

# A New Exact Method for Obtaining the Multifractal Spectrum of Multiscaled Multinomial Measures and IFS Invariant Measures

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## Abstract

In this paper we present a new exact method for obtaining the multifractal spectrum of multiscaled multinomial measures and invariant measures associated with Iterated Function Systems (IFS). A multinomial measure is shown to be generated as the invariant measure of an associated IFS. Then, the multifractal spectrum of the measure is determined by a couple of parametric implicit equations. This analysis generalizes some results previously obtained for the case of single-scaled multinomial measures (e.g., the binomial measure). A geometric interpretation of this new framework working in the space of codes of the IFS gives new insight into the nature of the multifractal formalism. This paper extends the results presented in [Gutiérrez et. al. *Fractals*, **4**, 17-27 (1996)].

## 1 Introduction

In recent years, many theoretical and experimental scientists have studied the so-called multifractal theory [1]-[6], which is concerned with the scaling properties of measures. The first stage on the multifractal analysis of a measure  $\mu$  is decomposing its support into sets,  $E(\alpha)$ , formed by points with the same local dimension  $\alpha$ , that is,

$$E(\alpha) = \left\{ x \in \text{supp}(\mu) : \lim_{\epsilon \rightarrow 0} \frac{\ln \mu(B_\epsilon(x))}{\ln \epsilon} = \alpha \right\}, \quad (1)$$

where  $B_\epsilon(x)$  is the ball of radius  $\epsilon$  about  $x$  and  $\text{supp}(\mu)$  denotes the support of the measure (i.e., the set of points where the measure does not vanish).

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The main objective of the analysis is estimating the size of each of these sets. This is done by introducing the function  $f(\alpha) = \dim_H(E(\alpha))$ , where  $\dim_H$  stands for the Hausdorff dimension. This function is generically known as the “dimension spectrum”, “singularity spectrum” or “multifractal spectrum” of the measure. Several multifractal formalisms have been proposed in the last few years as a result of an intensive research activity in this field. Some of these formalisms use a different concept of dimension, e.g., the box-counting or packing dimensions, to define the multifractal spectrum (a detailed introduction to these formalisms can be found in [6]).

The most classic and intuitive examples of multifractal measures are the so called multinomial measures. These measures are defined by an iterative process which fragments the support into smaller and smaller components according to a fixed rule, and at the same time fragments the measure of the components by another rule. In the first stage of this process, the unit mass supported by  $S$  is distributed on a selected number of subsets  $S_1, \dots, S_N$  (a fraction  $p_i$  of the measure is given to each of the sets  $S_i$ ). Single-scaled multinomial measures, originated when all the subsets  $S_i$  have the same size, are an important paradigm in the theory of multifractals and have been extensively studied in the literature [5,8]. In this case, it turns out that there exist numbers  $\alpha_{min}$  and  $\alpha_{max}$  such that  $f(\alpha) = 0$  for  $\alpha \in (0, \infty) \setminus [\alpha_{min}, \alpha_{max}]$  and  $f(\alpha)$  is concave and smooth on  $(\alpha_{min}, \alpha_{max})$ . Moreover, the multifractal spectrum can be easily obtained by using simple exact formulas (see, e.g., [7,8]).

In this paper we extend the above multifractal analysis to multiscaled multinomial measures showing that similar formulas also hold in this case. First we show that the multiplicative iterative process defining multinomial measures can be equivalently represented by an Iterated Function System with probabilities. Then, following the ideas in [9], we introduce an alternative multifractal analysis working in the space of codes of the resulting IFS. This will allow us to characterize the multifractal spectrum by a system of implicit parametric equations that can be numerically solved in an easy way (these equations extend those obtained in [7] for the case of single-scaled measures). Besides of obtaining the multifractal spectrum of the measure, this analysis allow us to divide the support into subsets with different scaling properties. The analysis of these subsets gives insight into the multifractal structure of the measure. Examples are introduced along the paper to illustrate the concepts and methods.

The paper is organized as follows. In Sec.2 we introduce the basic notation and definitions concerning the multifractal formalism. In Sec. 3 we introduce both multinomial and IFS invariant measures and present the new algorithm to obtain the multifractal spectrum of these measures in an easy and straightforward way. In Sec. 4 we analyze the obtained multifractal spectrum identifying the subsets of the support corresponding to a given region of the spectrum.

Finally, in Sec. 5 we give some conclusions and further remarks.

## 2 Preliminars

The iterative multiplicative process defining a multinomial measure can be precisely defined in the language of Iterated Function Systems (IFS). Let  $\{w_1, \dots, w_n\}$  be a set of contraction mappings on  $\mathbb{R}^n$ . It is well known that there exists a unique set  $S$  satisfying [10,11]

$$S = \bigcup_{i=1}^N w_i(S).$$

This set is called the attractor of the IFS and has fractal structure for a wide class of IFS models. If non-zero probabilities are associated with each  $w_i$ , then the set  $S$  supports a unique measure  $\mu$  satisfying

$$\mu = \sum_{i=1}^N p_i \mu \circ w_i^{-1}.$$

by a multiplicative cascade with subsets  $\{w_1(S), \dots, w_N(S)\}$  and associated probabilities  $\{p_1, \dots, p_N\}$ , and reciprocally. The coding space formed by all infinite sequences of pairs belonging to  $\{1, \dots, N\}$  is a convenient framework for analyzing the structure of the set  $S$ . This space is denoted as

$$C = \{(\sigma_1, \sigma_2, \dots) : \sigma_i \in \{1, \dots, N\}\}.$$

If the IFS satisfies the open set condition (see [11]), i.e., if there is a open set  $U \subset \mathbb{R}^n$ , with  $\cup_i w_i(U) \subset U$  and  $w_i(U) \cap w_j(U) = \emptyset$ , then there is a bijection from  $C$  to  $S$ . The pair  $(\sigma_1, \sigma_2, \dots)$  associated with a point  $x = w_{\sigma_1} \circ w_{\sigma_2} \dots (y) \in S$  is called the address of  $x$  and denoted as  $\sigma(x)$ . The address of  $x$  indicates the sequence of mapping compositions needed to reach the point starting at any other point  $y \in S$ . This is possible due to the contractive character of the mappings.

## 3 Multifractal Analysis

The open set condition leads to a multiplicative process where the different subsets do not overlap. Then, many interesting properties of  $S$  can be concluded in terms of its "basic iterators", i.e., the sets  $I_{i_1 i_2 \dots i_m} = w_{i_1} \circ w_{i_2} \dots \circ$

$w_{i_m}(U)$ , where  $U$  is the open set given by the open set condition. Moreover, the size and measure associated with a basic iterator is given by frequencies the symbols  $\{1, \dots, N\}$  appear in the corresponding sequence of mappings. Then, the scaling properties of the basic iterators containing a point  $x$  only depend on the relative frequencies  $r_i$  the digits  $i$  appear in the address  $\sigma(x)$ . We introduce the set

$$E_r = \left\{ x \in S : \lim_{n \rightarrow \infty} \frac{\delta_i \sigma_n}{n} = r_i, i = 1, \dots, N \right\}. \quad (2)$$

If the IFS satisfies the open set condition, then the local dimension of a point  $x \in E_r$  and the Hausdorff dimension of  $E_r$  are directly given by [9]

$$\alpha_r = \frac{\sum_i r_i \log p_i}{\sum_i r_i \log s_i}, \quad (3)$$

$$f(\alpha_r) = \frac{\sum_i r_i \log r_i}{\sum_i r_i \log s_i} \quad (4)$$

These sets form a partition of the support into sets with different scaling properties. We have  $E_r \subset E(\alpha)$ , with  $\alpha = \alpha_r$ . Moreover, since  $\forall x \in E(\alpha)$ ,  $x$  is a point of the support of  $\mu$  it follows that:

$$E(\alpha) = \left\{ \bigcup E_r / \alpha = \alpha_r \right\}$$

Then, the Hausdorff dimension of  $E(\alpha)$  is related with that of  $E_r$  by:

$$\dim_H(E(\alpha)) \geq \sup \left\{ \dim_H(E_r), \alpha = \frac{\sum_i r_i \log p_i}{\sum_i r_i \log s_i} \right\} \quad (5)$$

It has been shown that some types of measures are concentrated on a single value of  $\alpha$ , i.e.,  $\mu(A) = 0$  for a.e. set  $A$  such that  $A \cap E(\alpha) = \emptyset$ . This is the case of IFS satisfying the open set condition, where the corresponding value of  $\alpha$  is given by [12]

$$\alpha' = \frac{\sum_i p_i \log p_i}{\sum_i p_i \log s_i}, \quad (6)$$

and satisfies  $\dim_H(\mu) = \dim_H(E(\alpha')) = f(\alpha') = \alpha'$ . This value is usually known as "information dimension". Using the above multifractal analysis, this value appears naturally as the dimension of the set  $E_r$ , where  $r_i = p_i$ ,  $i = 1, \dots, N$ .

Another well known result for multifractal measures is the existence of values  $\alpha_{min}$  and  $\alpha_{max}$  such that  $f(\alpha) = 0$  for  $\alpha \notin [\alpha_{min}, \alpha_{max}]$ . These values represent the minimum and maximum singularities, respectively, and are given by

$$\alpha_{min} = \min \{t_i, i = 1, \dots, N\}, \quad \alpha_{max} = \max \{t_i, i = 1, \dots, N\}, \quad (7)$$

where

$$t_i = \frac{\log p_i}{\log s_i}.$$

Note that these values also appear naturally from the above multifractal analysis (see Equn. 3).

In the case of two-scaled binomial measures, each pair  $(r_1, r_2)$  defines a unique  $E(\alpha) = E_r$  with  $\alpha_r = \alpha$  (see Equn. 3). Therefore, there is a one-to-one correspondence between singularities and pairs of index frequencies. Using (3), (4), and the condition  $r_1 + r_2 = 1$  we can obtain an explicit parameterization of the multifractal spectrum. If we introduce the parameter  $r = r_1$ , we obtain:

$$\begin{aligned} \alpha(r) &= \frac{r \log p_1 + (1-r) \log(1-p_1)}{r \log s_1 + (1-r) \log s_2}, \\ f(\alpha(r)) &= \frac{r \log r + (1-r) \log(1-r)}{r \log s_1 + (1-r) \log s_2}. \end{aligned} \quad (8)$$

For example, consider the two-scaled binomial measure defined in the unit interval  $(0, 1)$  by the multiplicative cascade which, starting with a uniformly distributed unit of mass on the unit interval  $I = (0, 1)$ , fragments this mass by distributing a fraction  $p_1$  on the interval  $I_1 = (0, \frac{2}{3})$ , and the remaining fraction,  $p_2$ , on the interval  $I_2 = (\frac{2}{3}, 1)$ . Note that this binomial measure can be obtained as the invariant measure of the IFS  $\{w_1(x) = \frac{2}{3}x, w_2(x) = \frac{1}{3}x + \frac{2}{3}\}$  with probabilities  $\{p_1, p_2\}$ . A problem with interesting practical applications is that of analyzing the structure of the multifractal spectrum  $f(\alpha)$  for different probability assignments in the multiplicative process. Figure 1 shows the multifractal spectrum for the above IFS with three different probability assignments:  $\{p_1 = 0.1, p_2 = 0.9\}$ ,  $\{p_1 = 0.3, p_2 = 0.7\}$ , and  $\{p_1 = 2/3, p_2 = 1/3\}$ . Note that the values  $\alpha_{min}$  and  $\alpha_{max}$  for each of the probability sets can be easily obtained from (7). The values of  $\alpha$  associated with the dimension of the support and the information dimension are shown in each case to illustrate the differences of the obtained spectra. It can be shown that there is a critical value of  $p_1$ ,  $(p_1 = 2/3, p_2 = 1/3)$ , where the singularity spectrum is delta-shaped,

that is, there is only one singularity value. This situation corresponds to the uniform measure distributed on the support. From (4) this value can be easily shown to be associated with the probabilities  $p_i = s_i^d$ , where  $d = \dim_H(S)$  is the only real number satisfying  $\sum_i s_i^d = 1$  (see [9] for further details).

In the case of multinomial measures, the calculation of the multifractal spectrum is more difficult, since there is no one-to-one relationship between  $r$  and  $\alpha_r$ . In fact, each  $r$ -set in  $(0, 1)^N$  corresponding to the local dimension  $\alpha$ , define a  $(N-1)$ -manifold given by equation  $\alpha_r = \alpha$  (see (3)). The case of single-scaled measures was addressed in [7], obtaining a system of explicit parametric equations. Here we analyze the case of multiscaled measures, and show that an equivalent result can be obtained in terms of an implicit systems of parametric equations. In this case, the value  $f(\alpha)$  is obtained by maximizing  $f(\alpha_r)$  subject to the constraint  $\alpha_r = \alpha$ . The classic theory of Lagrange multipliers gives a solution of this problem by introducing a parameter  $\lambda$  and the auxiliary function  $Q = f(\alpha_r) + \lambda(\alpha_r - \alpha)$ . Using (3) and (4), each of the conditions  $\partial Q / \partial r_k = 0$  leads to:

$$\begin{aligned} \left( \sum_i r_i \log s_i \right) (r_k \log r_k + \lambda r_k \log p_k) - r_k \log s_k \left( \sum_i r_i \log r_i + \lambda \sum_i r_i \log p_i \right) &= 0, \\ r_k &= s_k^{f(\alpha) + \lambda \alpha_r - \lambda} p_k^{-\lambda}. \quad \forall k = 1, \dots, n. \end{aligned} \quad (9)$$

for each  $k = 1, \dots, n$ . Dividing by  $\sum_i r_i \log s_i$  and substituting (3) and (4) in the resulting expression leads to the following basic relationship among the optimum  $r$ , the local dimension  $\alpha$ , and the corresponding spectrum value  $f(\alpha)$ .

Thus, introducing  $q = -\lambda$  and using the condition  $\sum_i r_i = 1$  we have

$$\sum_i s_i^{f(\alpha) - q\alpha} p_i^q = 1. \quad (10)$$

Now, by substituting (9) in equation (3) we obtain:

$$\alpha = \frac{\sum_i r_i \log p_i}{\sum_i r_i \log s_i} = \frac{\sum_i s_i^{f(\alpha) - q\alpha} p_i^q \log p_i}{\sum_i s_i^{f(\alpha) - q\alpha} p_i^q \log s_i} \quad (11)$$

For each value  $-\infty < q < \infty$ , equations (10) and (11) define a unique pair  $(\alpha(q), f(\alpha(q)))$  of the singularities spectrum of the measure. Thus, the multifractal spectrum of the measure can be obtained by numerically solving the system of equations (10)-(11) for different values of the parameter  $q$ . This reduces the problem of obtaining the exact multifractal spectrum of a multiscaled multinomial measure (or, equivalently, an IFS invariant measure) to

solving several systems of equations using some standard numerical method (see, for example, [13]). As opposed to other standard methods, this algorithm do not involve any transformation (such as the wavelet transform [14]), or approximation (such as sampled histograms [15] or coarse-grains [16]).

Note that by introducing  $\tau(q) = q\alpha(q) - f(\alpha(q))$  (the Legendre transform of  $f(\alpha)$ ) in (10), we recover the basic result for single-scaled multinomial measures:

$$\sum_i s_i^{-\tau(q)} p_i^q = 1.$$

In the following example we consider the multiscaled multinomial measure defined in the unit interval  $(0, 1)$  by the following recursive multiplicative cascade. This cascade starts ( $n=0$ ) with a uniformly distributed unit of mass on the unit interval. The next stage of the process ( $n=1$ ) fragments this mass by distributing a fraction  $p_1$  on the interval  $I_1 = (0, \frac{3}{10})$ , a fraction  $p_2$  on the interval  $I_2 = (\frac{3}{10}, \frac{8}{10})$ , and the remaining fraction,  $p_3$ , on the interval  $I_3 = (\frac{8}{10}, 1)$ . Figure 2 shows the measure obtained when considering the probabilities  $p_1 = 0.2, p_2 = 0.3, p_3 = 0.5$ . Note that this multinomial measure can be obtained as the invariant measure of the IFS  $\{w_1, w_2, w_3\}$  with probabilities  $\{0.2, 0.3, 0.5\}$ , where

$$\left\{ \begin{aligned} w_1(x) &= \frac{3}{10}x, w_2(x) = \frac{5}{10}x + \frac{3}{10}, w_3(x) = \frac{2}{10}x + \frac{8}{10} \end{aligned} \right\}$$

Figure 3 shows the multifractal spectrum obtained by solving the system of equations (10) and (11) for a given set of  $q$  values in the interval  $(-20, 20)$ . Two important points in the spectrum, the dimension of the support  $(\alpha(0), f(\alpha(0))) = (1.188, 1.0)$ , and the information dimension,  $(\alpha(1), f(\alpha(1))) = (0.821, 0.821)$ , are given by the parameter value  $q = 0$  and  $q = 1$  respectively. These points are indicated by circles in the  $\alpha$ -axis in the above figure.

In the above example, we have  $f(\alpha_{min}) = f(\alpha_{max}) = 0$ . However, in general, the values  $f(\alpha_{min})$  and  $f(\alpha_{max})$  can be obtained from (10) and (7) as [3]

$$\sum_{t_i=t} s_i^f = 1, \quad (12)$$

where  $t = \alpha_{min} = \min\{t_i, i = 1, \dots, N\}$  for  $f = f(\alpha_{min})$  and  $t = \alpha_{max}$  for  $f = f(\alpha_{max})$ , respectively. Note that if there is only one value  $t_i = t$ , then the only solution of (12) will be  $f = 0$ .

In the following example, and with the aim of illustrating the situations that may appear for multiscaled measures, we analyze a case where all the singularities have positive dimension. Consider the multiscaled multinomial measure defined in the unit interval  $(0, 1)$  by the multiplicative cascade defined by the intervals  $(0, \frac{1}{10})$ ,  $(\frac{1}{10}, \frac{2}{10})$ ,  $(\frac{2}{10}, \frac{6}{10})$ , and  $(\frac{6}{10}, 1)$ , where the unit mass is fragmented according to the probabilities  $\frac{2}{10}, \frac{2}{10}, \frac{3}{10}$ , and  $\frac{3}{10}$ . The multifractal spectrum of this measure is shown in Figure 4. In this case, we have  $t_1 = t_2 = 0.699$  and  $t_3 = t_4 = 1.314$ . Therefore, the value  $f(\alpha_{min})$  is given by:

$$\sum_{i=1}^2 s_i^{f(\alpha_{min})} = 1 \Rightarrow f(\alpha_{min}) = 0.301. \quad (13)$$

Similarly, we have  $f(\alpha_{max}) = 0.756$ .

The system of equations given by (10) and (11) can be further simplified in some special cases, leading to an explicit parametric formula for the multifractal spectrum. For example, if all the scaling factors are equal ( $s_i = s$ ,  $i = 1, \dots, N$ ), i.e., if we have a single-scaled multinomial measure, then (10) and (11) lead to:

$$\alpha(q) = \sum \frac{p_i^q \log p_i}{\sum p_i^q \log s} \quad (14)$$

$$f(\alpha(q)) = \sum \frac{p_i^q \log(p_i / \sum p_i^q)}{\sum p_i^q \log s} \quad (15)$$

as obtained before by Mandelbrot [7] for the case of single-scaled multinomial measures.

#### 4 Analyzing the Multifractal Spectrum

The above multifractal analysis provides an intuitive framework for understanding the multifractal decomposition of a measure. The multifractal spectrum of a measure is obtained as the maximum Hausdorff dimension of the subsets  $E_r$ , where the vector  $r$  indicates the frequencies of the different mappings appear in the address of the points contained in the set  $E_r$ . Since there is no one-to-one relationship between  $r$  and  $\alpha_r$ , each  $r$ -set in  $(0, 1)^N$  corresponding to the local dimension  $\alpha$ , define a  $(N-1)$ -manifold given by equation  $\alpha_r = \alpha$  (see (3)). For example, Figure 5 shows the complete dimension spectrum, i.e., the dimensions associated with all the sets  $E_r$ , of the multifractal measure shown in Figure 2 (a similar figure for the case of single-scaled measures can

also be found in [7]. Note that each point  $(\alpha, f(\alpha))$  in the multifractal spectrum of this measure was obtained by maximizing the dimension of all the sets associated with a given local dimension,  $\alpha_r = \alpha$  (see Figure 3).

With the aim of illustrating the meaning of the multifractal spectrum, Considering the multifractal measure given in Figure 2, Figure 6 shows the dimension curves generated by fixing the values of  $N - 2$  components of the vector  $r$ . In this case, since  $N = 3$ , we only need to fix one of the values  $r_i$ . Then, we obtain a curve that allow us to analyze the contribution to the spectrum of small or high frequencies of the mapping  $w_i$ . For example, from Figure 6 we can see that values  $r_3 < \alpha_3$  define the part of the spectrum corresponding to large singularities ( $\alpha > \alpha_0$ ) (see Figure 6(c)) whereas small singularities are associated with  $r_3 > \alpha_3$ . The curves shown in Figures 6 (a), (b), and (c) illustrate the contribution to the multifractal spectrum of the mappings  $w_1$ ,  $w_2$ , and  $w_3$ , respectively.

## 5 Conclusions

In this paper we introduced a new framework for generating the multifractal spectrum of multiscaled multinomial measures and invariant measures of IFS. Although this spectrum can be simplified for some special cases (e.g., single-scaled multinomial measures), in the general case it can be obtained by solving a  $2 \times 2$  system of implicit parametric equations. This method provides a quick, easy and intuitive way for generating the spectrum of a measure. Using an illustrative example, we have shown how the relationship between the space of codes of the resulting IFS and the multifractal spectrum obtained with this new framework gives new insight into the nature of the multifractal formalism. A further analysis of this relationship will be the scope of a future paper.

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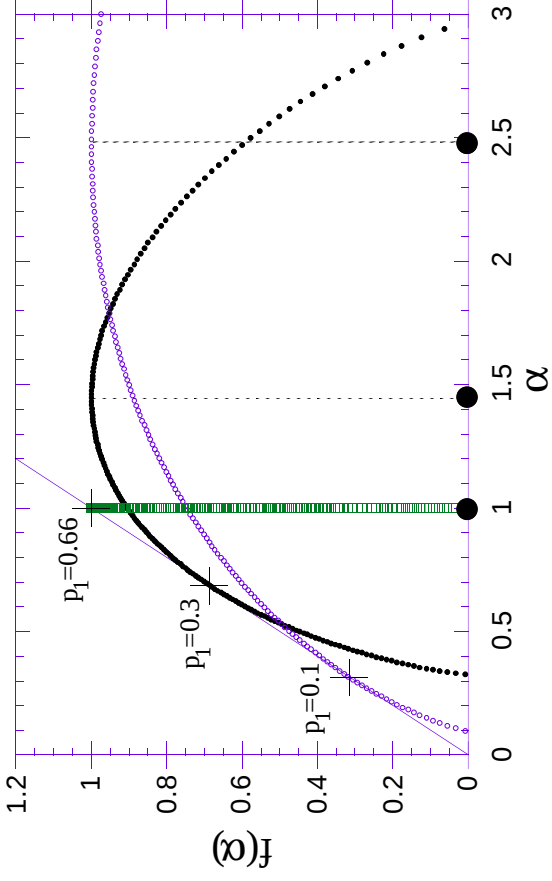


Fig. 1. Multifractal spectra for the IFS  $\{w_1(x) = \frac{2}{3}x, w_2(x) = \frac{1}{3}x + \frac{2}{3}\}$  with three different probability assignments:  $\{p_1 = 0.1, p_2 = 0.9\}$ ,  $\{p_1 = 0.3, p_2 = 0.7\}$ , and  $\{p_1 = 2/3, p_2 = 1/3\}$ . The values of  $\alpha$  associated with the dimension of the support,  $\alpha_0$ , and the information dimension,  $\alpha_1$ , are shown in each case to illustrate the differences of the obtained spectra.

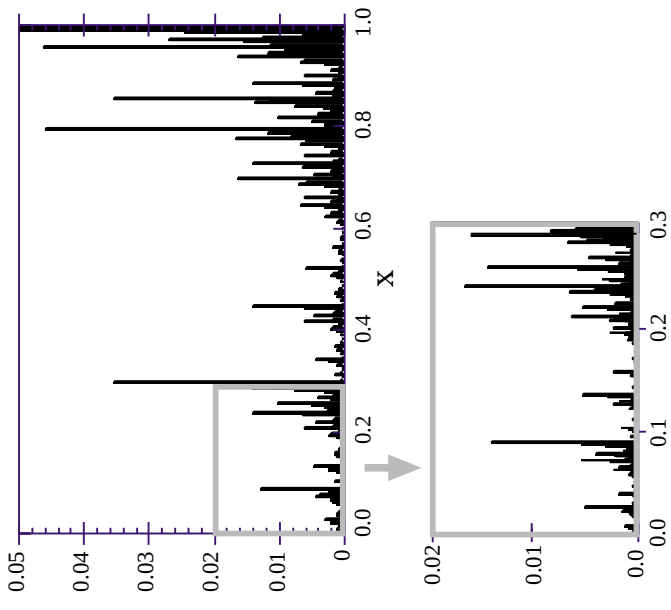


Fig. 2. Multifractal measure for the IFS  $\{w_1(x) = \frac{3}{10}x, w_2(x) = \frac{5}{10}x + \frac{3}{10}, w_3(x) = \frac{2}{10}x + \frac{8}{10}\}$ , with probabilities  $\{p_1 = 0.2, p_2 = 0.3, p_3 = 0.5\}$ . The enlarged region shows the self-similar structure of this measure.

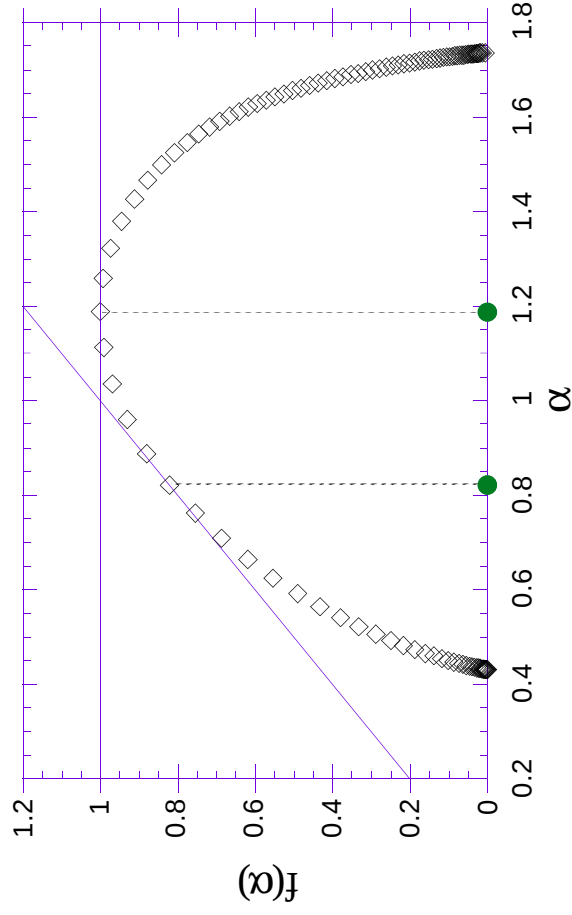


Fig. 3. Multifractal spectrum of the measure given in Figure 2 obtained by solving the system of equations (10) and (11) for a given set of  $q$  values in the interval  $(-20, 20)$ .

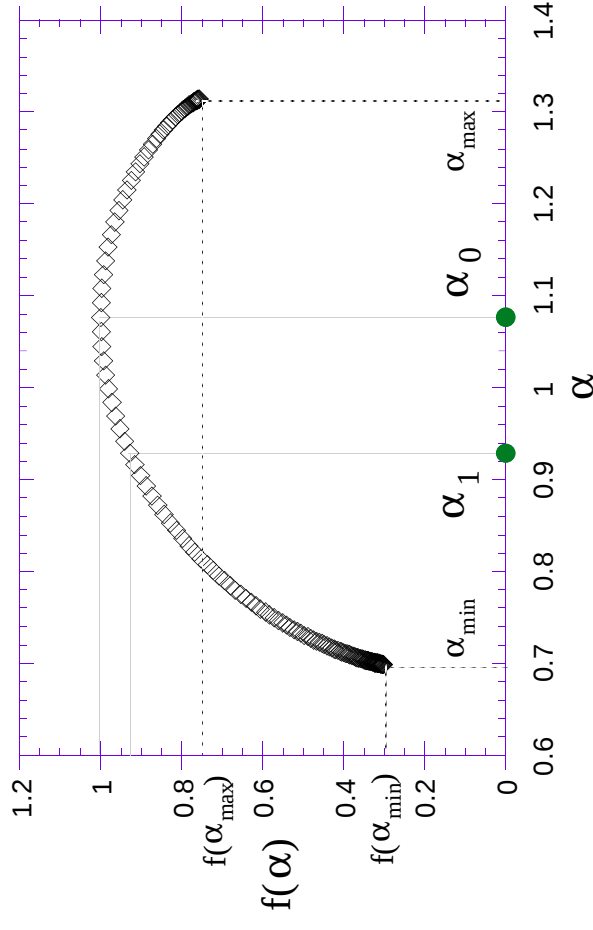


Fig. 4. Multifractal spectrum of the multiscaled multinomial measure in the unit interval  $(0, 1)$  defined by the multiplicative cascade with intervals  $(0, \frac{1}{10})$ ,  $(\frac{1}{10}, \frac{2}{10})$ ,  $(\frac{2}{10}, \frac{6}{10})$  and  $(\frac{6}{10}, 1)$ , and probabilities  $\frac{2}{10}$ ,  $\frac{3}{10}$ , and  $\frac{3}{10}$ .

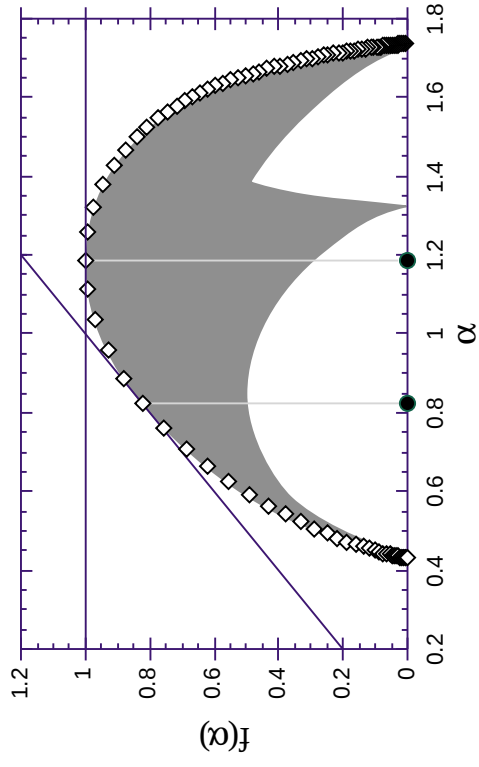


Fig. 5. Complete dimension spectrum of the multifractal measure given in Figure 2.

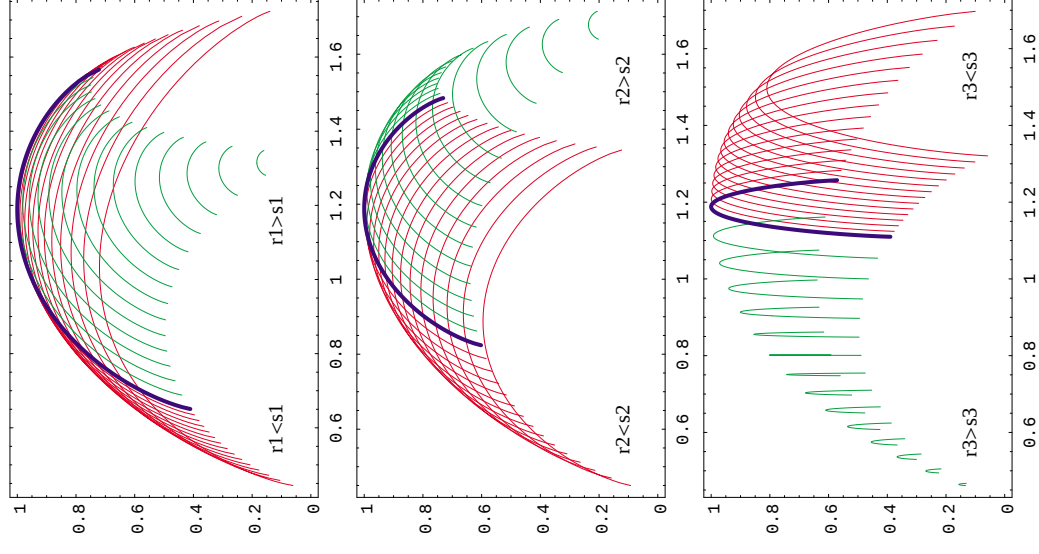


Fig. 6. Dimension curves for the measure given in Figure 2 generated by fixing the values of all the components of the vector  $r$ , but one of them  $r_i$ .