M. Kapranov, A. Zelevinsky, Discriminants, *Resultants and Multi-dimensional Determinants*, Modern Birkhäuser Classics, Springer, 2008.
[31] N.R. Goodman, *Statistical Analysis based on certain multi-variate complex Gaussian distribution (An Introduction).* Annals of Math. Statistics **34** (1963), 152–177.
[32] N.R. Goodman, *The Distribution of the Determinant of a Complex Wishart distributed matrix.* Annals of Math. Statistics **34** (1963), 178–180.
[33] P. Graczyk, G. Letac, H. Massam, *The Complex Wishart Distribution and the Symmetric Group*. The Annals of Stat. **31** (2003), 287–309.
[34] R. Howard, *Analysis on homogeneous spaces*, Class notes spring 1994. Royal institute of technology, Stockholm.
[35] O.H. Ibarra, S. Moran, *Probabilistic Algorithms for Deciding Equivalence of Straight-Line Programs.* Journal of the Assoclatton for Computing Machinery, **30** (1983), 217-228.
[36] J. P. Jouanolou, *Formes d'inertie et résultant: un formulaire.* Adv. in Math. **126** (1997), 119–250.
[37] S. Ji, J. Kollar, B. Shiffman, *A Global Łojasiewicz Inequality for Algebraic Varieties*, Trans. of the Amer. Math. Soc. **329** (1992), 813–818.
[38] T. Krick and L. M. Pardo, *A computational method for Diophantine approximation*, in Algorithms in Algebraic Geometry and Applications (Santander, 1994), Progr. Math. **143**, Birkhäuser, Basel, 1996, pp. 193-253.
[39] T. Krick, L. M. Pardo and M. Sombra, *Sharp estimates for the arithmetic Nullstellensatz.* Duke Math. J. **109** (2001), 521–598.
[40] N. Kurokawa, M. Lalín, H. Ochiai, *Higher Mahler measures and zeta functions.* Acta Arithmetica **153** (2008), 269– 297.
[41] P. Lelong, *Mesure de Mahler et calcul des constantes universelles pour les polynômes de N variables.* Math. Ann. **299** (1994), 673–695.
[42] D. Lind, K. Schmidt, T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups.* Inventiones mathematicae **101** (1990), 593–629.
[43] L. Mattner, *Complex differentiation under the integral.* Nieuw Arch voor Wiskunde **5/2** (2001),s 32–35.
[44] R.J. Muirhead, *Aspects of Multivariate Statistical Theory*, Wiley, New York, 1982.
[45] D. K. Nagar, A. K. Gupta, *Expectations of Functions of Complex Wishart Matrix.* Acta Applicandae Mathematicae **113** (2011), 265–288.
[46] S.Yu. Orevkov, *The fundamental group of the complement of a plane algebraic curve*, (Russian) Mat. Sb. (N.S.) 137(179) (1988), 260–270, 272; translation in Math. USSR-Sb. **65** (1990), 267–277.
[47] P. Philippon, *Critères pour l'indépendance algébrique* Inst. Hautes Études Sci. Publ. Math. **64** (1986), 5–52.
[48] P. Philippon, *Sur des hauteurs alternatives, I.* Math. Ann. **289** (1991), 255–283.
[49] P. Philippon, *Sur des hauteurs alternatives, II.* Ann. Inst. Fourier (Grenoble) **44** (1994), 1043–1065.
[50] P. Philippon, *Sur des hauteurs alternatives, III.* J. Math. Pures appl. **74** (1995), 345–365.
[51] G. Rémond, *Élimination multihomogène*, Chapter 5 of "Introduction to algebraic independence theory", Lecture Notes in Math. **1752**, Springer, 2001, pp. 53–81.
[52] G. Rémond, *Géométrie diophantienne multiprojective*, Chapter 7 of "Introduction to algebraic independence theory", Lecture Notes in Math. **1752**, Springer, 2001, pp 95–131.
[53] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Modern Birkhäuser Classics, Springer, 1995.
[54] M. S. Srivastava, *On the complex Wishart Distribution.* The Annals of Math. Statistics **36** (1965), 313–315.
[55] M. Shub, S. Smale, *Complexity of Bezout's Theorem I: Geometric Aspects.* J. of the Amer. Math. Soc. **6** (1993), 450–501.
[56] M. Shub, S. Smale, *Complexity of Bezout's Theorem II: Volumes and Probabilities.* Computational algebraic geometry (Nice, 1992), Progr. Math. **109**, Birkhäuser Boston, Boston, MA, 1993, pp. 267–285.
[57] M. Shub and S. Smale, *Complexity of Bezout's Theorem III: Condition number and packing.* J. Complexity **9** (1993), no.1, 4–14.

[58] M. Shub, S. Smale, *Complexity of Bezout's Theorem IV. Probability of success; extensions.* SIAM J. Numer. Anal. **33** (1996), 128–148.

[59] M. Shub, S. Smale, *Complexity of Bezout's Theorem V. Polynomial time.* Theoret. Comput. Sci. **133** (1994), 141–164,

[60] M. Shub, *Complexity of Bézout's Theorem VI. Geodesics in the condition (number) metric*. Found. Comput. Math. **9** (2009), 171–178.

[61] S. Smale, *Mathematical Problems for the Next Century*, Mathematics: Frontiers and Perspectives, Amer. Math. Soc., Providence, RI (2000) 271–294.

[62] T. Tao, *Topics in random matrix theory*, Graduate Studies in Mahtematics **132**, American Mathematical Society, 2012.

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