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Suppression of chaos through changes in the system variables through Poincaré and Lorenz return maps

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Abstract

A new chaos suppression method, able to stabilize flows by perturbing the system variables through both proportional and additive changes in a selected Poincaré section is presented. The method can use both Poincaré and Lorenz return maps. As an application, deterministic chaos has been suppressed in two different three variable flows: Rössler spiral chaos model and an isothermal three-variable autocatalator model introduced by Peng *et al.* Differences and similarities with other previous chaos stabilization methods are discussed.

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1 Introduction

In the last two decades many researchers have studied the behavior of many non-linear dynamical systems (see *e.g.* [1]), including the appearance of deterministic chaos, that is characterized by the sensitive dependence on the initial conditions of a trajectory of the system. In the case of dissipative systems described through nonlinear ordinary differential equations and exhibiting deterministic chaos, the system evolves in phase space over a fractal structure called strange or chaotic attractor, that consists of a continuous stretching and folding process. Deterministic chaos is usually associated with wild oscillations in the variables that may be inconvenient in practical applications of the systems exhibiting these features and, thus, for many applications one would like the system to have a more regular behavior. Although a chaotic system may appear at first sight to exhibit stochastic behavior, its deterministic nature implies that it has some regularity in its inner structure. In fact, it has been shown [2] that a chaotic system may bee seen as the superposition of a large number of unstable periodic orbits (also called cycles). The idea of controlling chaos [3] consists precisely in the stabilization of any of these orbits embedded in the strange attractor. As a result, this flexibility is changing the bad reputation of chaotic systems [4, 5].

Different classes of methods have been suggested in the literature for the suppression of chaotic behavior in non-linear systems. Within the framework of the original method, due to Ott, Grebogi, and Yorke (OGY) [3, 6], one uses a Poincaré section of the system in order to reduce in one the dimensionality of the system, skipping the direction having Lyapunov exponent equal to zero. The fixed point of the return map corresponding to the desired orbit of the flow is then stabilized by means of a linear approximation in a trust ball around this point. In practice, this is carried out by exploiting the saddle character of these unstable periodic orbits, namely by making the trajectory to lie at each crossing on the stable direction, called stable manifold, of the target orbit. In any case, these new ideas yield a regular behavior with less deformation of the target system than more standard control techniques [7, 8] are able to achieve. In other varieties of the OGY method, one takes advantage of the dissipative character of the dynamics to work with a one-dimensional return map [9, 10], while more refined techniques like the OPF method [11] also exist. In another set of methods one uses a harmonic signal to stabilize a periodic orbit in a resonant fashion [12, 13], although other signals have been also used [14]. It is also possible to perform non-resonant chaos suppression through fast modulation of the driving signal [15]. Other techniques that are more difficult to classify have been also suggested [16, 17].

All the above mentioned methods work by applying perturbations on some system parameter. However, in recent work [18] a new method that works by performing changes in the system variables has been suggested and applied to the case of some typical three-dimensional flows, although its capabilities to deal with maps, *i.e.* difference equations, have been also proved [19, 20]. An implementation of the OGY method in which time-dependent changes are applied on the system variables has been recently suggested [21].

In the case of flows one of the useful features of the OGY method is that the perturbations are applied in a natural time scale of the flow, namely each time that a crossing with the Poincaré plane occurs. Instead, in the case of the method introduced in [18] an arbitrary time scale for the perturbations is introduced. Thus, one may wonder whether the application of still non-specific changes in the system variables through a return map will be more efficient in achieving chaos suppression, in the sense that smaller perturbations need to be applied, and this is the aim of the present publication. If the method proves to be robust enough, its main virtue can be that no prior knowledge of the system is needed, unless the OGY method (although a variant of this latter method working directly with a time series has been suggested [22]).

The present paper is organized as follows. In Section 2 the method to be used in the rest of the work will be defined in two different varieties: the original proportional method introduced in [18] and a new version, that works by applying additive changes in the system variables, and that needs no information about the values of the variables. Section 3 will contain the application to two different three-dimensional flows: Rössler's model of spiral chaos [23] and the three-variable isothermal autocatalator introduced by Peng *et al.* [24]. Finally, the main conclusions are gathered in Section 5.

2 Chaos suppression methods

As already discussed in the Introduction, the aim of this work is to analyze the performance of the recently introduced chaos suppression method [18] acting on the system variables, but now every time the flow crosses a conveniently chosen Poincaré cross section. Some applications have been already presented for the case of one-dimensional [19] and two-dimensional [20] iterated maps. Due to the success of this method in the discrete case, one is tempted to try apply these ideas to chaotic flows of nonlinear differential equations taking into account a natural periodicity of the system, rather than in the initial application [18] in which the time interval in which the perturbations are applied is quite arbitrary.

The method applied in the present work is based on the reduction of the dimensionality of the system by choosing an appropriate Poincaré cross section that is orthogonal to the flow, such that the non-interesting direction in which the system neither expands nor contracts is skipped and the continuous flow is transformed into a simpler difference equation, although the latter is not defined analytically, but only numerically. This idea is analogous to the one used within the OGY procedure, and like this method it can only be applied to the stabilization of unstable orbits immersed in the strange attractor, that are transformed into fixed points of a difference equation. Thus, the method presented here cannot stabilize fixed points of the flow, that appear in the case of the Lorenz model [25], in opposition to the method introduced in [18]. In fact, for the Lorenz model a Poincaré section dividing the domains of attraction of the two unstable fixed points cannot be defined [26], because there is no upper bound to the time spent by a trajectory in each of these regions.

We shall consider the chaos suppression method in two different varieties, depending on the form in which the perturbations on the system variables are applied. In the first method proportional pulses are applied [20], having the form,

$$x_i' = x_i\,(1 + \lambda_i) \tag{1}$$

where x_i represents the value of one of the variables entering in the map and λ_i defines the strength of the pulse for variable x_i, that can be either positive or negative. In this method the strength of the perturbation depends on the position of the system in phase space. The second method tries to offer a simpler alternative through the use of additive pulses,

$$x_i' = x_i + \gamma_i \tag{2}$$

independent of the position of the system in phase space, with analogous meaning for the symbols. A final note to say that in the original applications to difference equations the possibility of not applying the perturbations at every cycle was considered, although applying them at each cycle one could stabilize virtually every different periodic behavior, while in the present work the algorithm is applied at every crossing with the Poincaré map. All the numerical work has been performed by employing a fourth-order Runge-Kutta algorithm [30]. The timestep length has been carefully chosen in order to avoid spurious behavior.

3 Applications

3.1 Rössler model of spiral chaos

In this Section we shall consider two different models written as a set of coupled differential equations and that exhibit deterministic chaos for certain values of the parameters. Rössler's model of spiral chaos [23]

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \tag{3}$$

is one of the simplest representations of deterministic chaos, as it has a single quadratic non-linearity that couples x with z in $\dot{z}$. When z takes a low value the nonlinear term is not important and the system can be transformed in the second-order linear oscillator $\ddot{x} - a\dot{x} + x = 0$, that for the values $a > 0$ used in this work presents negative damping, being the system an unstable focus for $0 < a < 2$ (see *e.g.* [27]). This implies that initial conditions spiral out of the center, being the aim of the non-linear term to break this explosion when x reaches a critical value $x_c = c$), and in this moment the flow is reinjected to the inner part (see *e.g.* [28]), having this system all the typical ingredients of stretching and folding in chaotic systems. In this work the following parameter values exhibiting deterministic chaos have been chosen: $a = 0.2$, $b = 0.2$, and $c = 4.6$, while the plane chosen to define the Poincaré section has the coordinates $y = 0, x < 0$. Fixing a and b to these values and considering c as the control parameter the model exhibits a Feigenbaum's period-doubling route to chaos.

The results of the application of the chaos suppression algorithm based on proportional perturbations on the system variables (1) are shown in Figure 1. The Poincaré section has been defined in such a way that $x = 0$ and $\dot{x} > 0$, *i.e.* only one of the possible crossings of the flow with the Poincaré cross-section $x = 0$ is considered, and the perturbation (1) is applied on both y and z. Depending on the strength of the applied perturbation different stable orbits can be obtained, and, in particular, in Figure 1 a period–2 and a period–4 orbit, respectively, are represented. The vertical dashed line signals the moment at which the chaos suppression algorithm starts being applied. Other orbits belonging to the subharmonic cascade, *e.g.* period–8, can be obtained, being this in agreement with the route to chaos exhibited by this system and the correspondence of the stabilized orbits with stable orbits of the system with different values of the parameters [20]. Moreover, it is also possible to obtain orbits with different periodicities, like 3 and 5, that may correspond to periodic windows of the original system born in saddle-node bifurcations.

A more detailed view of the way in which the method works can be seen from Fig. 2, in which a view of the way in which the flow crosses the Poincaré section is shown for the stabilized period–4 orbit of Fig. 1(b). The discrete character of the stabilized orbits at the crossing can be easily noticed. Fig. 3(a) shows the Poincaré map itself $(y_{n+1}, z_{n+1}) = f(y_n, z_n)$, that is a very thin structure, although it comprises many layers created in the stretch and fold process. The squares in the map represent the location of the points of the stabilized orbit. The essentially one-dimensional character of the map allows one to consider a true one-dimensional return map, that taken into account the narrow range comprised by variable z, is defined in the form $y_{n+1} = g(y_n)$, as shown in Fig. 3(b).

This type of map, whose validity does not rely on the previous definition of a Poincaré cross section is also called Lorenz map, as it was first introduced by this author [25]. Thus, it is natural to think of a variant of the previously introduced method that acts on both variables of the Poincaré map, namely by applying perturbations on y, once at each return of the map. Fig. 4 contains an example of a stabilized period–4 orbit, by representing the value of y at each crossing with the cross section. This orbit is represented with squares in Fig. 3(b). It can be seen how the points of the stabilized periodic orbit do not belong to the original chaotic set, but rather are somehow shifted. This is compatible with the explanation anticipated in previous work [20], that pointed out the fact that the stabilized periodic behavior is equivalent to that obtained for different parameters of the system, presenting this situation analogies to the nonresonant stabilization method introduced in Ref. [15]. In the previous situation, represented in Fig. 3(a), this shift would be only in the line defined by the map.

On the other hand, Figure 5 displays results corresponding to the application of the second method based on the application of additive perturbations to the system variables (2), and, in particular, to the case of a stabilized period–4 orbit. The large disparity among the values for which the y and z dynamics take place would imply that if one chooses the same value of γ for both variables the performance of the method would not be quite good.

Thus, the perturbations are only applied to y (with $\gamma = 0.9$). Fig. 6 shows the Poincaré and Lorenz maps for the additive method (2), with the squares representing again the position of the stabilized periodic orbits, being the results analogous to those shown in Fig. 3.

3.2 Autocatalator model

As a second application we have considered the isothermal three-variable autocatalator model put forward by Peng, Scott, and Showalter [24], that can be written in the following reduced form,

$$\begin{aligned} \dot{x} &= \kappa + \mu z - xy^2 - x \\ \sigma\, \dot{y} &= xy^2 + x - y \\ \delta\, \dot{z} &= y - z \end{aligned} \tag{4}$$

Following [24] μ has been chosen as the control parameter, taking the other parameters the following values: $\kappa = 10, \sigma = 0.005$ and $\delta = 0.02$. For the value $\mu = 0.154$ deterministic chaos appears in the form of a remerging tree [24], being the model flexible enough to exhibit mixed-mode oscillations, isolas, etc. [29]. The Poincaré plane that has been used in this work in order to stabilize periodic behavior is the same one already used by Petrov *et al.* [10], namely $z = 15, \dot{z} > 0$.

As a first application to this model, Fig. 7 contains the result corresponding to the application of the algorithm based on proportional perturbations (1) to the stabilization of a period–2 orbit. The corresponding Poincaré return map $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$ can be seen in Fig. 8(a). Analogously to the situation observed for the Rössler model in Sec. 3.1, the strongly dissipative nature of the flow makes that the Poincaré a very thin layer, and this is also implied by the narrow range spanned by variable x. For this reason, it is a sensible approach to consider only variable y, and this allows one to draw the corresponding one–dimensional return (Lorenz) map found in Fig. 8(b). Thus, the additive method (2) has been also applied to this model, but only variable y. Fig. 9 shows a period–4 orbit obtained with this method.

4 Effects of noise

In order that it can be effectively considered that a given chaos suppression is useful having in mind practical applications, one must first check whether the method under study is robust against the presence of several sources of external noise. In this section results are presented for the study carried out for the Rössler model [23]. External noise has been generated through a stochastic variable with gaussian distribution having zero mean and standard deviation equal to one, $\sigma[0,1] = N(0,1)$, generated by using the Box–Müller method [31], and introduced in the form,

$$x' = x\,(1 + \nu\,\sigma[0,1]) \tag{5}$$

for the case of multiplicative noise, where the noise is applied at each Runge–Kutta integration step.

Fig. 10 shows the results obtained by considering the presence of this type of noise for the case of both the multiplicative (1) and additive methods (2). In this sense, this Figure should be considered the counterpart of Figs. 1(a) and 5(a) in the presence of multiplicative noise. Regarding the effect of the presence of additive noise in both methods, and in agreement with the results reported in [18], the additive method is stronger under the presence of this noise. However, notice that this latter method (2) has been only applied to variable y, because of the narrow range of values in the other variable in the Poincaré map. If the noise intensity is increased the outcome of the stabilization method starts being less regular and the system starts travelling again to the chaotic region, in line with the well known effects of external noise [32], as can be seen in Fig. 11 for both methods.

To complete the study, a different source of noise has been considered, namely by allowing the stabilization algorithm to vary randomly within some defined range. Or in other words, by allowing the quoted values of λ, for the case of the proportional method (1), to be given by the expression

$$\lambda' = \lambda\,(1 + \nu\,\sigma[0,1]) \tag{6}$$

and analogously for γ in the case of the additive method (2). Examples of the robustness of the method in the presence of this type of noise can be found in Fig. 12 for both the proportional and additive methods. In particular, Fig. 12(c) shows an example of a burst for the case of the additive method, *i.e.* an example of transient behavior, after which the method is able to recover and continue with the periodic behavior.

5 Conclusions

In this work several versions of the recently introduced chaos suppression method through changes in the system variables [18] have been discussed and applied to the case of two different chaotic flows of three–dimensional differential equation systems. The methods work by reducing the system dimensionality through a suitably chosen Poincaré cross section of the system, *i.e.* a plane that is perpendicular to the flow. Several versions of the method have been implemented, namely by considering both proportional and additive changes in the system variables. The strongly dissipative nature of the chaotic flows studied in this work implies that most of the dynamics takes place on one dimension, that corresponds to the unstable manifold of a fixed point from which the strange attractor is born, being this the dimension in which the stretching part of the flow takes place. Instead, the attractor is only a superposition of very thin layers in the other direction, corresponding to the folding part of the flow. Thus, in addition to the two–dimensional Poincaré map, a one–dimensional Lorenz return map has been also considered, and, thus, the perturbations are applied only to a single variable.

The flexibility of the method to select different periodic behaviors has been shown, and also its robustness under the presence of some source of external noise. The application of the perturbations taking into account a natural timescale of the system makes the method more efficient from the viewpoint of the total perturbations applied along a whole cycle with the respect to the Poincaré plane. Moreover, the relationship of the stabilized periodic orbits to orbits corresponding to slightly different parameters of the original system has been shown. The present method might be useful in the stabilization of periodic behavior of some chemical and biological systems, for which the introduction of perturbations in the system variables may advantageous compared to the application of the usual techniques based on parametric perturbations.

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Figure 1: Stabilized orbits for the Rössler model (3) with $a = 0.2$, $b = 0.2$, and $c = 4.6$, obtained by applying the proportional method (1) to both y and z each time that the system crosses the Poincaré section $x = 0$ and $\dot{x} > 0$ by using: (a) period–2 orbit obtained with $\lambda = -0.12$; (b) period–4 orbit obtained $\lambda = -0.07$. The application of the algorithm is signaled by a vertical dashed line.

Figure 2: Side view of the effect of the chaos suppression method (1) on the condition represented in Fig. 1(b). The noncontinuous character of the stabilized flow at the crossing can be clearly appreciated.

Figure 3: (a) Poincaré 2-D return map $(y_{n+1}, z_{n+1}) = f(y_n, z_n)$ for the Rössler model (see Fig. 1 for further details). The squares represent the stabilized period–4 orbit of Fig. 1(b) ($\lambda = -0.07$). (b) Lorenz 1-D return map $y_{n+1} = g(y_n)$ for the same situation. The squares represent the stabilized period–4 orbit obtained by applying the same perturbations only on y (see also Fig. 4).

Figure 4: y_n vs. n for the one-dimensional Lorenz return map of the Rössler model (see Fig. 1 for further details). A stabilized period–4 orbit obtained by applying perturbations only on y (displayed in a different form also in Fig. 3(b)) is represented ($\lambda = -0.07$).

Figure 5: Stabilized period–4 orbit for the Rössler model (see Fig. 1 for further details), by using the additive method (2), and by applying the perturbation to y each time that the system crosses the Poincaré section $x = 0$ and $\dot{x} > 0$: (a) period–2 orbit obtained with $\gamma = 1.5$; (b) period–4 orbit obtained with $\gamma = 0.9$.

Figure 6: (a) Poincaré 2-D return map $(y_{n+1}, z_{n+1}) = f(y_n, z_n)$ for the Rössler model (see Fig. 1 for further details). The squares represent the stabilized period–4 orbit of Fig. 5. (b) Lorenz 1-D return map $y_{n+1} = g(y_n)$ for the same situation. The squares represent the stabilized period–4 orbit obtained by applying additive perturbations only to y with $\gamma = 0.9$.

Figure 7: A stabilized period–2 orbit of the three-variable isothermal autocatalator model (4) with $\kappa = 10$, $\delta = 0.02$, $\sigma = 0.005$, and $\mu = 0.154$ obtained with method (1) by using $\lambda = -0.2$, applied to both x and y and the Poincaré section defined by $z = 15$, $\dot{z} > 0$. The application of the algorithm is signaled by a vertical dashed line.

Figure 8: (a) Poincaré 2-D return map $(y_{n+1}, z_{n+1}) = f(y_n, z_n)$ for the autocatalator model (see Fig. 7 for further details). The squares represent the stabilized period–2 orbit of Fig. 7(a) ($\lambda = -0.2$). (b) Lorenz 1-D return map $y_{n+1} = g(y_n)$ for the same situation, excepting that the perturbations, also with $\lambda = -0.2$, are applied only to y (the squares represent the stabilized period–2 orbit).

Figure 9: A stabilized period–4 orbit of the autocatalator model (see Fig. 7 for further details), obtained by using method (2) with $\gamma = 3.9$. The application of the algorithm is signaled by a vertical dashed line.

Figure 10: Stabilized period–2 orbits for the Rössler model (see Fig. 1 for further details) in the presence of multiplicative noise (5) with $\nu = 0.001$: (a) proportional method (1) with $\lambda = -0.12$ (see Fig. 1(a)); (b) additive method (2) with $\gamma = 1.5$ (see Fig. 5(a)).

Figure 11: Stabilized period–2 orbits for the Rössler model (see Fig. 1 for further details) in the presence of multiplicative noise (5) with $\nu = 0.0015$: (a) proportional method (1) with $\lambda = -0.12$ (see Fig. 1(a)); (b) additive method (2) with $\gamma = 1.5$ (see Fig. 5(a)).

Figure 12: Stabilized period–2 orbits in the presence of multiplicative noise in the parameter given by (6) for periodic orbits of the Rössler model (3) (see Fig. 7 for further details), obtained with the method acting on the y Lorenz return map: (a) proportional method (1), $\mu = 0.12$; (b) additive method $\mu = 0.15$; (c) additive method (2) $\mu = 0.4$.