

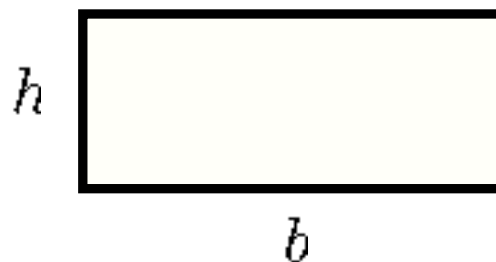


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## AREA OF A RECTANGLE

Assume that the expression giving the area of a rectangle is unknown, but we know that it is a function  $f(b, h)$  of the base and the height of the rectangle:

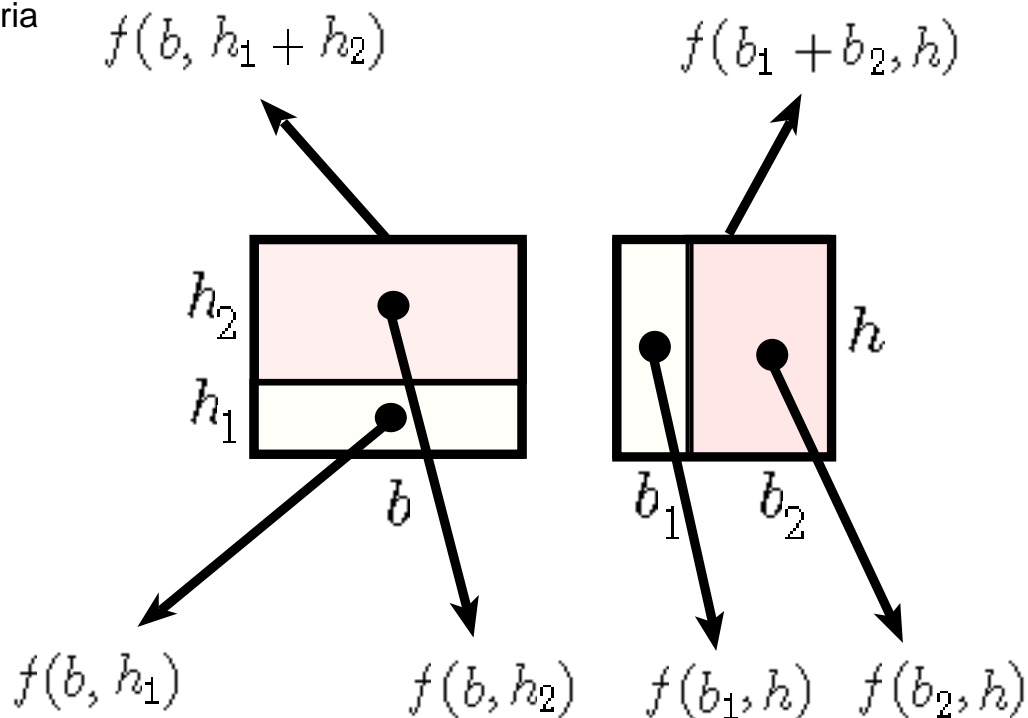
$$\text{Area} = f(b, h)$$



Is it possible to obtain this expression using some rectangle properties and functional equations?

**The answer is “YES”**

## AREA OF A RECTANGLE



According to the Figure, we have

$$\begin{aligned} f(b, h_1 + h_2) &= f(b, h_1) + f(b, h_2) \\ f(b_1 + b_2, h) &= f(b_1, h) + f(b_2, h). \end{aligned}$$

The solution of this system of functional equations is:

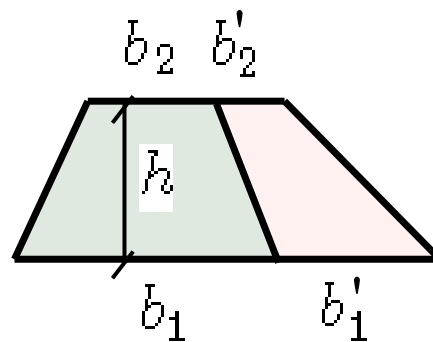
$$f(b, h) = cbh,$$

where  $c$  is an arbitrary non-negative constant.

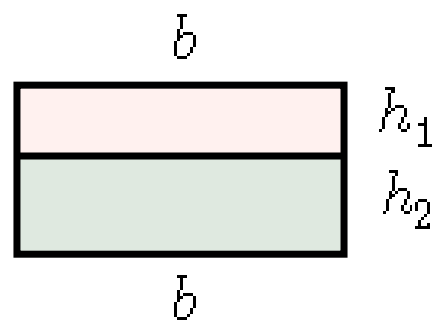
This proves that the area of a rectangle is not the well known “base  $\times$  height”, but “a constant  $\times$  its base  $\times$  its height”. The constant takes care of the units we use for base, height and area.

## AREA OF A TRAPEZOID

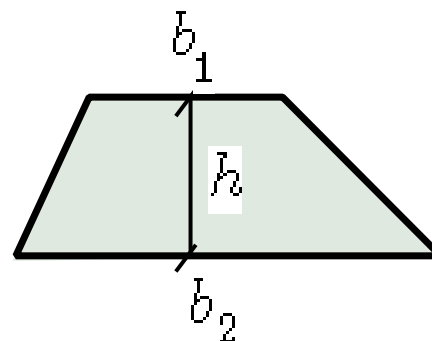
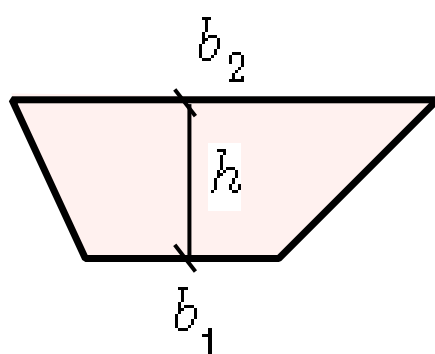
$$\text{Area} = f(b_1, b_2, h)$$



$$f(b_1 + b_1', b_2 + b_2', h) = f(b_1, b_2, h) + f(b_1', b_2', h)$$



$$f(b, b, h_1 + h_2) = f(b, b, h_1) + f(b, b, h_2)$$



$$f(b_1, b_2, h) = f(b_2, b_1, h)$$



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## AREA OF A TRAPEZOID

### SYSTEM OF EQUATIONS

$$\begin{aligned}f(b_1 + b'_1, b_2 + b'_2, h) &= f(b_1, b_2, h) + f(b'_1, b'_2, h) \\f(b, b, h_1 + h_2) &= f(b, b, h_1) + f(b, b, h_2) \\f(b_1, b_2, h) &= f(b_2, b_1, h)\end{aligned}$$

### SOLUTION

The general solution of this system of functional equations is:

$$\text{Area} = f(b_1, b_2, h) = c(b_1 + b_2)h,$$

where  $c$  is a positive arbitrary constant, which considers the measurement units used for the bases  $b_1$  and  $b_2$ , the height  $h$ , and the area.



## SIMPLE INTEREST

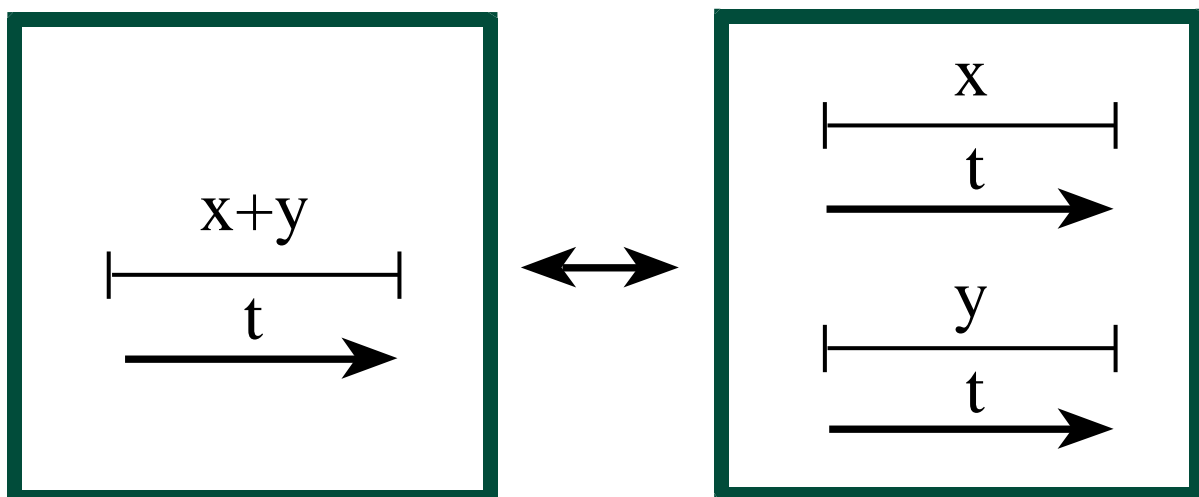
Let  $f(x, t)$  be the interest we get from the bank when we deposit an amount  $x$  during a time period  $t$ .

In the case of the simple interest, we have the following assumptions:

1 At the end of the time period  $t$ , we receive the same interest in the following two cases:

- (a) We deposit the amount  $x + y$  in one account.
- (b) We deposit the amount  $x$  in one account, and the amount  $y$  in another account.

$$f(x + y, t) = f(x, t) + f(y, t).$$

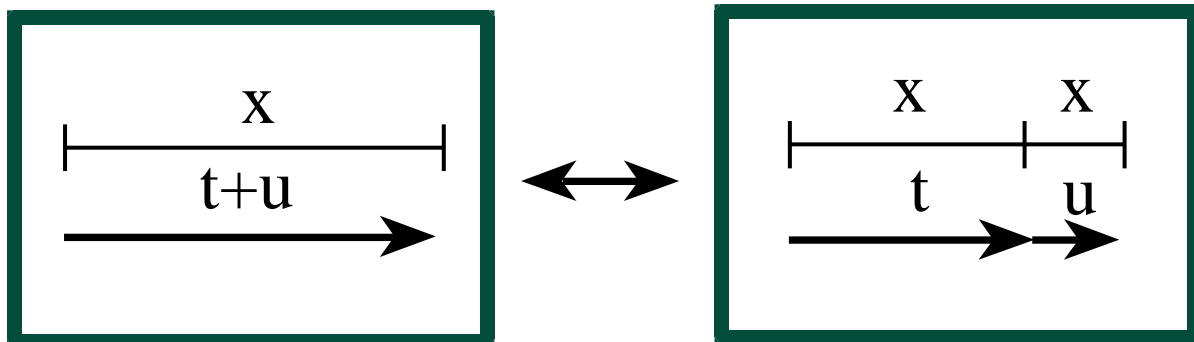


## SIMPLE INTEREST

2 At the end of the time period  $t + u$ , you receive the same interest in the following two cases:

- (a) You deposit the amount  $x$  during a period of duration  $t + u$ , or
- (b) You deposit the amount  $x$  first during a period of duration  $t$  and later for a period of duration  $u$ .

$$f(x, t + u) = f(x, t) + f(x, u).$$



### SYSTEM OF EQUATIONS

$$\left. \begin{aligned} f(x, t + u) &= f(x, t) + f(x, u) \\ f(x, t + u) &= f(x, t) + f(x, u) \end{aligned} \right\} x, y, t, u \in \mathbb{R}_+$$

### SOLUTION

$$f(x, t) = cxt,$$

where  $c$  is an arbitrary non-negative constant.



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## A COMPARISON OF TWO SOLUTIONS

### AREA OF A RECTANGLE

$$\begin{aligned}f(b, h_1 + h_2) &= f(b, h_1) + f(b, h_2) \\f(b_1 + b_2, h) &= f(b_1, h) + f(b_2, h).\end{aligned}$$

#### Solution

$$f(b, h) = cbh$$

### SIMPLE INTEREST

$$\left. \begin{aligned}f(x, t + u) &= f(x, t) + f(x, u) \\f(x, t + u) &= f(x, t) + f(x, u)\end{aligned} \right\} x, y, t, u \in \mathbb{R}_+$$

#### Solution

$$f(x, t) = cxt$$



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## DEFINITIONS OF FUNCTIONAL EQUATION

**Definition 1** *A functional equation is an equation in which the unknowns are functions. We exclude differential and integral equations.*

### EXAMPLES

#### CAUCHY'S FUNCTIONAL EQUATION

$$f(x + y) = f(x) + f(y)$$

**Solution**

$$f(x) = cx,$$

where  $c$  is an arbitrary constant.

#### PEXIDER'S FUNCTIONAL EQUATION

$$f(x + y) = g(x) + h(y)$$

**Solution**

$$f(x) = cx + a + b$$

$$g(x) = cx + a$$

$$h(x) = cx + b$$



## ASSOCIATIVITY EQUATION

$$F[F(x, y), z] = F[x, F(y, z)]$$

### Solution

$$F(x, y) = f^{-1}[f(x) + f(y)]$$

### The case of the sum

$$\begin{aligned} f(x) &= x \\ F(x, y) &= x + y \end{aligned}$$

### The case of the product

$$\begin{aligned} f(x) &= \log(x) \\ F(x, y) &= \exp[\log(x) + \log(y)] = xy \end{aligned}$$



## SUMS OF PRODUCTS EQUATION

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All solutions of the functional equation

$$\sum_{k=1}^n f_k(x)g_k(y) = 0 \quad (1)$$

can be written in the form:

$$\begin{aligned} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \dots \\ f_n(x) \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nr} \end{bmatrix} \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \dots \\ \varphi_r(x) \end{bmatrix} \\ \begin{bmatrix} g_1(y) \\ g_2(y) \\ \dots \\ g_n(y) \end{bmatrix} &= \begin{bmatrix} b_{1r+1} & b_{1r+2} & \dots & b_{1n} \\ b_{2r+1} & b_{2r+2} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{nr+1} & b_{nr+2} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} \psi_{r+1}(y) \\ \psi_{r+2}(y) \\ \dots \\ \psi_n(y) \end{bmatrix} \end{aligned} \quad (2)$$

where  $\{\varphi_1(x), \varphi_2(x), \dots, \varphi_r(x)\}$ , on one hand, and  $\{\psi_{r+1}(x), \psi_{r+2}(x), \dots, \psi_n(x)\}$ , on the other hand, are arbitrary systems of functions which are mutually linearly independent,  $0 < r < n$  is an integer, and the constants  $a_{ij}$  and  $b_{ij}$  satisfy

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1r} & a_{2r} & \dots & a_{nr} \end{bmatrix} \begin{bmatrix} b_{1r+1} & b_{1r+2} & \dots & b_{1n} \\ b_{2r+1} & b_{2r+2} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{nr+1} & b_{nr+2} & \dots & b_{nn} \end{bmatrix} = \mathbf{0}$$



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## NORMAL CONDITIONALS

Consider a two-dimensional random variable  $(X, Y)$  with joint, marginal and conditionals densities  $f_{(X,Y)}(x, y)$ ,  $g(x)$ ,  $h(y)$ ,  $f_{X|Y}(x|y)$  and  $f_{Y|X}(y|x)$ , respectively. Then we have

$$f_{(X,Y)}(x, y) = f_{X|Y}(x|y)h(y) = f_{Y|X}(y|x)g(x)$$

If we assume normal conditionals we have

$$f_{Y|X}(y|x) \propto \frac{\exp \left\{ -\frac{1}{2} \left[ \frac{y-a(x)}{b(x)} \right]^2 \right\}}{b(x)}$$

$$f_{X|Y}(x|y) \propto \frac{\exp \left\{ -\frac{1}{2} \left[ \frac{x-d(y)}{c(y)} \right]^2 \right\}}{c(y)}$$

Taking logarithms and letting

$$u(x) = \log[g(x)/b(x)]; v(y) = \log[h(y)/c(y)]$$

we get

$$\begin{aligned} & [2u(x)b^2(x) - a^2(x)]c^2(y) - y^2c^2(y) \\ & + x^2b^2(x) + b^2(x)[d^2(y) - 2v(y)c^2(y)] \\ & + 2a(x)yc^2(y) - 2xb^2(x)d(y) = 0, \end{aligned}$$

which is a functional equations of the form

$$\sum_{k=1}^n f_k(x)g_k(y) = 0.$$



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## NORMAL CONDITIONALS

### Solution

$$\begin{bmatrix} 2u(x)b^2(x) - a^2(x) \\ b^2(x) \\ 1 \\ x^2b^2(x) \\ 2a(x) \\ xb^2(x) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \\ 0 & 0 & 1 \\ a_{51} & a_{52} & a_{53} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b^2(x) \\ xb^2(x) \\ x^2b^2(x) \end{bmatrix}$$

and

$$\begin{bmatrix} c^2(y) \\ d^2(y) - 2v(y)c^2(y) \\ -y^2c^2(y) \\ 1 \\ yc^2(y) \\ -2d(y) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b_{24} & b_{25} & b_{26} \\ 0 & 0 & -1 \\ b_{44} & b_{45} & b_{46} \\ 0 & 1 & 0 \\ b_{64} & b_{65} & b_{66} \end{bmatrix} \begin{bmatrix} c^2(y) \\ yc^2(y) \\ y^2c^2(y) \end{bmatrix}$$

where

$$\begin{bmatrix} a_{11} & 1 & a_{31} & 0 & a_{51} & 0 \\ a_{12} & 0 & a_{32} & 0 & a_{52} & 1 \\ a_{13} & 0 & a_{33} & 1 & a_{53} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ b_{24} & b_{25} & b_{26} \\ 0 & 0 & -1 \\ b_{44} & b_{45} & b_{46} \\ 0 & 1 & 0 \\ b_{64} & b_{65} & b_{66} \end{bmatrix} = \mathbf{0}$$



## NORMAL CONDITIONALS

### Solution

$$a(x) = \frac{-(A+Bx+Cx^2)}{(D+2Ex+Fx^2)}$$

$$d(y) = \frac{-(H+By+Ey^2)}{(J+2Cy+Fy^2)}$$

$$b^2(x) = \frac{1}{(D+2Ex+Fx^2)}$$

$$c^2(y) = \frac{1}{(J+2Cy+Fy^2)}$$

$$f(x, y) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{G}{2}\right\} \times \\ \times \exp\left\{-\frac{1}{2}[2Hx + 2Ay + Jx^2 + Dy^2 + 2Bxy]\right\} \\ \times \exp\left\{-\frac{1}{2}[2Cx^2y + 2Exy^2 + Fx^2y^2]\right\}$$

where the constant must satisfy one of the following two conditions:

- **Normal Model:**

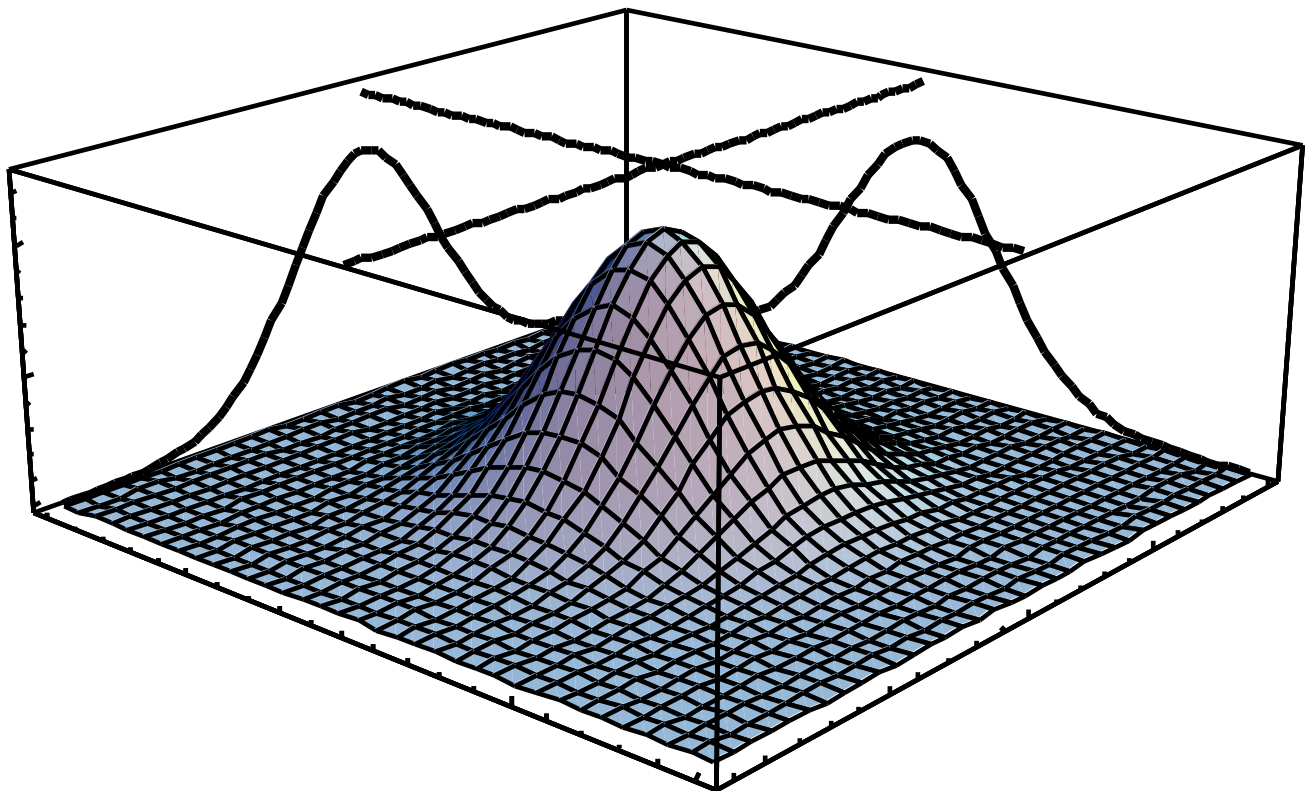
$$F = E = C = 0; \quad D > 0; \quad J > 0; \quad B^2 < DJ$$

- **Non-Normal Model:**

$$F > 0; \quad FD > E^2; \quad JF > C^2$$

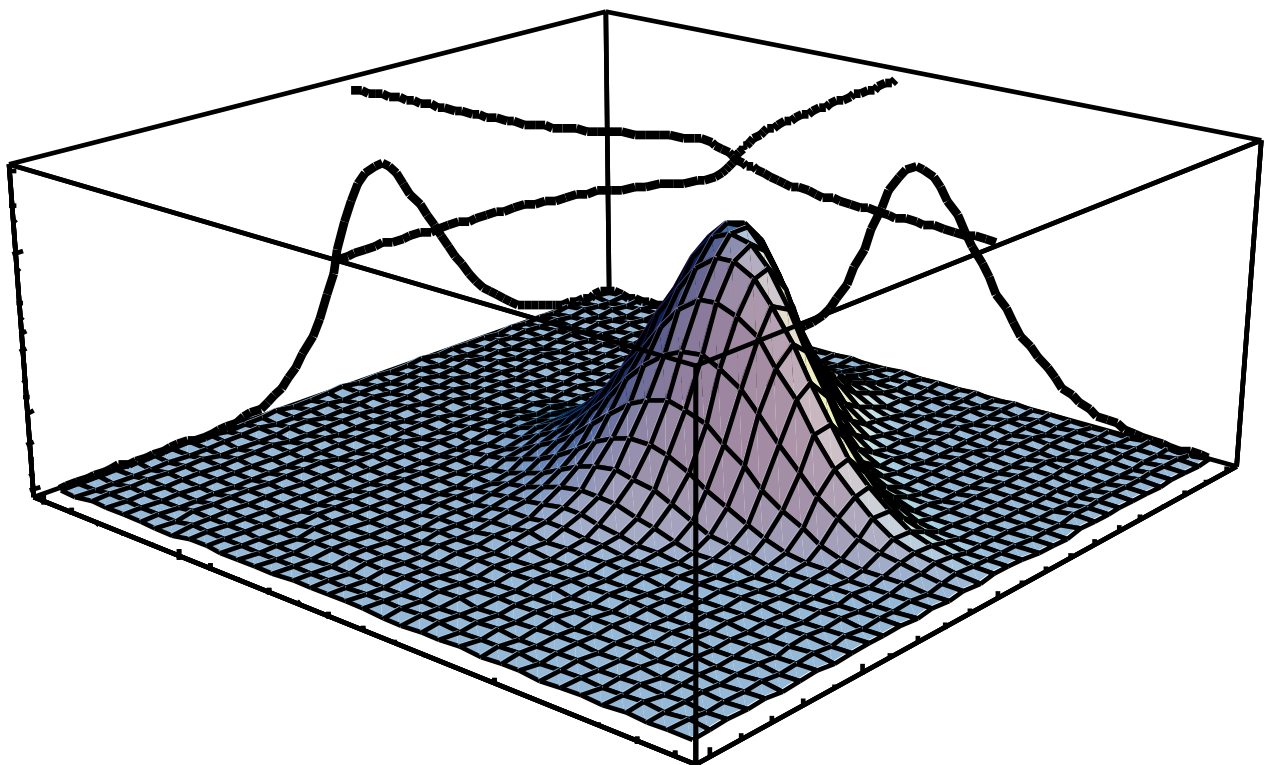
## Normal Model

- Regression lines are straight lines.
- Marginal distributions are normal.
- Mode is in the intersection of regression lines.



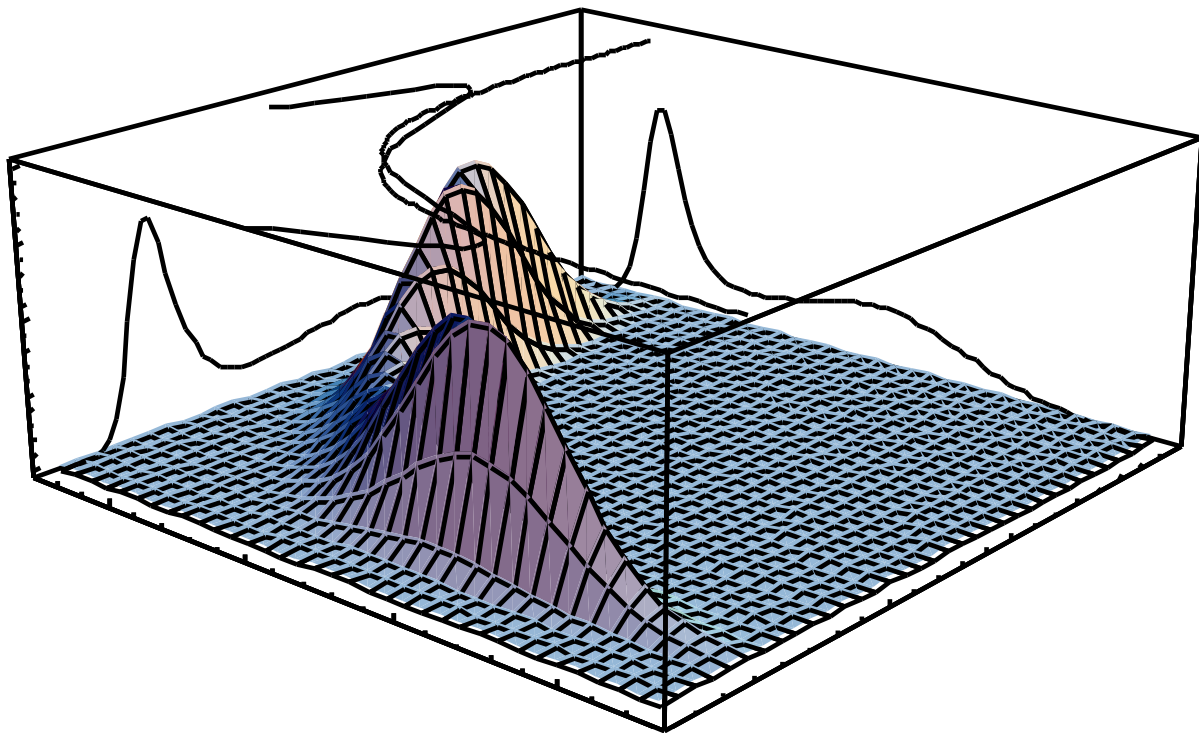
## Non-Normal Model

- Regression lines are not straight lines.
- Marginal distributions are not normal.
- Mode is in the intersection of regression lines



## Non-Normal Model Two Modes

- Regression lines are not straight lines.
- Marginal distributions are not normal.
- Modes are in the intersection of regression lines





## COVER WITH POLYNOMIAL CROSS SECTIONS

We look for the most general surface of the form  $Z = z(x, y)$  such that all of its cross-sections or intersections with planes parallel to the coordinate planes are of the form

$$\begin{aligned} z(x, y) &= a(y)x^2 + b(y)x + c(y) \\ z(x, y) &= d(x)y^2 + e(x)y + f(x), \end{aligned} \quad (3)$$

where  $a(y), b(y)$  and  $c(y)$ , on one hand, and  $d(x), e(x)$  and  $f(x)$ , on the other, are the coefficients of the two polynomials intersection curves associated with planes  $Y = y$  and  $X = x$ , respectively.

The equations state that we obtain second degree polynomials when we intersect planes  $X = \text{constant}$  and  $Y = \text{constant}$ .

From (3) we get

$$a(y)x^2 + b(y)x + c(y) - d(x)y^2 - e(x)y - f(x) = 0,$$

which is of the form  $\sum_{k=1}^n f_k(x)g_k(y) = 0$ .



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## COVER WITH POLYNOMIAL CROSS SECTIONS

### SOLUTION

$$\begin{bmatrix} x^2 \\ x \\ 1 \\ d(x) \\ e(x) \\ f(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a(y) \\ b(y) \\ c(y) \\ -y^2 \\ -y \\ -1 \end{bmatrix} = \begin{bmatrix} b_{14} & b_{15} & b_{16} \\ b_{24} & b_{25} & b_{26} \\ b_{34} & b_{35} & b_{36} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y^2 \\ y \\ 1 \end{bmatrix}$$

where

$$\begin{bmatrix} 1 & 0 & 0 & a_{41} & a_{51} & a_{61} \\ 0 & 1 & 0 & a_{42} & a_{52} & a_{62} \\ 0 & 0 & 1 & a_{43} & a_{53} & a_{63} \end{bmatrix} \begin{bmatrix} b_{14} & b_{15} & b_{16} \\ b_{24} & b_{25} & b_{26} \\ b_{34} & b_{35} & b_{36} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \mathbf{0}$$



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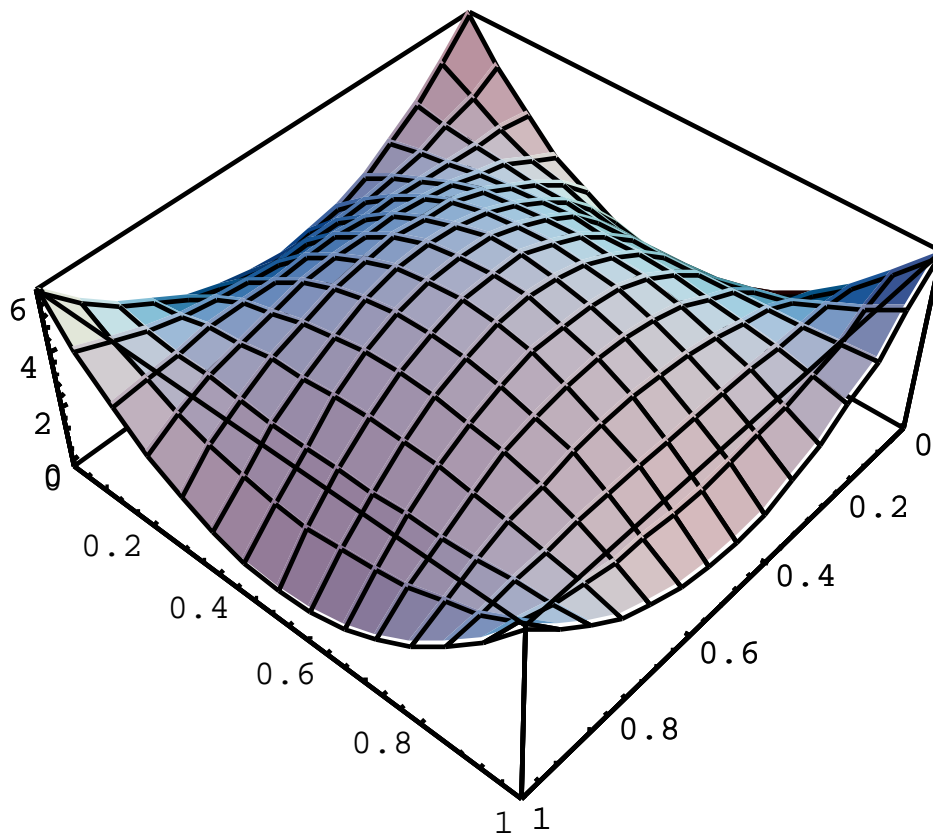
## COVER WITH POLYNOMIAL CROSS SECTIONS

### SOLUTION

$$\begin{aligned} a(y) &= A + By + Cy^2 & ; & & b(y) &= D + Ey + Fy^2 \\ c(y) &= G + Hy + Iy^2 & ; & & d(x) &= I + Fx + Cx^2 \\ e(x) &= H + Ex + Bx^2 & ; & & f(x) &= G + Dx + Ax^2 \end{aligned}$$

$$\begin{aligned} z(x, y) &= Cx^2y^2 + Bx^2y + Fxy^2 \\ &+ Ax^2 + Exy + Iy^2 + Dx + Hy + G, \end{aligned}$$

where  $A, B, C, D, E, F, G, H$  and  $I$  are arbitrary constants.





## SYNTHESIS OF JUDGEMENTS

Suppose that we have  $n$  quantifiable judgements  $x_1, \dots, x_n$  which we want to synthesize into a consensus judgement  $f(x_1, \dots, x_n)$ .

We make the following assumptions:

1. **Separability:** The function  $f$  is separable:

$$f(x_1, \dots, x_n) = g_1(x_1) \Delta g_2(x_2) \Delta \dots \Delta g_n(x_n).$$

where  $\Delta$  is an associative, commutative and cancelative operation.

2. **Equality :** All members in the jury have the same weight in the final decision.
3. **Unanimity :** When the judges coincide in the same result  $x$ , the consensus decision must be the same  $x$ :

$$f(x, \dots, x) = x$$



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## SYNTHESIS OF JUDGEMENTS

The associativity assumption implies:

$$y_1 \Delta y_2 = \varphi^{-1}[\varphi(y_1) + \varphi(y_2)],$$

that is,

$$f(x_1, \dots, x_n) = \varphi^{-1}\left\{\sum_{i=1}^n \varphi[g_i(x_i)]\right\}$$

Because of equality, all  $g_i$  must coincide:

$$f(x_1, \dots, x_n) = \varphi^{-1}\left\{\sum_{i=1}^n \varphi[g(x_i)]\right\}$$

From unanimity we get

$$f(x, \dots, x) = x \Rightarrow g(x) = \varphi^{-1}\left[\frac{\varphi(x)}{n}\right]$$

and, finally we obtain:

$$f(x_1, x_2, \dots, x_n) = \varphi^{-1}\left\{\frac{\sum_{i=1}^n \varphi(x_i)}{n}\right\}$$

where  $\varphi$  is an arbitrary invertible function.

### PARTICULAR CASES

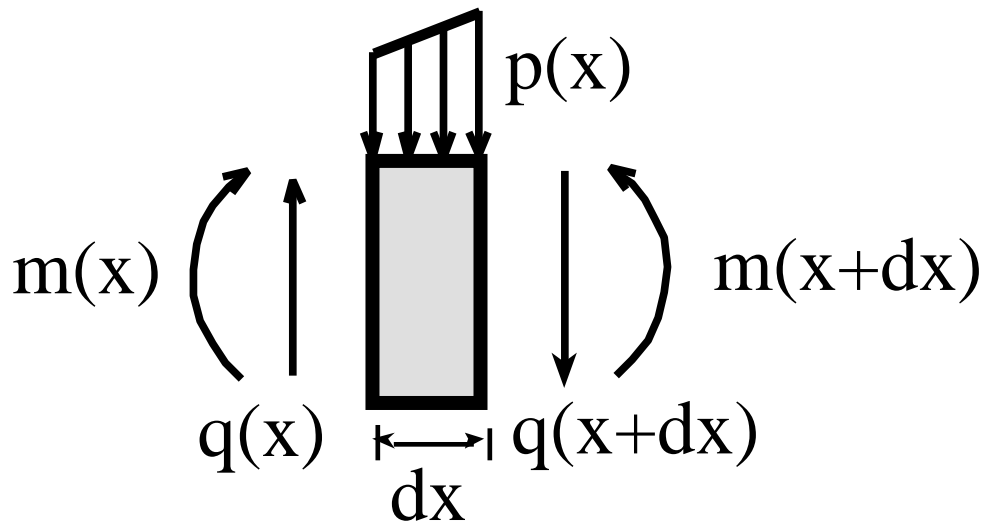
- The arithmetic mean:  $\varphi(x) = x$ .
- The geometric mean:  $\varphi(x) = \log x$ .
- The  $L_p$  mean:  $\varphi(x) = x^p$ .

## THE BEAM EQUATION

### CLASSICAL APPROACH

#### Differential equations

The equilibrium equations are stated for differential pieces.



The equilibrium of vertical forces leads to

$$q(x + dx) = q(x) + p(x)dx \Rightarrow q'(x) = p(x), \quad (4)$$

where  $q(x)$  and  $p(x)$  are the shear and the load at the point  $x$ , respectively, and the equilibrium of moments

$$m(x + dx) = m(x) + q(x)dx + p(x)dx dx/2 \quad (5)$$

which implies

$$m'(x) = q(x), \quad (6)$$

where  $m(x)$  is the bending moment at  $x$ .



## THE BEAM EQUATION

Using the well known strength of materials relation

$$m(x) = EIz''(x), \quad (7)$$

where  $z(x)$  is the deflection of the beam and from (4), (6) and (7) we get the well known differential equation

$$EIz^{(iv)}(x) = p(x). \quad (8)$$

Letting  $w(x) = z'(x)$  be the rotation of the beam at point  $x$ , from Equations (4), (6) and (7) we get the system of differential equations

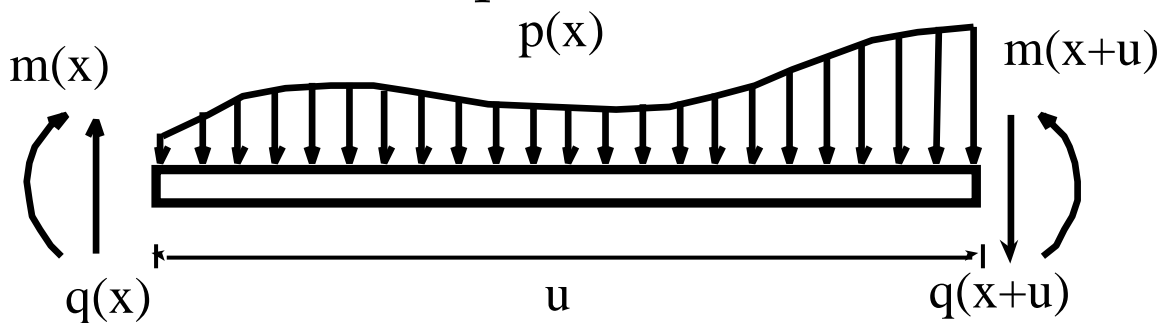
$$\begin{aligned} q'(x) &= p(x) \\ m'(x) &= q(x) \\ w'(x) &= \frac{m(x)}{EI} \\ z'(x) &= w(x), \end{aligned} \quad (9)$$

which is the usual mathematical model in terms of differential equations.

## NEW APPROACH

### Functional equations

In the new approach, the equilibrium is analyzed for discrete pieces.



The equilibrium of vertical forces leads to

$$\boxed{q(x+u) = q(x) + A(x, u),} \quad (10)$$

where

$$A(x, u) = \int_x^{x+u} p(s) ds. \quad (11)$$

The equilibrium of moments gives

$$\boxed{m(x+u) = m(x) + uq(x) + B(x, u),} \quad (12)$$

where

$$B(x, u) = \int_x^{x+u} (x+u-s)p(s) ds. \quad (13)$$



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## THE BEAM EQUATION

Using Equation (7) we get

$$\begin{aligned} w(x+u) &= w(x) + \frac{1}{EI} \int_x^{x+u} m(s) ds \\ &= w(x) + \frac{1}{EI} \left[ m(x)u + q(x)\frac{u^2}{2} + C(x, u) \right]. \end{aligned} \quad (14)$$

In addition we have

$$\begin{aligned} z(x+u) &= z(x) + \int_x^{x+u} w(s) ds \\ &= z(x) + w(x)u + \frac{1}{EI} \left[ m(x)\frac{u^2}{2} + q(x)\frac{u^3}{6} + D(x, u) \right]. \end{aligned} \quad (15)$$

Thus, we get the system of functional equations

$$\begin{aligned} q(x+u) &= q(x) + A(x, u) \\ m(x+u) &= m(x) + uq(x) + B(x, u) \\ w(x+u) &= w(x) + \frac{1}{EI} \left[ m(x)u + q(x)\frac{u^2}{2} + C(x, u) \right] \\ z(x+u) &= z(x) + w(x)u + \frac{1}{EI} \left[ m(x)\frac{u^2}{2} + q(x)\frac{u^3}{6} + D(x, u) \right] \end{aligned} \quad (16)$$

where

$$\begin{aligned} A(x, u) &= \int_x^{x+u} p(s) ds \\ B(x, u) &= \int_x^{x+u} (x+u-s)p(s) ds \\ C(x, u) &= \int_x^{x+u} B(x, s-x) ds \\ D(x, u) &= \int_x^{x+u} C(x, s-x) ds \end{aligned} \quad (17)$$

Equation (16) is equivalent to the system of functional equations (9).



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## CAUCHY'S EQUATIONS

**Theorem 1** If the equation

$$f(x + y) = f(x) + f(y) \quad ; \quad x, y \in \mathbf{R} \quad (18)$$

is satisfied for all real  $x, y$ , and if the function  $f(x)$  is (a) continuous at a point, or (b) nonnegative for small  $x$ , or (c) bounded in an interval or (d) integrable or (e) measurable, then

$$f(x) = cx \quad , \quad x \in \mathbf{R} \quad (19)$$

where  $c$  is an arbitrary constant.

**Theorem 2** The most general solutions, which are continuous-at-a-point, of the functional equation

$$f(xy) = f(x) + f(y) \quad x, y \in \mathbf{T} \quad (20)$$

are

$$f(x) = \begin{cases} c \log(x) & \text{if } \mathbf{T} = \mathbf{R}_{++} \\ c \log(|x|) & \text{if } \mathbf{T} = \mathbf{R} - \{0\} \\ 0 & \text{if } \mathbf{T} = \mathbf{R} \end{cases} \quad (21)$$



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## CAUCHY'S EQUATIONS

**Theorem 3** *The most general solutions of the functional equation*

$$f(x + y) = f(x)f(y) ; x, y \in \mathbf{R} \text{ or } x, y \in \mathbf{R}_{++} \quad (22)$$

*which are continuous-at-a-point are*

$$f(x) = \exp(cx) \text{ and } f(x) = 0. \quad (23)$$

**Theorem 4** *The most general solutions, which are continuous-at-a-point, of the functional equation*

$$f(xy) = f(x)f(y) \quad x, y \in \mathbf{T} \quad (24)$$

*are*

$$\left. \begin{aligned} f(x) &= 1 \\ f(x) &= \begin{cases} |x|^c & x \neq 0 \\ 0 & x = 0 \end{cases} \\ f(x) &= \begin{cases} |x|^c \operatorname{sgn}(x) & x \neq 0 \\ 0 & x = 0 \end{cases} \end{aligned} \right\} \text{ if } \mathbf{T} = \mathbf{R} \quad (25)$$

$$\left. \begin{aligned} f(x) &= |x|^c \\ f(x) &= |x|^c \operatorname{sgn}(x) \end{aligned} \right\} \text{ if } \mathbf{T} = \mathbf{R} - \{0\}$$

$$f(x) = x^c \text{ if } \mathbf{T} = \mathbf{R}_{++}$$

*where  $c$  is an arbitrary real number, together with*

$$f(x) = 0 ; f(x) = \begin{cases} 0 & |x| \neq 1 \\ x & |x| = 1 \end{cases} ; f(x) = \begin{cases} 0 & |x| \neq 1 \\ 1 & |x| = 1 \end{cases}$$

*which are common to the three domains.*



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## PEXIDER'S EQUATIONS

**Theorem 5** (*Pexider's main equation*) *The most general system of solutions of*

$$f(x+y) = g(x)+h(y) ; x, y \in \mathbf{R} \text{ or } [a, b] \text{ with } a, b \in \mathbf{R} \quad (26)$$

*with  $f$ : (a) continuous at a point, or (b) non-negative for small  $x$ , or (c) bounded in an interval, is*

$$f(x) = Ax + B + C ; g(x) = Ax + B ; h(x) = Ax + C \quad (27)$$

*where  $A, B$  and  $C$  are arbitrary constants.*

**Theorem 6** *The most general system of solutions of*

$$f(xy) = g(x) + h(y); x, y \in \mathbf{R} \text{ or } \mathbf{R}_{++} \text{ or } \mathbf{R} - \{0\} \quad (28)$$

*with  $f$  continuous at a point is*

$$\left. \begin{aligned} f(x) &= A \log(BCx) \\ g(x) &= A \log(Bx) \\ h(x) &= A \log(Cx) \end{aligned} \right\} ; x, y \in \mathbf{R}_{++}$$

$$\left. \begin{aligned} f(x) &= A \log(BC|x|) \\ g(x) &= A \log(B|x|) \\ h(x) &= A \log(C|x|) \end{aligned} \right\} x, y \in \mathbf{R} - \{0\} \quad (29)$$

$$f(x) = A + B; g(x) = A; h(x) = B;$$

*if  $x, y \in \mathbf{R} \text{ or } \mathbf{R} - \{0\} \text{ or } \mathbf{R}_{++}$*



## PEXIDER'S EQUATIONS

**Theorem 7** *The most general system of solutions of*

$$f(x + y) = g(x)h(y); x, y \in \mathbf{R} \quad (30)$$

*with  $f$  continuous at a point is*

$$f(x) = ABe^{Cx}; g(x) = Ae^{Cx}; h(x) = Be^{Cx} \quad (31)$$

*where  $A$ ,  $B$  and  $C$  are arbitrary non-zero constants, together with the trivial solutions*

$$\begin{aligned} f(x) = g(x) = 0; h(x) \text{ arbitrary} \\ f(x) = h(x) = 0; g(x) \text{ arbitrary.} \end{aligned} \quad (32)$$

**Theorem 8** *The most general system of solutions of*

$$f(xy) = g(x)h(y); x, y \in \mathbf{R} \text{ or } \mathbf{R}_{++} \text{ or } \mathbf{R} - \{0\} \quad (33)$$

*with  $f$  continuous at a point is*

$$\begin{aligned} f(x) = AB; g(x) = A; h(x) = B \\ \text{if } x, y \in \mathbf{R} \text{ or } \mathbf{R} - \{0\} \text{ or } \mathbf{R}_{++} \end{aligned} \quad (34)$$

$$\left. \begin{aligned} f(x) &= ABx^C \\ g(x) &= Ax^C \\ h(x) &= Bx^C \end{aligned} \right\} \text{ if } x, y \in \mathbf{R}_{++} \quad (35)$$

$$\begin{aligned} f(x) = g(x) = 0; h(x) \text{ arbitrary} \\ f(x) = h(x) = 0; g(x) \text{ arbitrary.} \end{aligned} \quad (36)$$



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## PEXIDER'S EQUATIONS

$$\left. \begin{aligned} f(x) &= AB |x|^C \\ g(x) &= A |x|^C \\ h(x) &= B |x|^C \end{aligned} \right\} \text{ or } \left. \begin{aligned} f(x) &= AB |x|^C \operatorname{sgn}(x) \\ g(x) &= A |x|^C \operatorname{sgn}(x) \\ h(x) &= B |x|^C \operatorname{sgn}(x) \end{aligned} \right\} \\ \text{if } x, y \in \mathbf{R} - \{0\},$$

$$\left. \begin{aligned} f(x) &= \begin{cases} AB |x|^C & x \neq 0 \\ 0 & x = 0 \end{cases} \\ g(x) &= \begin{cases} A |x|^C & x \neq 0 \\ 0 & x = 0 \end{cases} \\ h(x) &= \begin{cases} B |x|^C & x \neq 0 \\ 0 & x = 0 \end{cases} \end{aligned} \right\} \text{ or}$$

$$\left. \begin{aligned} f(x) &= \begin{cases} AB |x|^C \operatorname{sgn}(x) & x \neq 0 \\ 0 & x = 0 \end{cases} \\ g(x) &= \begin{cases} A |x|^C \operatorname{sgn}(x) & x \neq 0 \\ 0 & x = 0 \end{cases} \\ h(x) &= \begin{cases} B |x|^C \operatorname{sgn}(x) & x \neq 0 \\ 0 & x = 0 \end{cases} \end{aligned} \right\} \text{ if } x, y \in \mathbf{R},$$

(37)

where  $A$ ,  $B$  and  $C$  are arbitrary constants.



## TRANSLATION EQUATION

**Theorem 9** The general continuous solution of the translation equation

$$F[F(x, u), v] = F(x, u + v) \quad (38)$$
$$x, F(x, u) \in (a, b) ; u, v \in (-\infty, \infty)$$

is

$$F(x, y) = f[f^{-1}(x) + y] \quad (39)$$

where  $f$  is an arbitrary function which is continuous and strictly monotonic in  $(-\infty, \infty)$ , if one of the following conditions holds:

- (a)  $F(x, u)$  is strictly monotonic for each value of  $x$  with respect to  $u$  and for uncountably many values of  $u$  with respect to  $x$ .
- (b)  $F(x, u)$  is continuous for each value of  $u$  with respect to  $x$  and for  $x = x_0$  with respect to  $u$  and nonconstant for every fixed value of  $x$ .



## ITERATIVE METHODS

The translation equation has many interesting applications. Let us define the  $n$ -th iterate  $F(x, n) = g_n(x)$ :

$$\begin{cases} g_0(x) = x \\ g_n(x) = g[g_{n-1}(x)] \text{ for } n > 0 \end{cases}$$

Then  $F(x, n)$  satisfies translation equation:

$$\begin{aligned} g_n[g_m(x)] &= g_{m+n}(x) \Leftrightarrow \\ F[F(x, m), n] &= F(x, m + n) \end{aligned}$$

Thus:

$$F(x, y) = f[f^{-1}(x) + y] ; F(x, 1) = g(x) \quad (40)$$

which implies

$$g[f(x)] = f(x + 1) \quad (41)$$

Thus, the problem of finding the  $n$ -th iterate can be solved by solving the equivalent functional equation  $g[f(x)] = f(x + 1)$  or  $f^{-1}[g(x)] = f^{-1}(x) + 1$ , which is a particular case of the Abel equation.



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## SOME METHODS TO SOLVE FUNCTIONAL EQUATIONS

The most common methods for solving functional equations are:

1. Replacing variables by given values.
2. Transforming one or several variables.
3. Transforming one or several functions.
4. Using a more general functional equation.
5. Treating variables as constants.
6. Inductive methods.
7. Iterative methods.
8. Separation of variables.
9. Analytical techniques (differentiation, integration, etc.).
10. Mixed methods.



## REPLACING VARIABLES BY GIVEN VALUES

**Example 1 (The homogeneous functions)** The general solution of the functional equation

$$f(yx) = y^k f(x) \quad ; \quad x, y \in \mathbf{R}_+, \quad (42)$$

where  $f$  is real and  $k$  is constant, is

$$f(x) = cx^k \quad (43)$$

where  $c$  is an arbitrary constant. ■

**Proof:** Letting  $x = 1$  in (42) we get  $f(y) = cy^k$ , where  $c = f(1)$ . ■

**Example 2 (Sincov's equation).** The general solution of the functional equation

$$f(x, y) + f(y, z) = f(x, z) \quad (44)$$

is

$$f(x, y) = g(y) - g(x) \quad (45)$$

where  $g$  is an arbitrary function. ■

**Proof:** Letting  $z = 0$  in (44) and  $g(x) = f(x, 0)$ , we get (45), which satisfies (44). ■



## TRANSFORMING ONE OR SEVERAL VARIABLES

**Example 3** The general solution of the functional equation

$$G(x + z, y + z) = G(x, y) + z \quad (46)$$

is

$$G(x, y) = x + g(y - x) \quad (47)$$

where  $g$  is an arbitrary function. ■

**Proof:** Letting  $z = -x$  in (46) and  $g(x) = G(0, x)$ , we get (47). ■



## TRANSFORMING ONE OR SEVERAL FUNCTIONS

**Example 4** If the functional equation

$$f(x + y) = f(x) + f(y) + K, \quad (48)$$

where  $K$  is a real constant, is satisfied for every pair of real numbers  $x$  and  $y$  and if the function  $f(x)$  is (a) continuous in at least one point, or (b) bounded by  $K$  for small values of  $x$ , or (c) bounded in a given interval, then

$$f(x) = cx - K, \quad (49)$$

where  $c$  is an arbitrary constant. ■

**Proof:** Letting

$$f(x) = g(x) - K, \quad (50)$$

the functional equation (48) transforms to

$$g(x + y) = g(x) + g(y)$$

which is the Cauchy functional equation, with solution  $g(x) = cx$ . Replacing this in (50) we can check that (48) holds. ■



## USING A MORE GENERAL FUNCTIONAL EQUATION

**Example 5** The general solution of the functional equation

$$F[G(x, y), G(u, v)] = K[x + u, y + v] \quad (51)$$

can be obtained from the solution

$$\begin{aligned} F(x, y) &= k[f(x) + g(y)] \quad , \quad G(x, y) = f^{-1}[p(x) + q(y)], \\ K(x, y) &= k[l(x) + m(y)] \quad , \quad H(x, y) = g^{-1}[r(x) + s(y)], \\ M(x, y) &= l^{-1}[p(x) + r(y)] \quad , \quad N(x, y) = m^{-1}[q(x) + s(y)], \end{aligned}$$

of the functional equation

$$F[G(x, y), H(u, v)] = K[M(x, u), N(y, v)] \quad (52)$$

taking into account that

$$H(x, y) = G(x, y); M(x, u) = x+u; N(y, v) = y+v. \blacksquare$$

**Example 6** The two functional equations

$$\begin{aligned} F(x + y, u + v) &= K(M(x, u), N(y, v)) \\ F(F(x, y), z) &= F(x, F(y, z)), \end{aligned}$$

and

$$F[G(x, y), H(u, v)] = K[M(x, u), N(y, v)].$$

can be solved as particular cases of the above functional equation (52).  $\blacksquare$



**Example 7** The continuous (with respect to its first argument) general solution of the functional equation

$$f(x + y, z) = f(x, z)f(y, z) \quad ; \quad x, y, z \in \mathbf{R} \quad (53)$$

is

$$f(x, z) = \exp c(z)x \quad (54)$$

where  $c$  is an arbitrary function. ■

**Proof:** For each value  $z$ , (53) is Cauchy II, the solution of which is (54) (the constant depends on  $z$ ). ■

**Example 8** The general solution of

$$f(ux, y, z) = u^k f(x, y, z) \quad (55)$$

is

$$f(x, y, z) = x^k c(y, z) \quad (56)$$

where  $c$  is an arbitrary constant. ■

Since the general solution of  $f(ux) = u^k f(x)$  is  $f(x) = cx^k$ , and for each fixed  $y$  and  $z$ , Equation (55) is of this form, then (56) holds.

**Example 9** The general solution of the functional equation

$$f^{-1}(g(x) + h(y)) = \exp(x) \quad x, y \in \mathbf{R} \quad (57)$$

is

$$g(x) = f(\exp(x)) - c \quad ; \quad h(y) = c \quad (58)$$

where  $f$  is an invertible arbitrary function and  $c$  is an arbitrary constant. ■

**Proof:** Equation (57) can be written as

$$g(x) + h(y) = f(\exp(x)) \quad \forall x, y \in \mathbf{R}$$

and then we get

$$-g(x) + f(\exp(x)) = h(y) = c,$$

from which we finally obtain (58). ■



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## SOLVING FUNCTIONAL EQUATIONS BY DIFFERENTIATION

**Example 10** To solve the equation

$$f(x, y) + f(y, z) = f(x, z), \quad (59)$$

we differentiate with respect to  $x$ ,  $y$  and  $z$ , independently, and we get

$$\begin{cases} f'_1(x, y) = f'_1(x, z) \\ f'_2(x, y) + f'_1(y, z) = 0 \\ f'_2(y, z) = f'_2(x, z), \end{cases} \quad (60)$$

where the subindices refer to partial derivatives with respect to the indicated arguments. From (60), we obtain

$$\begin{aligned} f'_1(x, y) = s'(x) &\Rightarrow f(x, y) = s(x) + g(y) \\ f'_2(x, y) = -f'_1(y, z) &\Rightarrow g'(y) = -s'(y) \\ \Rightarrow g(y) &= -s(y) + k \end{aligned} \quad (61)$$

and we get  $f(x, y) = s(x) - s(y) + k$ , but substitution into (59) leads to  $k = 0$ . Thus the general differentiable solution of (59) becomes

$$f(x, y) = s(x) - s(y) \quad (62)$$