A Multifractal Analysis of IFSP Invariant Measures With Application to Fractal Image Generation[1]

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Abstract

In this paper, we focus on invariant measures arising from Iterated Function System with Probabilities (IFSP). We show the equivalence between an IFSP and a linear dynamical system driven by a white noise. Then, we use a multifractal analysis to obtain scaling properties of the resulting invariant measures, working in the framework of dynamical systems. Finally, as an application to fractal image generation, we show how this analysis can be used to obtain the most efficient choice for the probabilities to render the attractor of an IFS by applying the probabilistic algorithm known as "chaos game".

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1 Introduction

In recent years, it has been shown that fractals appear naturally in the mathematical formulation of many nonlinear phenomena in many fields. The Iterated Function System (IFS) model provides a convenient framework for the description, classification and generation of fractals. IFS models are particularly interesting due to their mathematical soundness [2] and simplicity: an IFS model is generated by a finite number of contractive lineal transformations. For a comprehensive introduction to this field we refer to [2, 3, 5].

In this paper, we introduce a multifractal analysis to characterize and classify the invariant measure arising from an IFS with Probabilities (IFSP), working in the framework of dynamical systems. To do this we show the equivalence between an IFSP and a linear dynamical system driven by a white noise. For example, the Markov operator of the IFSP is equivalent to the evolution operator of the master equation of the dynamical system but, in general, all properties of the IFSP can be translated to properties of random dynamical systems and vice versa. Finally, we show an application of this analysis in the field of fractal image generation. There are several well known methods for rendering fractal images based on IFS. Among them, the probabilistic algorithm, also known as "chaos game", is gaining more and more popularity due to its efficiency and conceptual simplicity [3]. We show that the efficiency of this algorithm is closely related to the multifractal properties of the invariant measure arising on the corresponding IFSP. Thus, the best choice for the probabilities can be calculated as the values that lead a narrow multifractal spectrum.

The paper is organized as follows. In Sec.2 we introduce the basic notation and definitions concerning IFS and IFSP. In Sec.3 we show the equivalence between an IFSP and a linear dynamical system driven by a white noise. We then construct a set of measures, which we call restricted measures arising from conditioned processes in the evolution of the dynamical system. In Sec.4, we deal with the multifractal analysis of the invariant measure of an IFSP. With the use of restricted measures, a complete multifractal analysis can be performed in the case in which the functions forming the IFSP are similitudes satisfying the open set condition. Finally, in Sec.5 we show a direct application of the multifractal formalism to fractal image generation. The possibility of choosing the most efficient set of probabilities of an IFSP for generating a fractal image is discussed.

2 Basic Notation and Framework

Before introducing the random dynamical system, we need to consider the background of IFS and IFSP [1]-[3]. An IFS consists of a complete metric space (X, d) and a finite set of linear transformations $w_i : X \longrightarrow X,\ i = 1, \ldots, n$. We refer to the IFS as $\mathcal{W} = \{X; w_1, \ldots, w_N\}$. Let us define a transformation, T, in the compact subsets of X, $\mathcal{H}(X)$, by

$$T(A) = \bigcup_{i=1}^{N} w_i(A). \tag{1}$$

If all the w_i are contractions, T is also a contraction in $\mathcal{H}(X)$ with the induced Hausdorff metric. Then, T has a unique fixed point, $|\mathcal{W}|$, called the *attractor of the IFS*. Considering a set of probabilities $p_1, \ldots, p_N \in (0,1)$, with $\sum_{i=1}^{n} p_i = 1$, $|\mathcal{W}|$ supports several measures in a natural way. We refer to $\{X; w_1, \ldots, w_N; p_1, \ldots, p_N\}$ as an IFS with Probabilities (IFSP). Given a set $\{p_1, \ldots, p_N\}$, there exists an unique Borel regular measure $\nu \in \mathcal{M}(X)$, called the *invariant measure of the IFSP*, such that

$$\nu(S) = \sum_{i=1}^{N} p_i \nu(w_i^{-1}(S)), \quad S \in \mathcal{B}(X),$$

where $\mathcal{B}(X)$ denotes the Borel subsets of X. This fact can be easily shown [2] through the introduction of the Markov operator. The Markov operator associated to the IFSP is a transformation, $M : \mathcal{M}(X) \longrightarrow \mathcal{M}(X)$ defined by:

$$(M\nu)(S) = \sum_{i=1}^{N} p_i \nu(w_i^{-1}(S)), \tag{2}$$

where S is a Borel subset of X. Using the Hutchinson metric on $\mathcal{M}(X)$, it is possible to show that M is a contraction with a unique fixed point, $\nu \in \mathcal{M}(X)$. Furthermore, *support*$(\nu) = |\mathcal{W}|$. Thus, given an arbitrary initial measure $\nu_0 \in \mathcal{M}(X)$ the sequence $\{\nu_n\}_{n=0,1,2,\ldots}$ constructed as $\nu_{n+1} = M(\nu_n)$ converges to the invariant measure of the IFSP. Also, a similar iterative deterministic scheme can be derived from Eq.(1) to obtain $|\mathcal{W}|$. However, there exists a more efficient method, known as probabilistic algorithm, for the generation of the attractor of an IFS. This algorithm follows from the result $\overline{\{x_n\}}_{n>0} = |\mathcal{W}|$ provided that $x_0 \in |\mathcal{W}|$, where $x_n = w_i(x_{n-1})$ with probability $p_i > 0$ (see, for example, [4]). In Sec.5 we analyze the efficiency of this algorithm by using the multifractal analysis developed in the next sections.

3 Equivalence Between an IFSP and a Random Dynamical System

Going back to the introduction of IFSP as a random dynamical system, we first introduce a discrete stochastic process formed by a sequence of i.i.d. random variables, ξ_n, $n \in \mathbb{N}$, which can take values in the set $\{1, \ldots, N\}$ with probability distribution $P(\xi_n = j) = p_j$, $\forall n \in \mathbb{N}$, where p_i are the probabilities of the IFSP; i.e. $P(\xi_n) = \sum_{i=1}^{N} p_i \delta_{i,\xi_n}$. The probability measure, P, associated with this white process can be determined by defining the joint probabilities $P(\xi_1, \xi_2, \ldots)$. These joint probabilities factorize as:

$$P(\xi_1, \xi_2, \ldots, \xi_n) = P(\xi_1)P(\xi_2)\ldots P(\xi_n)$$

Once the process has been defined we can proceed to formulate the stochastic equations of motion. The random dynamical system equivalent to the IFSP is governed by the stochastic

equation:

$$x_n = w_{\xi_n}(x_{n-1}) \tag{3}$$

where w_n are the contractive applications of the IFSP and ξ_n is the stochastic process defined above. Some equivalences between the IFSP and the random process are evident. The stochastic orbit $\{x_n\}_{n=0}^{\infty}$ is equivalent to the sequence obtained from the IFSP applying the random algorithm to x_0. Given an initial probability density $P_0(x)$, the probability at time n, $P_n(x)$, follows a forward master equation:

$$P_n(x) = \sum_{i=1}^{N} p_i \int \delta(x - w_i(y)) P_{n-1}(y) dy \tag{4}$$

which with invertible applications can be integrated to give:

$$P_n(x) = \sum_{i=1}^{N} p_i c_i^{-1} P_{n-1}(w_i^{-1}(x)), \tag{5}$$

where c_i is the jacobian of the transformation $x = w_i(y)$. The contractive character of the IFSP leads to a dissipative character of the dynamical system. This property and the linearity of the applications w_i guarantee the existence and unicity of the invariant density as the limit of P_n as $n \to \infty$. Hence the invariant density $P(x)$ always exists and can be computed as the solution of the stationary master equation:

$$P(x) = \sum_{i=1}^{N} p_i c_i^{-1} P(w_i^{-1}(x)). \tag{6}$$

This density is highly singular. Therefore, it is more convenient to use a less singular object as the measure. Given an initial measure $\nu_0 \in \mathcal{M}(X)$, the measure at time n is given by:

$$\nu_n(S) = \sum_{i=1}^{N} p_i \nu_{n-1}(w_i^{-1}(S)), \tag{7}$$

and the invariant measure, if it exists, is obtained in the stationary limit, $n \to \infty$, as:

$$\nu = \sum_{i=1}^{N} p_i \, \nu \circ w_i^{-1}. \tag{8}$$

From Eq.(7) and Eq.(2) the equivalence between the Markov operator and the master equation is evident. From these equations for densities and measures it is possible to calculate moments, and also to establish ergodic properties, relating time averaging over realizations of the process to measure of event sets. This has been made by several authors [3, 4]. In the following sections, we will show the possibility of using conditioned master equations, or restricted measures, in a multifractal analysis. To this end we introduce here the equivalent to a conditioned master equation in terms of measures.

Suppose we are interested in realizations restricted to some condition. For example, to maintain fixed the rate of apparition for each contraction applied in a given period of time. Let $\mathbf{m} = (m_1, \ldots, m_N)$, with $m_i \in \mathbb{N}$ and $\sum_{i=1}^N m_i = M$, and let us construct a restricted sequence of measures, $\{\nu_n^{\mathbf{m}}\}$, taking into account realizations such that the application w_i appears m_i times every time period of length M. From Eq.(7) it is easy to write a master equation for this restricted measure as:

$$\nu_{(l+1)M}^{\mathbf{m}}(S) = \sum_{i_M}\sum_{i_{M-1}}\ldots\sum_{i_1}\delta(\mathbf{m})p_{i_M}p_{i_{M-1}}\ldots p_{i_1}\nu_{lM}^{\mathbf{m}}(w_{i_M}^{-1}w_{i_{M-1}}^{-1}\ldots w_{i_1}^{-1}(S)), \tag{9}$$

where $\delta(\mathbf{m})$ takes into account the restricted process:

$$\delta(\mathbf{m}) = \prod_j \delta_{m_j, \sum_{l=1}^M \delta_{j,i_l}}. \tag{10}$$

Obviously, with the normalized set of probabilities $\{p_i\,;\ \sum_{i=1}^N p_i = 1\}$ the limit $l \to \infty$ leads to a trivial null measure, since we have restricted the number of events without increasing the probability. To have a nontrivial invariant measure $\nu^{\mathbf{m}}$ we must take into account the possible existence of a modified set of probabilities without normalization $\{\pi_i\,;\ \sum_{i=1}^N \pi_i \geq 1\}$. If for some set $\{\pi_i\}$ the limit $l \to \infty$ of Eq.(9) exists then, the invariant measure $\nu^{\mathbf{m}}$ for this restrictive process of period M is given by:

$$\nu^{\mathbf{m}} = \sum_{i_M}\sum_{i_{M-1}}\ldots\sum_{i_1}\delta(\mathbf{m})\pi_{i_M}\pi_{i_{M-1}}\ldots\pi_{i_1}\,\nu^{\mathbf{m}} \circ w_{i_M}^{-1} \circ w_{i_{M-1}}^{-1} \ldots \circ w_{i_1}^{-1} \tag{11}$$

with some conditions over $\{\pi_i\}$ to be determined. It is not difficult to see that the support of the measure $|\nu^{\mathbf{m}}|$ corresponds to points $x \in |\mathcal{W}|$, with addresses having the same restriction as the process, i.e., the symbol i appears m_i times in $address(x) = \sigma_1\sigma_2\sigma_3\ldots$ every M symbols. As we will show in the next section, these measures can be advantageously used in a multifractal analysis.

4 Multifractal Analysis

The term multifractal is applied when many fractal subsets with different scaling properties coexist simultaneously. In this sense, the invariant measure, ν, generated by an IFSP has a multifractal character. There exists a widespread recent interest in multifractals [5]-[9] and their applications [10, 11]. Our aim in this paper is to introduce a multifractal analysis in the context of IFSP, working in the framework of dynamical systems, and show its applicability to fractal image generation. Our work has not the rigor of a mathematical theory since we need to connect objects, as images, with processes, keeping as much information as possible.

4.1 Basic results

First we recall some basic results and definitions [5]-[9]. Consider $B_{i,n}$ a ρ_n-cover of X, with $\lim_{n\to\infty}\rho_n = 0$. Consider also the set E_α defined as:

$$E_\alpha := \left\{x \in X/ \lim_{n\to\infty}\frac{log\ \nu(B_n(x))}{log(\rho_n)} = \alpha\right\}, \tag{12}$$

where $B_n(x)$ is the ball of the ρ_n-cover containing x. The main objective of a multifractal analysis of the measure ν is the analysis of the dimension and structure of these sets, as a function of the parameter α. In general, a calculation of these dimensions is not possible and some indirect methods based on the power law dependence of

$$S_n(q) = \sum_i^* \nu(B_{i,n})^q, \tag{13}$$

as $\rho_n \to 0$ are used, where $\sum^*$ means that the summation runs through those indices i such that $\nu(B_{i,n}) \neq 0$. For instance, assuming that the indicated limits exist and following the notation in [5] we define:

$$\tau_q := \lim_{n\to\infty}\frac{log(S_n(q))}{-log(\rho_n)} \tag{14}$$

$$N_n^\epsilon(\alpha) := \#\left\{B_{i,n}/ \frac{log(\nu(B_{i,n}))}{log(\rho_n)} \in [\alpha-\epsilon, \alpha+\epsilon]\right\} \tag{15}$$

$$f(\alpha) := \lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{log(N_n^\epsilon(\alpha))}{log(\rho_n)}. \tag{16}$$

$f(\alpha)$ is known as the multifractal spectrum or spectrum of singularities. Usually it is easier to calculate the scaling properties of $S_n(q)$ than those of $N_n(\alpha)$ and then the multifractal spectrum $f(\alpha)$ can be obtained indirectly through the expression [5]:

$$\tau_q = \sup_{0\leq\alpha<\infty}(f(\alpha) - q\alpha)). \tag{17}$$

It is known [5] that under very general assumptions

$$f_H(\alpha) \leq f(\alpha), \tag{18}$$

where $f_H(\alpha)$ is the Hausdorff dimension of E_α. Only for special classes of measures, such as multinomial measures [7], the equality $f_H(\alpha) = f(\alpha)$ has been proved. Then, in the general case f is not the dimension of any particular set. This fact rests physical insight to a multifractal analysis, since in practical applications one deals with particular sets. In the following we can prove that for certain kinds of IFSP the equality $f_H(\alpha) = f(\alpha)$ also holds.

4.2 Analysis of similitudes

First we consider a class of IFSP that admits an analytical treatment, those formed by similitudes. In this section we consider and IFS, $\mathcal{S} = \{w_1, \ldots, w_n\}$, where each of the w_i is a similitude with contraction factor s_i. This special class of contractions have interesting properties that are very useful in a multifractal analysis. For example, a similitude w_i maps closed balls of size ρ onto closed balls of size $s_i\rho$. On the other hand, imposing *the open set condition* (see [2]) two different applications map sets onto sets which intersection has a null measure. Rigorously, there exists an open set $U \subset X$, with $T(U) \subset U$ and $w_i(U) \bigcap w_j(U) = \emptyset$, with $i \neq j$. Then, the following properties hold:

$$\nu(w_j^{-1} w_i(S)) = \delta_{i,j}\nu(S) \tag{19}$$

$$\nu(w_{j_1}^{-1} \ldots w_{j_l}^{-1} w_{i_l} \ldots w_{i_1}(S)) = \delta_{i_1,j_1} \ldots \delta_{i_l,j_l}\nu(S), \tag{20}$$

where $S \subset U$. In an IFSP satisfying open set condition each point of the fractal has an unique address [3]. In the following, we deal with $\mathcal{S}$, an IFS formed by similitudes and satisfying the open set condition. In the special case when all similitudes have the same contraction factor, the measure of any associated IFSP reduces to a multinomial measure. In this sense an IFSP formed by similitudes satisfying open set condition is the simplest generalization of multinomial measures. The Hausdorff dimension of $|\mathcal{S}|$, D, which coincides in this case with the box counting dimension, has been calculated by Hutchinson [2] as the unique solution of

$$\sum_{n=1}^{N} |s_n|^D = 1. \tag{21}$$

Here we extend the calculations to τ_q and introduce the possibility of a multifractal analysis. We use the fact that the attractor $|\mathcal{S}|$ is the disjoint union $|\mathcal{S}| = \bigcup_{i=1}^N w_i(|\mathcal{S}|)$. Then a ρ-covering of $|\mathcal{S}|$ can be always performed by the union of $s_i\rho$-covers of $w_i(|\mathcal{S}|)$ and if we use this property in the calculation of $S_n(q)$, we have

$$S_n(q) = \sum_{i=1}^{N} S_n^i(q) \tag{22}$$

where $S_n^i(q) = \sum_i \nu(B_{i,n}^l)^q$ and $B_{i,n}^l$ is a ball of the subpartition that covers $w_l(|\mathcal{S}|)$. Now using Eq.(8) and the open set condition, we obtain:

$$S_{s_l\rho_n}^l(q) = \sum_i \nu(w_l(B_{i,n}))^q = p_l^q \sum_i \nu(B_{i,n})^q = p_l^q S_{\rho_n}(q) \tag{23}$$

and

$$S_{\rho_n}(q) = \sum_l S_{\rho_n}^l(q) = \sum_l p_l^q S_{s_l^{-1}\rho_n}(q). \tag{24}$$

Finally, τ_q can be calculated taking into account the scaling law $S_{\rho_n}(q) \sim \rho_n^{-\tau_q}$ as:

$$\rho_n^{-\tau_q} \sim \sum_l p_l^q s_l^{\tau_q} \rho_n^{-\tau_q},$$

which implies:

$$1 = \sum_l p_l^q s_l^{\tau_q}. \tag{25}$$

The knowledge of τ_q allows, from Eq.(17) the calculation of $f(\alpha)$ which is a upper bound of $f_H(\alpha)$.

4.2.1 Structure of singularities

The open set condition leads to a multiplicative character in the iterative process of building the attractor of an IFS. Thus, several of the most important properties of $|\mathcal{S}|$ can be concluded in terms of its "basic iterators" (see, for example [5]). Using the notation in Sec.3, and defining $\mathbf{x} = (x_1, \ldots, x_N)$, where $x_i \in \mathbb{R}$ and $\sum_{i=1}^N x_i = 1$, we introduce a new set:

$$E_{\mathbf{x}} = \left\{ y \in |\mathcal{W}| / \lim_{m\to\infty} n_i(y|m)/m = x_i \ ; \ i = 1, \ldots, N \right\}, \tag{26}$$

where $n_i(y|m)$ is the number of times the symbol i appears in $\{\sigma_j / j = 1, \ldots, m\}$ with $address(y) = \sigma_1\sigma_2\sigma_3\ldots$. This set corresponds to the points of the attractor whose directions have a rate of repetition x_i for the symbol i. It is not difficult to see that $E_{\mathbf{x}} \subset E_\alpha$, being

$$\alpha = \frac{\sum_i x_i log(p_i)}{\sum_i x_i log(s_i)}$$

Let be $y \in E_{\mathbf{x}}$ and consider the basic iterators of $|\mathcal{S}|$, we have

$$\lim_{n\to\infty} \frac{log\nu(I_n(y))}{log(|I_n(y)|)} = \lim_{n\to\infty} \frac{\sum_i n_i(y|m) log(p_i)}{\sum_i n_i(y|m) log(s_i)} = \frac{\sum_i x_i log(p_i)}{\sum_i x_i log(s_i)}$$

being $I_n(y)$ the basic iterator containing y. Moreover, since $\forall y \in E_\alpha$, y is a point of the support of ν it follows that:

$$E_\alpha = \left\{ \bigcup E_{\mathbf{x}} \ / \ \alpha = \frac{\sum_i x_i log(p_i)}{\sum_i x_i log(s_i)}, x_i \in \mathbb{R} \right\}$$

Then, the Hausdorff dimension of E_α is related with that of $E_{\mathbf{x}}$ by:

$$\dim_H(E_\alpha) \geq sup \left\{ \dim_H(E_{\mathbf{x}}), \ \alpha = \frac{\sum_i x_i log(p_i)}{\sum_i x_i log(s_i)} \right\} \tag{27}$$

4.3 Restricted measures

We introduce a multifractal analysis of the restricted measures defined in the second section. This provides insight into the structure of the multifractal and allows a physical interpretation of the singularity spectrum. The analysis is also restricted to IFS satisfying the open set condition. In this case, it is possible to obtain the condition of the modified set of probabilities $\{\pi_i\}$ for the existence of $\nu^{\mathbf{m}}$. Hence, let us consider a covering of $|\nu^{\mathbf{m}}|$ by means of a set of basic iterators $\{I_{Ml}\}$. Recall that $\nu^{\mathbf{m}}$ can always be covered by basic iterators such that $\forall x \in |\nu^{\mathbf{m}}|$:

$$I_{Ml}(x) = \delta(\mathbf{m}) w_{l_1} w_{l_2} \dots w_{l_M} \left(I_{M(l-1)}(y) \right)$$

where $address(x) = \sigma_{l_1}\sigma_{l_2}\dots\sigma_{l_M} address(y)$. Supposing that $\nu^{\mathbf{m}}$ exists for some sets $\{\pi_i\}$, the measure and size of the basic iterators can be calculated from Eq.(11) as:

$$\nu^{\mathbf{m}}\left(I_{M_l}(x)\right) = \delta(\mathbf{m})\pi_{l_1}\pi_{l_2}\dots\pi_{l_M}\nu^{\mathbf{m}}\left(I_{M(l-1)}(y)\right) = [\pi_1^{m_1}\dots\pi_N^{m_N}]^l \nu^{\mathbf{m}}(I_0)$$

Every basic iterator of the $(l-1)th$ generation defines a set $\{I_{Ml}/I_{Ml} = \delta(\mathbf{m}) w_{l_1}\dots w_{l_M}(I_{M(l-1)})\}$ of $\frac{M!}{m_1!\dots m_N!}$ iterators of the lth generation, with sizes

$$|I_{Ml}| = s_1^{m_1}\dots s_N^{m_N}|I_{M(l-1)}| = [s_1^{m_1}\dots s_N^{m_N}]^l |I_0|$$

From these expressions we can obtain the condition for the existence of $\nu^{\mathbf{m}}$:

$$\frac{M!}{m_1!\dots m_N!}\pi_1^{m_1}\dots\pi_N^{m_N} = 1$$

and the box counting dimension of $|\nu^{\mathbf{m}}|$:

$$Dim_B(|\nu^{\mathbf{m}}|) = \frac{-log(\frac{M!}{m_1!\dots m_N!})}{\sum m_i log(s_i)} = \frac{\sum m_i log(\pi_i)}{\sum m_i log(s_i)}.$$

On the other hand, applying the mass distribution principle for the calculation of lower bounds of the Hausdorff dimension, $dim_H(|\nu^{\mathbf{m}}|)$, with measures $\nu^{\mathbf{m}}$, we have

$$\dim_H(|\nu^{\mathbf{m}}|) \geq s$$

with

$$s = \frac{\sum m_i \log(\pi_i)}{\sum m_i \log(s_i)}$$

Taking into account that the box counting dimension is always greater than the Hausdorff dimension we can conclude:

$$Dim_B(|\nu^{\mathbf{m}}|) = Dim_H(|\nu^{\mathbf{m}}|) = \frac{-log(\frac{M!}{m_1!\dots m_N!})}{\sum m_i log(s_i)} = \frac{\sum m_i log(\pi_i)}{\sum m_i log(s_i)} \tag{28}$$

Consider now the limit $M \to \infty$, taking $\mathbf{x} = (x_1 \dots x_N)$, where $x_i = \lim_{M\to\infty} \frac{m_i}{M}$, and supposing that $|\nu^{\mathbf{x}}| = \lim_{M\to\infty} |\nu^{\mathbf{m}}|$ exists, we obtain from Eq.(28) and from the definition of $E_{\mathbf{x}}$ in Eq.(26):

$$|\nu^{\mathbf{x}}| = E_{\mathbf{x}}$$

$$dim_B(E_{\mathbf{x}}) = dim_H(E_{\mathbf{x}}) = \frac{\sum x_i log(x_i)}{\sum x_i log(s_i)}$$

In this way we have related the sets $E_{\mathbf{x}}$ with supports of restricted measures. Note that the whole fractal is now partitioned into fractal subsets $E_{\mathbf{x}}$ of well known scaling properties.

Finally let us connect this multifractal analysis based on sets $E_{\mathbf{x}}$ with the usual one based on sets E_α. In order to compare with the usual analysis we define:

$$f(\mathbf{x}) = dim_B(E_{\mathbf{x}}) = dim_H(E_{\mathbf{x}}) = \frac{\sum x_i log(x_i)}{\sum x_i log(s_i)} \tag{29}$$

$$\alpha(\mathbf{x}) = \frac{\sum x_i log(p_i)}{\sum x_i log(s_i)} \tag{30}$$

$$\tau_q(\mathbf{x}) = f(\mathbf{x}) - q\alpha(\mathbf{x}) \tag{31}$$

$$\overline{\tau_q} = \sup_{\mathbf{x}}\{\tau_q(\mathbf{x})\} \tag{32}$$

$$\overline{f(\alpha)} = \sup_{\alpha=\alpha(\mathbf{x})}\{f(\mathbf{x})\} \tag{33}$$

From Eq.(32) and Eq.(31) we obtain a definition equivalent to Eq.(17):

$$\overline{\tau_q} = \sup_{\alpha\geq 0}\{\overline{f(\alpha)} - \alpha q\} \tag{34}$$

Now we can see that $\overline{\tau_q}$ and $\overline{f(\alpha)}$ have the same value as τ_q and $f(\alpha)$. It is enough to demostrate the identity $\tau_q = \overline{\tau_q}$, since with Eq.(34) and Eq.(17) the identity $f(\alpha) = \overline{f(\alpha)}$ follows. Note that $\tau_q \geq \overline{\tau_q}$ because $E_{\mathbf{x}} \subset |\mathcal{W}|$. The opposite relation $\overline{\tau_q} \geq \tau_q$ can be obtained by substituting $x_i = p_i^q s_i^{\tau_q}$ in Eq.(32) and using Eq.(31) and Eq.(30). Hence, we conclude $\overline{\tau_q} = \tau_q$ and $\overline{f(\alpha)} = f(\alpha)$. From this last identity and from Eq.(18) and Eq.(27) we finally have:

$$dim_H(E_\alpha) = f(\alpha) = \overline{f(\alpha)}$$

5 Application to Fractal Image Generation

As an application of multifractal analysis, we analyze in this section the problem of fractal image generation using IFSP. One concern of the computer graphic community has been the efficiency of rendering algorithms. There exists, apart from sophisticated methods like ray tracing [12], etc., two main algorithms to render the attractor of an IFS. Equation (1) provides a computational algorithm known as the *deterministic algorithm* for approximating the attractor of an IFS. In Sec.3 we introduced the probabilistic algorithm, also know as *chaos game*, by choosing an associated set of probabilities (IFSP). The role that probabilities play to generate the image of the attractor can be investigated using a multifractal analysis. In this section, we determine the most efficient choice of probabilities to render $|\mathcal{S}|$ in the least time possible. The main goal is to optimize the generation time at a given resolution scale.

The efficiency of the random algorithm to reder the attractor of an IFS using a set of probabilities $\{p_1, \ldots, p_N\}$ can be related to the spectrum of singularities of the multifractal measure associated with the corresponding IFSP. In order to illustrate this assertion we use the analysis developed in the previous section. Hence, let us restrict our attention to similitudes satisfying the open set condition and consider the subsets $E_{\mathbf{x}}$ as the limit of subsets $E_{\mathbf{m}}$ which can be covered by basic iterators. Roughly speaking, the fractal dimension of these sets

$$\frac{\sum x_i log(x_i)}{\sum x_i log(s_i)} \tag{35}$$

represents the logarithmic ratio of the relative number of iterators over its size. With the same criterion the singularity

$$\frac{\sum x_i log(p_i)}{\sum x_i log(s_i)} \tag{36}$$

would be the logarithmic ratio of the probability of filling such basic iterators with the stochastic orbit over the size of the iterators. Obviously, if a set $E_{\mathbf{x}}$ has a singularity greater than its fractal dimension, it is being overestimated in the process of generation since it has assigned more probability than the corresponding ratio of occurrence. Similarly, a set with a singularity lower than the fractal dimension is underestimated. Hence, the spectrum of singularities provides a good criterion for estimating the efficiency of a given IFSP. Sets E_α with the greatest singularities are overestimated in detriment of sets with lower singularities. Moreover, when the dispersion of singularities grows, the corresponding algorithm will be more inefficient since there will be sets of the fractal with a small probability to be filled and others in which the probability of overlapping would be appreciable.

Consider now the problem of the choice of probabilities of the corresponding IFSP for generating a fractal image. The standard criterion for this choice is the proportionally with respect to the contractive factor (see, for example, Barnsley [3], pag. 85)

$$p_i = \frac{s_i}{\sum_{j=1}^{N} s_j}\,;\ i = 1, \ldots, N. \tag{37}$$

As we shall show, this is not the most efficient choice because it provides a broad spectrum of singularities bounded by

$$\alpha_{max} = \max_{s_j}\{1 - \frac{\log(\sum s_i)}{\log s_j}\}$$

and

$$\alpha_{min} = \min_{s_j}\{1 - \frac{log(\sum s_i)}{\log s_j}\}.$$

Note that these boundaries are obtained by substituting Eq.(37) into Eq.(36). The most efficient set of probabilities is given by:

$$p_i = s_i^D\,;\ i = 1, \ldots, N \tag{38}$$

where D denotes the *similarity dimension* or fractal dimension. It is easy to see that this set of probabilities leads to a trivial multifractal invariant measure with a unique singularity $\alpha = D$. Consequently, with this choice all subsets have the same singularity which is the fractal dimension and the probability of filling coincides with the occupation rate.

We now propose a numerical method to obtain this set of probabilities. From Eq.(38) we obtain:

$$D = \frac{lnp_j}{lns_j} = \frac{lnp_i}{lns_i},\ \forall i \neq j,\ \ i, j \in \{1, \ldots, N\}. \tag{39}$$

Selecting the index i with maximum s_i and using (39), we obtain:

$$p_j = p_i^{\frac{logs_j}{logs_i}},\ \forall j \neq i, \tag{40}$$

then we have

$$p_i + \sum_{j \neq i} p_i^{\frac{lns_j}{lns_i}} = 1. \tag{41}$$

Thus we obtain an equation of the form $\sum_i x^{e_i} = 1$, with $e_i \geq 1$, which has a unique solution in $(0, 1)$. In these conditions there exists a unique value of p_i satisfying Eq.(41) that can be obtained with some standard numerical algorithms: bisection, Newton-Raphson, etc. (see [13]). Finally, the optimal value of the probabilities can be obtained by substituting p_i in Eq.(40).

As a first example we have considered a two-scaled Cantor IFSP defined in the interval $u = (0, 1)$ by:

$$\left\{w_1(x) = \frac{1}{5}x,\ \ w_2(x) = \frac{1}{2}x + \frac{1}{2};\ \ p_1,\ \ p_2 = 1 - p_1\right\}$$

We can check that the corresponding IFS is formed by two similitudes which satisfy the open set condition. The standard choice given by Eq.(37) provides a value $p_1 = 0.2857$, and our proposal gives $p_1 = 0.3577$ using Eq.(41). In Figs.1-3 we show the differences between both cases. In Fig.1 we plot the width of the spectrum of singularities $\alpha_{max} - \alpha_{min}$ versus the probability p_1. The two dashed lines correspond to the standard value $p_1 = 0.2857$ (which exhibits a broad spectrum) and to the best choice $p_1 = 0.3577$ (that gives a zero value for the width). We can also calculate the fractal dimension of the attractor by using Eq.(38):

$$D = \frac{ln(p_1)}{ln(s_1)} = 0.6387$$

In Fig.2, we show the fractal images generated by the random algorithm with 5000 iterations in both cases. In Fig.2.a we show the standard case with $p_1 = 0.2857$ and in Fig.2.b our choice. The difference can be established visually. In order to quantify this difference we represent in Fig.3 the ratio of black pixels appearing at early times (2500, 5000 and 10000 iterations) with respect to those in a practically infinite time (5.10^5 iterations) versus p_1. This gives an idea of what the filling rate of the corresponding IFSP could be. The dashed lines correspond to the standard and best choices respectively. This figure shows clearly the importance of choosing an adequate set of probabilities.

Finally, in a general IFSP with intersecting applications, the analytical calculation of the singularity spectrum is not possible and then the best choice must be done with a fitting method. We have seen that, even in this situation, our choice is always better than the standard choice, and very close to the best choice calculated numerically. This fact can be observed in Fig.4 and Fig.5. which show the fractal attractor of two IFS models formed by applications which are not similitudes and do not satisfy the open set condition.

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Table Captions

Table 1. Differences between the probabilities of the standard choice and of our choice for the Cantor, ZigZag and Fern IFS.

IFS	Probs.	Standard	Best Choice
Cantor	p_1	0.286	0.356
	p_2	0.724	0.654
ZigZag	p_1	0.889	0.815
	p_2	0.111	0.185
Fern	p_1	0.753	0.701
	p_2	0.123	0.150
	p_3	0.104	0.129
	p_4	0.020	0.020

Figure Captions

Figure 1. Width of the spectrum of singularities $\alpha_{max} - \alpha_{min}$ for different values of the probability of p_1. The dashed lines correspond to the standard and best choices respectively.

Figure 2. Filling two-scaled Cantor set using random algorithm with standard (a), and our proposal (b) probabilities.

Figure 3. Filling rate versus p_1.

Figure 4. The ZigZag IFS given by the applications $\{w_1(x, y) = (-0.63x - 0.61y + 3.84, -0.55x + 0.66y + 1.28), w_2(x, y) = (-0.04x + 0.44y + 2.07, 0.21x + 0.04y + 8.33)\}$. (a) the standard choice, (b) our choice (see Table 1).

Figure 5. The fern IFS given by the applications $\{w_1(x, y) = (0.81x+0.07y+0.12, -0.04x+0.84y + 0.195), w_2(x, y) = (0.18x - 0.25y + 0.12, 0.27x + 0.23y + 0.02), w_3(x, y) = (0.19x + 0.275y + 0.16, 0.238x - 0.14y + 0.12), w_4(x, y) = 0.0235x + 0.087y + 0.11, 0.045x + 0.1666y)\}$. (a) the standard choice, (b) our choice (see Table 1).