# Synchronizing Chaotic Systems with Positive Conditional Lyapunov Exponents by Using Convex Combinations of the Drive and Response Systems

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**Abstract**

In this paper we introduce a new method for synchronizing chaotic systems with positive conditional Lyapunov exponents, i.e., systems that do not synchronize in the Pecora-Carroll sense. This method works by considering a convex combination of the drive and response systems as a new driving signal. In this combination, the component associated with the response system acts as a chaos suppression method stabilizing the dynamics of the response system. This allows the chaotic component from the drive signal to synchronize both systems. The method is applied to synchronize some connections of the Rössler, Lorenz and Van der Pol-Duffing systems that do not synchronize using the Pecora-Carroll scheme.

## 1 Introduction

In the last decade many advances have been made in the field of nonlinear dynamics (see *e.g.* Jackson [1] for a survey), including the possibility of synchronizing chaotic systems. At first sight, this is not an obvious result, since these systems are very sensitive to small perturbations on the initial conditions and, therefore, close orbits of the system quickly become uncorrelated. However, in their seminal contribution Pecora and Carroll [2] showed that certain subsystems of chaotic systems can be synchronized by linking them with common signals. They consider the situation of unidirectional coupling, in which one has a drive-response coupling such that the chaotic signal from the drive system is injected into the response system in such a way that both systems become synchronized. They first considered the case of synchronizing two exact replicas of a given system (homogeneous driving) started from different initial conditions. Then, they also showed that synchronization is robust to small perturbations on the parameters of the drive or response systems. This



is an important result for experimental settings, where one does not usually have two exact replicas of a chaotic system. This situation is usually referred to as inhomogeneous driving.

Given a couple of autonomous $n$-dimensional chaotic systems, $\dot{x_1} = f(x_1)$ and $\dot{x_2} = f(x_2)$, as a drive and response system, the basic idea of the Pecora-Carroll (PC) scheme is decomposing the drive system in two subsystems,

$$\left.\begin{aligned} \dot{u_1} &= g(u_1, v_1) \\ \dot{v_1} &= h(u_1, v_1) \end{aligned}\right\} \text{ drive,} \tag{1}$$

where $x = (u, v)$, and considering one of the decomposed subsystems as drive signal, say $u_1$, to be injected into the response system. This reduces the dimensionality of the response becoming

$$\dot{v_2} = h(u_1, v_2) \;\} \text{ response,} \tag{2}$$

where $u_1$ is the set of connecting variables. Note that the system (1) is independent of the response system, whereas (2) is driven by $u_1(t)$ (unidirectional coupling). Then, the question is whether or not the subsystems $v_1$ and $v_2$ will synchronize, i.e., whether $|v_1(t) - v_2(t)| \to 0$, as $t \to \infty$. The answer to this question is given by the Lyapunov exponents of the difference system, $\dot{\Delta v} = h(u_1, v_1) - h(u_1, v_2)$, since they indicate if small displacements of trajectories are along stable or unstable directions. Lyapunov exponents of the $v_2$-subsystem for a particular drive trajectory are called *conditional Lyapunov exponents*. In the PC scheme, synchronization occurs if the conditional Lyapunov exponents are all negative.

The above scheme for a couple of systems can also be extended for a chain, or network, of chaotic systems. In this case, the response system (with respect to a given drive) acts as the drive of a second response system, and so on. Thus, a cascade of synchronized systems can be obtained. This setup has very interesting applications in the field of secure communications, as shown in [3] and [4].

One of the shortcomings of the PC scheme is the limited number of different connections, since not all the possible system decompositions lead to stable subsystems from the viewpoint of synchronization. In this paper we present a modification of this method that allows synchronizing drive-response connections that do not synchronize in the PC sense, that is, connections with positive conditional Lyapunov exponents. The key idea of the method is considering a convex combination of the drive and the response subsystems as the new driving signal. In this combination, the component associated with the response system acts as a chaos suppression method stabilizing the dynamics of this system. Then, the component associated with the drive imposes the behavior of the drive into the stabilized response synchronizing both systems.

In this paper, we shall consider the case of homogeneous driving, but similar results have been obtained when synchronizing non-identical systems. In Section 2, we introduce the convex combination method and compare it with the PC scheme. In Section 3, we apply the method to some of the chaotic systems originally considered by Pecora and Carroll [2]. We show that the convex combination method can synchronize some of the connections of the Rössler, Van der Pol-Duffing, and Lorenz systems having positive conditional Lyapunov exponents. Finally, Section 4 closes with the main conclusions and further remarks for future work.

## 2 The Convex Combination Method

We start by considering the drive system in (1) and a copy of this system decomposed in the same form:

$$\left.\begin{aligned} \dot{u_2} &= g(u_2, v_2) \\ \dot{v_2} &= h(u_2, v_2) \end{aligned}\right\} \text{ response.} \tag{3}$$

Now, instead of injecting the $u$-signal from the drive into the response (as in the PC scheme), we connect both systems by considering the following combination of the drive and response $u$-systems as the new driving signal:

$$\bar{u_2}(u_1, u_2) = u_2 + \epsilon\, \delta_{\Delta t,t}(u_1 - u_2). \tag{4}$$

where $\Delta t$ indicates the time step when the above signal is injected into the response system, i.e., when $u_2$ is given the value $\bar{u_2}$, $\delta_{\Delta t,t}$ is a Kronecker's delta, and $\epsilon$ is the combination parameter. That is, the variables of the response $u$-subsystem are perturbed by a factor $\epsilon(u_1 - u_2)$ at time steps $\Delta t$. Thus, for $\epsilon = 0$ the dynamics of both systems are independent and for $\epsilon = 1$ we recover the PC scheme (total driving). Values of $\epsilon$ between 0 and 1 give combined dynamics where drive and response are weighted by the factor $\epsilon$.

Note that (4) is equivalent to the convex combination

$$\bar{u_2}(u_1, u_2) = (1 - \epsilon\, \delta_{\Delta t,t})u_2 + \epsilon\, \delta_{\Delta t,t} u_1, \tag{5}$$



and, therefore, this method can be seen as the combination of two different factors, one associated with the response system, $\bar{u_2}(0, u_2) = (1 - \epsilon\, \delta_{\Delta t,t})u_2$, and other associated with the drive system, $\bar{u_2}(u_1, 0) = \epsilon\, \delta_{\Delta t,t} u_1$, such that $\bar{u_2}(u_1, u_2) = \bar{u_2}(u_1, 0) + \bar{u_2}(0, u_2)$.

The component associated with the response system applies proportional pulses to the system variables at time steps $\Delta t$. It has been reported in a recent work that chaotic systems can be stabilized by applying proportional pulses to the system variables [5] (see [6] for a general survey about controlling chaos). Thus, $\bar{u_2}(0, u_2)$ acts as a chaos suppression method. As we shall see, for some values of the $\epsilon$ parameter these perturbations stabilize the dynamics of the system, obtaining a fixed point. On the other hand, the component associated with the drive, $\bar{u_2}(u_1, 0)$, is the connecting term that induces the behavior of the chaotic drive into the stabilized response system. The effect of this combination is illustrated in the next section with several chaotic systems.

All the numerical work has been performed by using a fourth-order Runge-Kutta algorithm [7]. The time step has been carefully chosen in each case and the use of smaller time steps lead to results indistinguishable from those presented in this work. For the sake of simplicity, in this paper we consider $\Delta t$ to be equal to the integration step. However, we have experimentally checked that the results obtained for a given value of $\epsilon$ can be also reproduced by considering another values $\Delta t'$ and $\epsilon'$ that keep constant the ratio $\Delta t/\epsilon = \Delta t'/\epsilon'$ (in agreement with the results found for the stabilization method in [5]).

## 3 Results

In this section, we describe a series of applications of the method introduced in Sec. 2. We consider some continuous dynamical systems that have been previously used in the literature related to the synchronization of nonlinear systems: the Rössler, Van der Pol-Duffing, and Lorenz systems. First, we analyze the different PC connections of these systems. Then, we apply the new method (4) to those connections that do not synchronize in the PC sense.

### A. The Rössler model.

The Rössler model is a simple nonlinear vector field defined by the equations:

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned} \tag{6}$$

where $a, b$ and $c$ are the system parameters [8]. By fixing $a = b = 0.2$ and considering $c$ as the control parameter of the system, this model exhibits a period-doubling route to chaos, with a period-1 for $c = 2.6$, period-2 for $c = 3.5$ and so on. Larger values for $c$ yield to deterministic chaos in the form of a single-scroll strange attractor, excepting for the presence of a number of periodic windows.

In this paper, we consider the parameter value $c = 9$, exhibiting deterministic chaos. The conditional Lyapunov exponents corresponding to the three possible PC separations of this system are shown in Table 1. The first column of this table indicates the system and the corresponding parameter values, whereas the second and third column establish the kind of connection by giving the drive and response subsystems, respectively. This table shows that the only possible synchronizing system separation is given by the response system $(x, z)$, with drive signal $y$, i.e., $u = y$, $v = (x, z)$. Each of the other two connections has, at least, one positive conditional Lyapunov exponent.

Let us consider one of these connections, for example the one given by the drive signal $u = x$ and apply the convex combination method to this case. We consider two replicas $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ of (6) and establish the connection $\bar{x_2} = (1 - \epsilon)x_2 + \epsilon x_1$. Note that for $\epsilon = 1$ we recover the PC scheme and, therefore, both systems do not synchronize. On the contrary, for $\epsilon = 0$ both systems have independent dynamics. However, as $\epsilon$ increases the component $(1 - \epsilon)x_2$ acts by stabilizing some of the periodic behaviors embedded in the period-doubling route to chaos. This phenomenon is shown in Fig. 1(a) for values of $\epsilon$ in the range $(0, 0.0004)$. The time step that has been used in the integration is $\Delta t = 0.001$. Figure 1(b) shows the difference between the drive and response systems when both components of the method are applied. It can be seen that when the response system is stabilized to a fixed point, then the method synchronizes both systems.

Figure 2(a) shows the $y$-signal from the drive system, $y_1$, Figure 2(b) shows the response signal, $y_2$, and Figure 2(c) shows the difference between both signals for the value $\epsilon = 0.0004$ of the combination parameter. Each of these figures is divided in three different parts. First, from time 0 to 200, both the drive and the response systems evolve independently. Then, after the first dotted line,



Table 1
Conditional Lyapunov exponents for the possible Pecora-Carroll connections for the Rössler, Van der Pol-Duffing, and Lorenz models, obtained using the method proposed in [13].

| System | Drive | Response | Conditional LE |
|---|---|---|---|
| Rössler | $x$ | $(y, z)$ | $(+0.20, -8.89)$ |
| $a = b = 0.2$ and $c = 9$ | $y$ | $(x, z)$ | $(-0.06, -8.81)$ |
| | $z$ | $(x, y)$ | $(+0.10, +0.10)$ |
| Van der Pol-Duffing | $x$ | $(y, z)$ | $(-0.42, -0.42)$ |
| $\nu = 100$, $\alpha = 0.35$ and $\beta = 300$ | $y$ | $(x, z)$ | $(-0.01, -48.4)$ |
| | $z$ | $(x, y)$ | $(+4.31, -53.8)$ |
| Lorenz | $x$ | $(y, z)$ | $(-1.81, -1.84)$ |
| $\sigma = 10$, $b = \frac{8}{3}$ and $r = 60$ | $y$ | $(x, z)$ | $(-8/3, -10)$ |
| | $z$ | $(x, y)$ | $(0, -11.01)$ |

the synchronization method is switched on by only considering the component associated with the response signal, i.e., $\bar{x_2} = (1-\epsilon)x_2$. From this figure we can see that, due to this perturbation, the system is stabilized into a fixed point. After the second dotted line, both components of the synchronization method are applied and the systems quickly become synchronized. It is important to remark here that in this example, the method is applied in two steps by illustrative purposes, but the same results are obtained if we initially apply the whole method, without first stabilizing the system into a fixed point.

### B. The Van der Pol-Duffing model.

The Van der Pol-Duffing (VPD) oscillator is given by

$$\begin{aligned}\dot{x} &= -\nu\left[x^3 - \alpha x - y\right] \\ \dot{y} &= x - y - z \\ \dot{z} &= \beta y\end{aligned} \tag{7}$$

This circuit can be easily implemented as an analog circuit resembling Chua's circuit (see [9] for a description of this system). The main difference between both systems consists of the replacement of the piecewise nonlinear element characteristic of Chua's circuit by a cubic nonlinearity, which can be physically constructed by using a set of diodes and an operational amplifier.

A numerical analysis of Eq. (7) shows that, fixing the values of $\nu$ and $\alpha$ ($\nu = 100$ and $\alpha = 0.35$), the system exhibits period-doubling bifurcation route to chaos as the parameter $\beta$ is decreased from $\beta \approx 500$. In particular, we have chosen the value to work at $\beta = 300$, beyond the period-doubling bifurcation cascade, which exhibits deterministic chaos.

The conditional Lyapunov exponents for the different PC connections of a couple of VPD systems are given in Table 1. It can be shown that the connections given by $u = x$ and $u = y$ synchronize in the PC sense. On the other hand, the connection $u = z$ has one positive conditional Lyapunov exponent. When we apply the convex combination method to this case, we obtain similar results as those for the Rössler system. For example, Figures 3(a) and 3(b) shows the $x$-signals from the drive and response systems, respectively, for the value $\epsilon = 0.001$ of the combination parameter. The integration step has been chosen $\Delta t = 0.0001$. As in Fig. 2, each of the figures is divided in three different parts. First, both systems evolve independently. Then, the synchronization component of the method is switched on, stabilizing the system into one of the fixed points (the negative one). Finally, the synchronization method is applied and the systems become synchronized.

**C. The Lorenz model.**
The Lorenz model is a three-dimensional flow defined by the equations [10]:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{8}$$

The dynamics of this system is determined by its steady-state solutions. For example, fixing the parameters $\sigma = 10$ and $b = \frac{8}{3}$ this system presents a number of different behaviors when varying the $r$ parameter [11]. For $0 < r < 1$ all initial conditions eventually decay to the equilibrium at the origin. For $1 < r < 24.74$, the origin becomes unstable and bifurcates into a pair of locally attracting solutions. Then, depending on the initial conditions, the solution of (8) settles into either a chaotic motion or into one of the stable equilibrium points. For $r > 24.74$ the orbit is chaotic for all initial conditions, although for very large values of $r$, there are alternating regimes of turbulent and periodic behavior.

For a chaotic regime corresponding to the value $r = 60$, this system admit two synchronizing PC separations given by $u = x$ and $u = y$, respectively



(see Table 1). However, the separation $u = z$ has associated a zero conditional Lyapunov exponent and no synchronization is achieved with this connection. Figure 4 shows the result of applying the synchronization method in (4) with $\epsilon = 0.0005$ and integration step $\Delta t = 0.0001$. Figures 4(a) and 4(b) show the time series of the drive and response system, and Figure 4(c) shows the difference of both signals. Each of these figures is divided in three different parts, corresponding to the independent, stabilizing and synchronizing zones. From this figure we can see that due to this perturbation the system is stabilized into one of the two fixed points (the negative one). Afterwards, both components of the synchronization method are applied and both systems quickly become synchronized.

The symmetry $(x, y, z) = (-x, -y, z)$ of the system (8) produces an interesting effect when synchronizing a couple of Lorenz systems by using the convex combination method with $u = z$ and $v = (x, y)$. Due to this symmetry synchronization may be achieved between the system $(x_1, y_1)$ and either $(x_2, y_2)$ (Fig. 4) or $(-x_2, -y_2)$. The last case is shown in Figure 5, that has been obtained as Fig. 4, but starting from different initial conditions. From this figure it can be seen that, now, $|x_1 + x_2| \to 0$ and $|y_1 + y_2| \to 0$.

## 4 Conclusions and Further Remarks

One of the shortcomings of the Pecora-Carroll (PC) scheme is the limited number of different connections, since not all the possible subsystems are stable from the viewpoint of synchronization. Then, an interesting idea consists of generalization of the PC method, increasing the number of synchronizing connections. In this paper we introduce a new synchronization method that combines both chaos suppression and unidirectional coupling strategies to synchronize chaotic systems. This idea of combining these two strategies has shown to be very efficient. Another interesting extension of the PC method has been considered by Güémez and Matías (GM) in [12]. In this modified method, the synchronization has been achieved by introducing the driving signal at a given place of the evolution equations of the response. As a consequence, it is possible to regenerate the input signal within a single connection. It implies the possibility of using twice the same connection without alternating with another one and, therefore, a number of different networks with different connectivities can be set up. Preliminary results show that the convex combination method also extends the applicability of this method by synchronizing subsystems that do not synchronize with the GM method.

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FIGURE 1. Chaotic synchronization of two Rössler models (6) with connection $\bar{x}_2(x_1, x_2)$ in (5) as a function of the parameter $\epsilon$: (a) Bifurcation diagram of the response system when only applying the stabilizing component of the method, $\bar{x}_2 = (1-\epsilon)\, x_2$. (b) Difference between drive and response systems. The parameters are $a = 0.2$, $b = 0.2$, and $c = 9$.

FIGURE 2. Time series for the $y$ variable of a drive-response couple of two identical Rössler systems started from different initial conditions and connected as shown in Fig. 1 with the parameter value $\epsilon = 0.0004$ and integration step $\Delta t = 0.001$. After the first dashed line, only the stabilizing component of the method, $\bar{x}_2 = (1-\epsilon)\, x_2$ is applied and the response system is stabilized to a fixed point. After the second dashed line the component associated with the drive is also applied, $\bar{x}_2 = (1-\epsilon)\, x_2 + \epsilon\, x_1$, and synchronization is rapidly achieved.

FIGURE 3. Time series for the $x$ variable of two identical Van der Pol-Duffing systems showing the result of applying the stabilizing component and the whole synchronization method in (5) with connection $\bar{z}_2(z_1, z_2)$, after the first and second dotted lines, respectively. The value of $\epsilon$ is 0.001 and the integration step has been chosen $\Delta t = 0.0001$.

FIGURE 4. Time series for the $x$ variable of two Lorenz systems coupled by using the synchronization method in (5) with the connection given by $\bar{z}_2(z_1, z_2)$. After the first dashed line, only the stabilizing component of the method is applied and the response system is stabilized to a fixed point. After the second dashed line the whole method is applied, $\bar{z}_2 = (1-\epsilon) z_2 + \epsilon z_1$), and synchronization is rapidly achieved. The value of $\epsilon$ is 0.0005 and the integration step has been chosen $\Delta t = 0.0001$.

FIGURE 5. Symmetric synchronizing of the response systems of two Lorenz models coupled in the same conditions as in Fig. 4. The signals of the drive, $z$, and response, $(x, y)$, systems are shown to illustrate how synchronization is achieved in this case.