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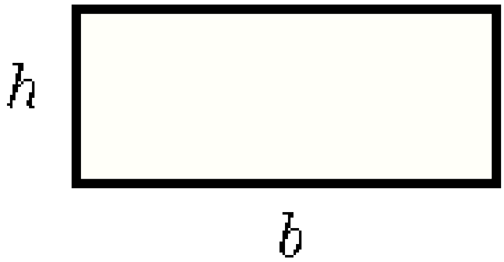
AREA OF A RECTANGLE

Assume that the expression giving the area of a rectangle is unknown, but we know that it is a function $f(b, h)$ of the base and the height of the rectangle:

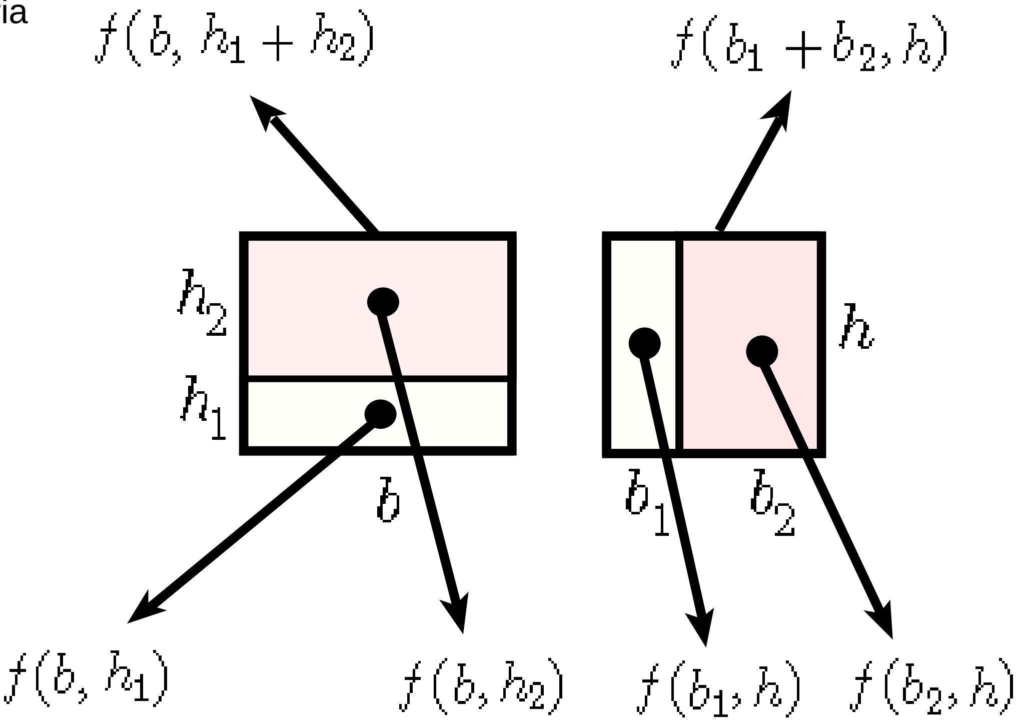
$$Area = f(b, h)$$



Is it possible to obtain this expression using some rectangle properties and functional equations?

The answer is "YES"

AREA OF A RECTANGLE



According to the Figure, we have

$$
\begin{aligned}
f(b, h_1 + h_2) &= f(b, h_1) + f(b, h_2) \\
f(b_1 + b_2, h) &= f(b_1, h) + f(b_2, h).
\end{aligned}
$$

The solution of this system of functional equations is:

$$\boxed{f(b, h) = cbh,}$$

where c is an arbitrary non-negative constant.
This proves that the area of a rectangle is not the well known "base × height", but "a constant × its base × its height". The constant takes care of the units we use for base, height and area.

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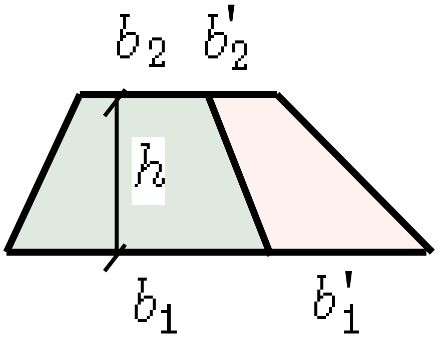
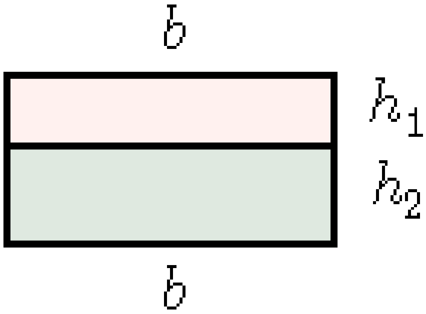
AREA OF A TRAPEZOID

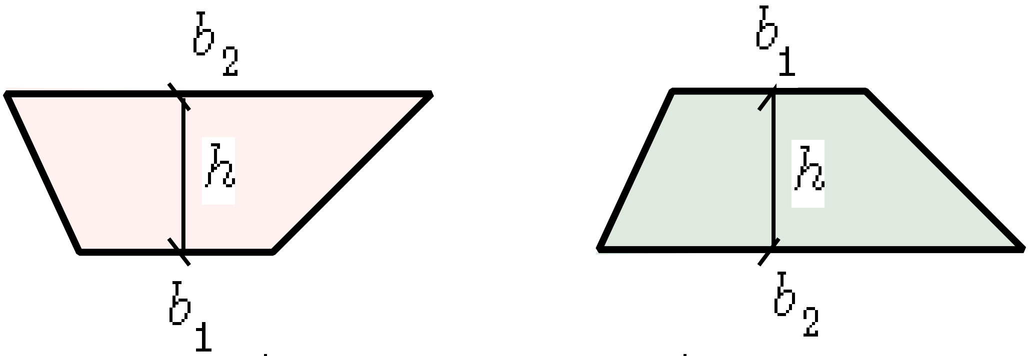
$$Area = f(b_1, b_2, h)$$



$$f(b_1+b'_1, b_2+b'_2, h) = f(b_1, b_2, h) + f(b'_1, b'_2, h)$$



$$f(b, b, h_1+h_2) = f(b, b, h_1) + f(b, b, h_2)$$



$$f(b_1, b_2, h) = f(b_2, b_1, h)$$

SYSTEM OF EQUATIONS

$$
\begin{aligned}
f(b_1+b_1', b_2+b_2', h) &= f(b_1, b_2, h) + f(b_1', b_2', h) \\
f(b, b, h_1+h_2) &= f(b, b, h_1) + f(b, b, h_2) \\
f(b_1, b_2, h) &= f(b_2, b_1, h)
\end{aligned}
$$

SOLUTION

The general solution of this system of functional equations is:

$$
\boxed{Area = f(b_1, b_2, h) = c(b_1+b_2)h,}
$$

where c is a positive arbitrary constant, which considers the measurement units used for the bases b_1 and b_2, the height h, and the area.

SIMPLE INTEREST

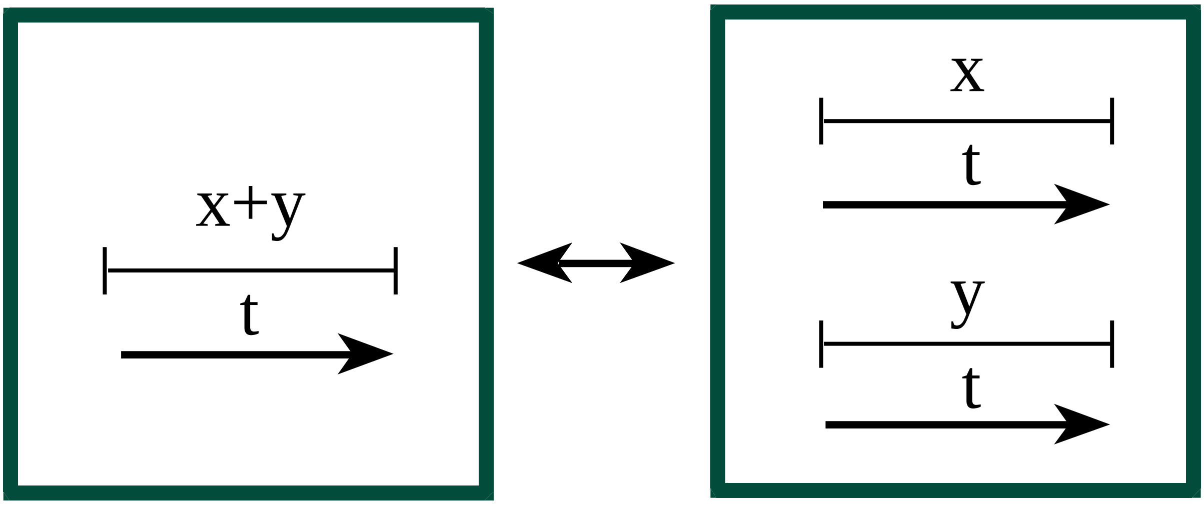
Let $f(x,t)$ be the interest we get from the bank when we deposit an amount x during a time period t.
In the case of the simple interest, we have the following assumptions:

1 At the end of the time period t, we receive the same interest in the following two cases:

(a) We deposit the amount $x+y$ in one account.

(b) We deposit the amount x in one account, and the amount y in another account.

$$f(x+y,t) = f(x,t) + f(y,t).$$



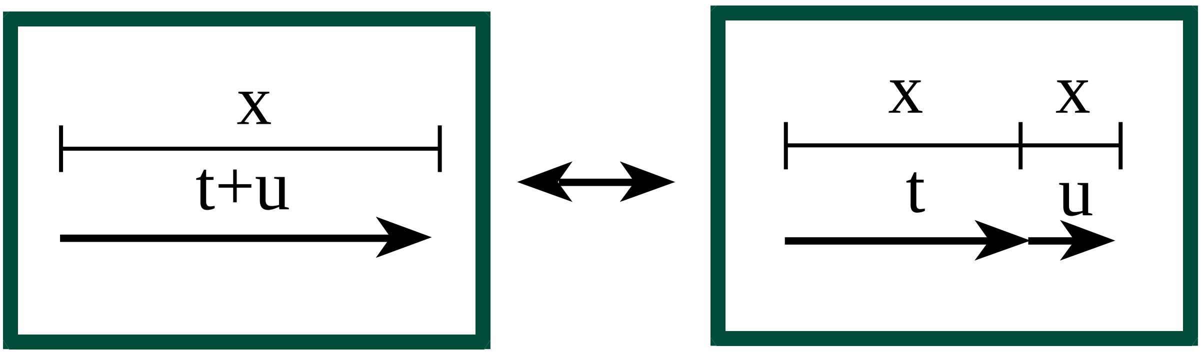
SIMPLE INTEREST

2 At the end of the time period $t+u$, you receive the same interest in the following two cases:

(a) You deposit the amount x during a period of duration $t+u$, or

(b) You deposit the amount x first during a period of duration t and later for a period of duration u.

$$f(x,t+u) = f(x,t) + f(x,u).$$



SYSTEM OF EQUATIONS

$$\left.\begin{array}{l} f(x,t+u) = f(x,t) + f(x,u) \\ f(x,t+u) = f(x,t) + f(x,u) \end{array}\right\} x,y,t,u \in \mathbb{R}_+$$

SOLUTION

$$\boxed{f(x,t) = cxt,}$$

where c is an arbitrary non-negative constant.

AREA OF A RECTANGLE

$$\begin{aligned} f(b, h_1 + h_2) &= f(b, h_1) + f(b, h_2) \\ f(b_1 + b_2, h) &= f(b_1, h) + f(b_2, h). \end{aligned}$$

Solution

$$\boxed{f(b, h) = cbh}$$

SIMPLE INTEREST

$$\left.\begin{aligned} f(x, t + u) = f(x, t) + f(x, u) \\ f(x, t + u) = f(x, t) + f(x, u) \end{aligned}\right\} x, y, t, u \in \mathbb{R}_+$$

Solution

$$\boxed{f(x, t) = cxt}$$

DEFINITIONS OF FUNCTIONAL EQUATION

Definition 1 *A functional equation is an equation in which the unknowns are functions. We exclude differential and integral equations.*

EXAMPLES

CAUCHY'S FUNCTIONAL EQUATION

$$f(x+y) = f(x) + f(y)$$

Solution

$$\boxed{f(x) = cx,}$$

where c is an arbitrary constant.

PEXIDER'S FUNCTIONAL EQUATION

$$f(x+y) = g(x) + h(y)$$

Solution

$$\boxed{\begin{array}{rcl} f(x) & = & cx + a + b \\ g(x) & = & cx + a \\ h(x) & = & cx + b \end{array}}$$

ASSOCIATIVITY EQUATION

$$F[F(x,y),z] = F[x,F(y,z)]$$

Solution

$$\boxed{F(x,y) = f^{-1}[f(x)+f(y)]}$$

The case of the sum

$$\begin{aligned} f(x) &= x \\ F(x,y) &= x+y \end{aligned}$$

The case of the product

$$\begin{aligned} f(x) &= \log(x) \\ F(x,y) &= \exp[\log(x)+\log(y)] = xy \end{aligned}$$

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SUMS OF PRODUCTS EQUATION

All solutions of the functional equation

$$\sum_{k=1}^{n} f_k(x) g_k(y) = 0 \qquad (1)$$

can be written in the form:

$$\begin{bmatrix} f_1(x) \\ f_2(x) \\ \dots \\ f_n(x) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nr} \end{bmatrix} \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \dots \\ \varphi_r(x) \end{bmatrix}$$

$$\begin{bmatrix} g_1(y) \\ g_2(y) \\ \dots \\ g_n(y) \end{bmatrix} = \begin{bmatrix} b_{1r+1} & b_{1r+2} & \dots & b_{1n} \\ b_{2r+1} & b_{2r+2} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{nr+1} & b_{nr+2} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} \psi_{r+1}(y) \\ \psi_{r+2}(y) \\ \dots \\ \psi_n(y) \end{bmatrix} \qquad (2)$$

where $\{\varphi_1(x), \varphi_2(x), \dots, \varphi_r(x)\}$, on one hand, and $\{\psi_{r+1}(x), \psi_{r+2}(x), \dots, \psi_n(x)\}$, on the other hand, are arbitrary systems of functions which are mutually linearly independent, $0 < r < n$ is an integer, and the constants a_{ij} and b_{ij} satisfy

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1r} & a_{2r} & \dots & a_{nr} \end{bmatrix} \begin{bmatrix} b_{1r+1} & b_{1r+2} & \dots & b_{1n} \\ b_{2r+1} & b_{2r+2} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{nr+1} & b_{nr+2} & \dots & b_{nn} \end{bmatrix} = \mathbf{0}$$

NORMAL CONDITIONALS

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Consider a two-dimensional random variable (X,Y) with joint, marginal and conditionals densities $f_{(X,Y)}(x,y), g(x), h(y), f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$, respectively. Then we have

$$f_{(X,Y)}(x,y) = f_{X|Y}(x|y)h(y) = f_{Y|X}(y|x)g(x)$$

If we assume normal conditionals we have

$$f_{Y|X}(y|x) \propto \frac{\exp\left\{-\frac{1}{2}\left[\frac{y-a(x)}{b(x)}\right]^2\right\}}{b(x)}$$

$$f_{X|Y}(x|y) \propto \frac{\exp\left\{-\frac{1}{2}\left[\frac{x-d(y)}{c(y)}\right]^2\right\}}{c(y)}$$

Taking logarithms and letting

$$u(x) = \log[g(x)/b(x)]; v(y) = \log[h(y)/c(y)]$$

we get

$$\begin{array}{l} [2u(x)b^2(x) - a^2(x)]c^2(y) - y^2c^2(y) \\ +x^2b^2(x) + b^2(x)[d^2(y) - 2v(y)c^2(y)] \\ +2a(x)yc^2(y) - 2xb^2(x)d(y) = 0, \end{array}$$

which is a functional equations of the form $\sum\limits_{k=1}^{n} f_k(x)g_k(y) = 0$.

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NORMAL CONDITIONALS

Solution

$$\begin{bmatrix} 2u(x)b^2(x)-a^2(x) \\ b^2(x) \\ 1 \\ x^2b^2(x) \\ 2a(x) \\ xb^2(x) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \\ 0 & 0 & 1 \\ a_{51} & a_{52} & a_{53} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b^2(x) \\ xb^2(x) \\ x^2b^2(x) \end{bmatrix}$$

and

$$\begin{bmatrix} c^2(y) \\ d^2(y)-2v(y)c^2(y) \\ -y^2c^2(y) \\ 1 \\ yc^2(y) \\ -2d(y) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b_{24} & b_{25} & b_{26} \\ 0 & 0 & -1 \\ b_{44} & b_{45} & b_{46} \\ 0 & 1 & 0 \\ b_{64} & b_{65} & b_{66} \end{bmatrix} \begin{bmatrix} c^2(y) \\ yc^2(y) \\ y^2c^2(y) \end{bmatrix}$$

where

$$\begin{bmatrix} a_{11} & 1 & a_{31} & 0 & a_{51} & 0 \\ a_{12} & 0 & a_{32} & 0 & a_{52} & 1 \\ a_{13} & 0 & a_{33} & 1 & a_{53} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ b_{24} & b_{25} & b_{26} \\ 0 & 0 & -1 \\ b_{44} & b_{45} & b_{46} \\ 0 & 1 & 0 \\ b_{64} & b_{65} & b_{66} \end{bmatrix} = \mathbf{0}$$

Solution

$$\begin{aligned} a(x) &= \frac{-(A+Bx+Cx^2)}{(D+2Ex+Fx^2)} \\ d(y) &= \frac{-(H+By+Ey^2)}{(J+2Cy+Fy^2)} \\ b^2(x) &= \frac{1}{(D+2Ex+Fx^2)} \\ c^2(y) &= \frac{1}{(J+2Cy+Fy^2)} \end{aligned}$$

$$\begin{aligned} & f(x,y) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{G}{2}\} \times \\ & \times \exp\left\{-\frac{1}{2}[2Hx + 2Ay + Jx^2 + Dy^2 + 2Bxy]\right\} \\ & \times \exp\left\{-\frac{1}{2}[2Cx^2y + 2Exy^2 + Fx^2y^2]\right\} \end{aligned}$$

where the constant must satisfy one of the following two conditions:

- **Normal Model**:

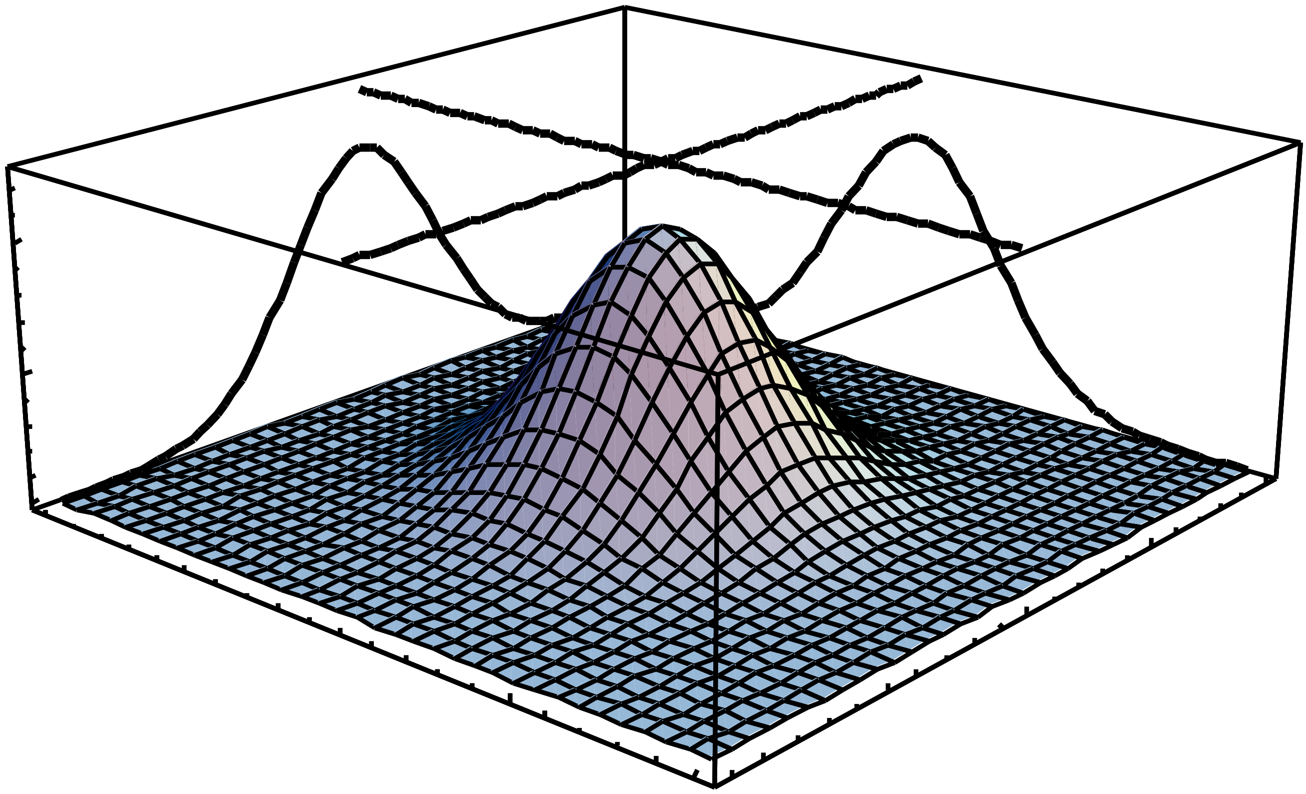
$$F = E = C = 0; \quad D > 0; \quad J > 0; \quad B^2 < DJ$$

- **Non-Normal Model**:

$$F > 0; \quad FD > E^2; \quad JF > C^2$$

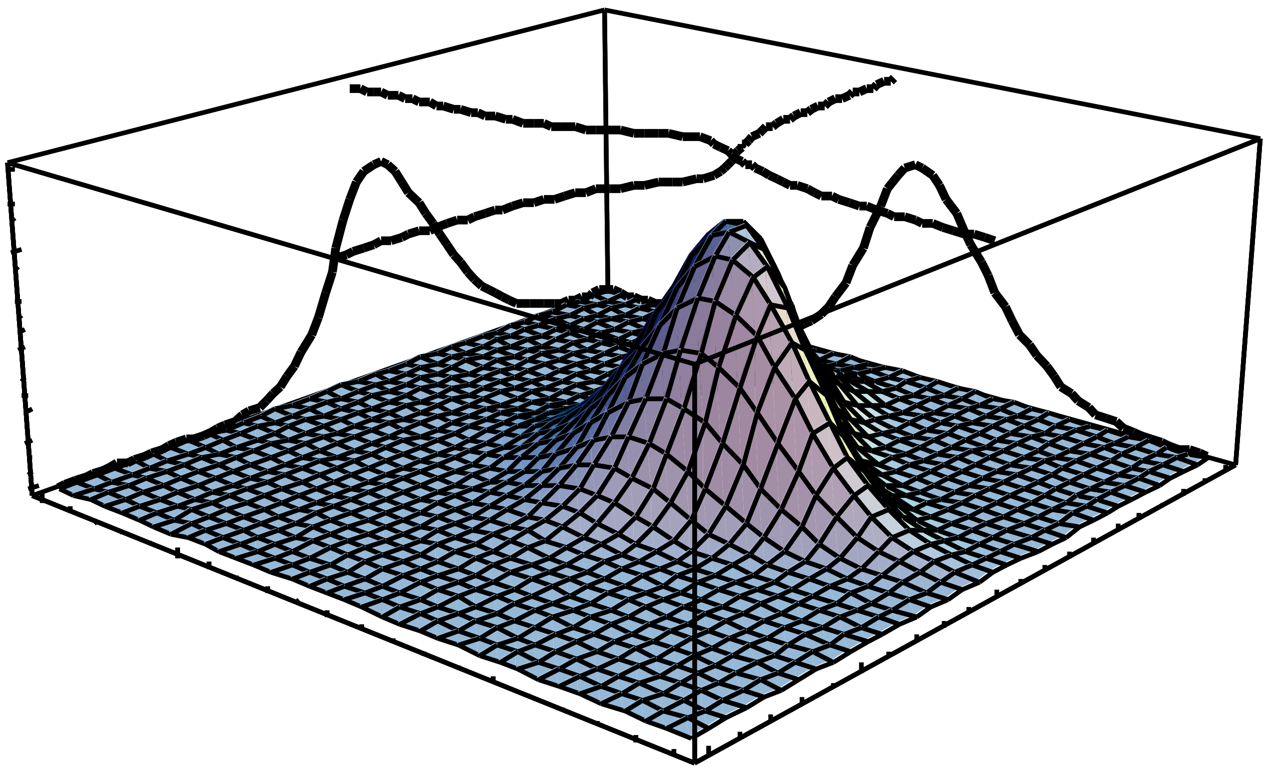
Normal Model

- Regression lines are straight lines.
- Marginal distributions are normal.
- Mode is in the intersection of regression lines.



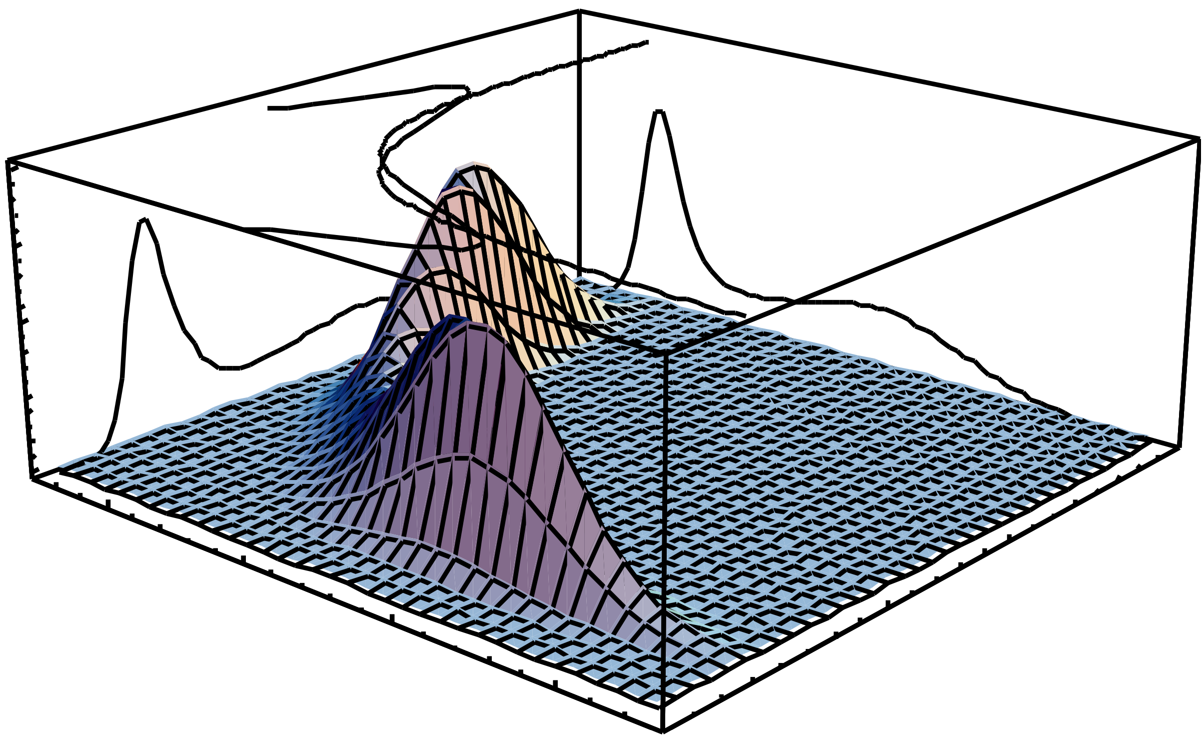
Non-Normal Model

- Regression lines are not straight lines.
- Marginal distributions are not normal.
- Mode is in the intersection of regression lines



Non-Normal Model
Two Modes

- Regression lines are not straight lines.
- Marginal distributions are not normal.
- Modes are in the intersection of regression lines



COVER WITH POLYNOMIAL CROSS SECTIONS

We look for the most general surface of the form $Z = z(x, y)$ such that all of its cross-sections or intersections with planes parallel to the coordinate planes are of the form

$$\begin{aligned} z(x, y) &= a(y)x^2 + b(y)x + c(y) \\ z(x, y) &= d(x)y^2 + e(x)y + f(x), \end{aligned} \quad (3)$$

where $a(y), b(y)$ and $c(y)$, on one hand, and $d(x), e(x)$ and $f(x)$, on the other, are the coefficients of the two polynomials intersection curves associated with planes $Y = y$ and $X = x$, respectively.
The equations state that we obtain second degree polynomials when we intersect planes $X = constant$ and $Y = constant$.

¿From (3) we get

$$a(y)x^2+b(y)x+c(y)-d(x)y^2-e(x)y-f(x) = 0,$$

which is of the form $\sum_{k=1}^{n} f_k(x)g_k(y) = 0.$

COVER WITH
POLYNOMIAL CROSS SECTIONS

SOLUTION

$$\begin{bmatrix} x^2 \\ x \\ 1 \\ d(x) \\ e(x) \\ f(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a(y) \\ b(y) \\ c(y) \\ -y^2 \\ -y \\ -1 \end{bmatrix} = \begin{bmatrix} b_{14} & b_{15} & b_{16} \\ b_{24} & b_{25} & b_{26} \\ b_{34} & b_{35} & b_{36} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y^2 \\ y \\ 1 \end{bmatrix}$$

where

$$\begin{bmatrix} 1 & 0 & 0 & a_{41} & a_{51} & a_{61} \\ 0 & 1 & 0 & a_{42} & a_{52} & a_{62} \\ 0 & 0 & 1 & a_{43} & a_{53} & a_{63} \end{bmatrix} \begin{bmatrix} b_{14} & b_{15} & b_{16} \\ b_{24} & b_{25} & b_{26} \\ b_{34} & b_{35} & b_{36} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \mathbf{0}$$

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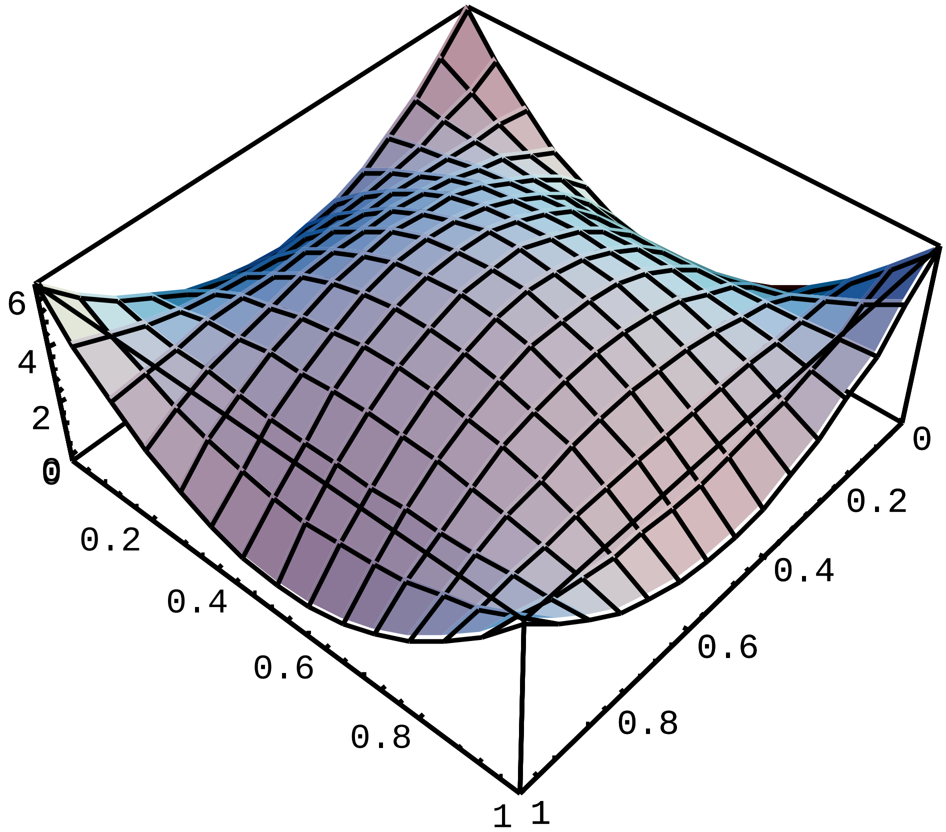
COVER WITH POLYNOMIAL CROSS SECTIONS

SOLUTION

$$
\begin{array}{ll}
a(y) = A + By + Cy^2 \ ; & b(y) = D + Ey + Fy^2 \\
c(y) = G + Hy + Iy^2 \ ; & d(x) = I + Fx + Cx^2 \\
e(x) = H + Ex + Bx^2 \ ; & f(x) = G + Dx + Ax^2
\end{array}
$$

$$
\begin{aligned}
z(x,y) \; = \; & Cx^2y^2 + Bx^2y + Fxy^2 \\
& + Ax^2 + Exy + Iy^2 + Dx + Hy + G,
\end{aligned}
$$

where A, B, C, D, E, F, G, H and I are arbitrary constants.



Suppose that we have n quantifiable judgements $x_1, \ldots, x_n$ which we want to synthesize into a consensus judgement $f(x_1, \ldots, x_n)$.
We make the following assumptions:

1. **Separability**: The function f is separable:

   $$f(x_1, \ldots, x_n) = g_1(x_1) \Delta g_2(x_2) \Delta \ldots \Delta g_n(x_n).$$

   where Δ is an associative, commutative and cancelative operation.

2. **Equality** : All members in the jury have the same weight in the final decision.

3. **Unanimity** : When the judges coincide in the same result x, the consensus decision must be the same x:

   $$f(x, \ldots, x) = x$$

The associativity assumption implies:

$$y_1 \Delta y_2 = \varphi^{-1}[\varphi(y_1) + \varphi(y_2)],$$

that is,

$$f(x_1, \ldots, x_n) = \varphi^{-1}\{\sum_{i=1}^{n} \varphi[g_i(x_i)]\}$$

Because of equality, all g_i must coincide:

$$f(x_1, \ldots, x_n) = \varphi^{-1}\{\sum_{i=1}^{n} \varphi[g(x_i)]\}$$

¿From unanimity we get

$$f(x, \ldots, x) = x \Rightarrow g(x) = \varphi^{-1}[\frac{\varphi(x)}{n}]$$

and, finally we obtain:

$$\boxed{f(x_1, x_2, \ \ldots \ , \ x_n) = \varphi^{-1}\{\frac{\sum_{i=1}^{n} \varphi(x_i)}{n}\}}$$

where φ is an arbitrary invertible function.

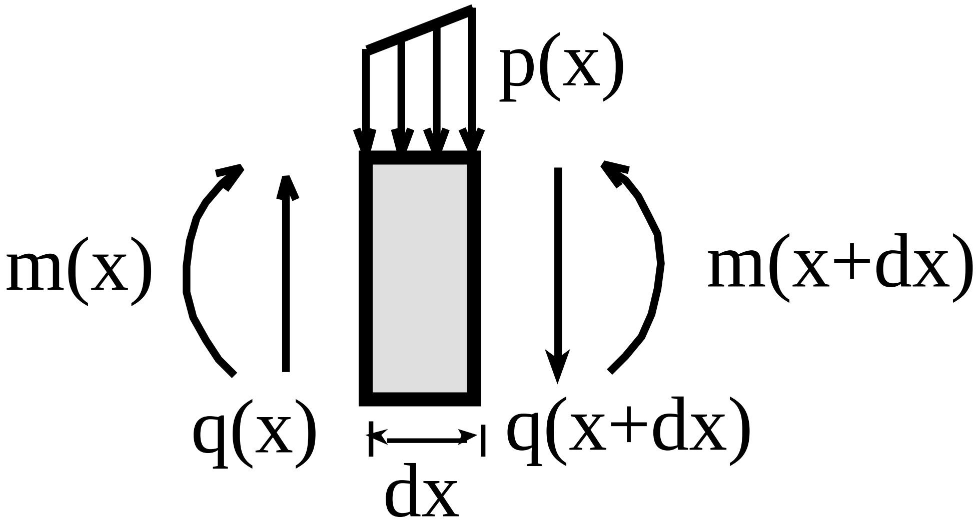
PARTICULAR CASES

- **The arithmetic mean**: $\varphi(x) = x$.
- **The geometric mean**: $\varphi(x) = \log x$.
- **The L_p mean**: $\varphi(x) = x^p$.

CLASSICAL APPROACH

Differential equations

The equilibrium equations are stated for differential pieces.



The equilibrium of vertical forces leads to

$$q(x+dx) = q(x) + p(x)dx \Rightarrow q'(x) = p(x), \qquad (4)$$

where $q(x)$ and $p(x)$ are the shear and the load at the point x, respectively, and the equilibrium of moments

$$m(x+dx) = m(x) + q(x)dx + p(x)dxdx/2 \qquad (5)$$

which implies

$$m'(x) = q(x), \qquad (6)$$

where $m(x)$ is the bending moment at x.

Using the well known strength of materials relation

$$m(x) = EIz''(x), \tag{7}$$

where $z(x)$ is the deflection of the beam and from (4), (6) and (7) we get the well known differential equation

$$\boxed{EIz^{(iv)}(x) = p(x).} \tag{8}$$

Letting $w(x) = z'(x)$ be the rotation of the beam at point x, from Equations (4), (6) and (7) we get the system of differential equations

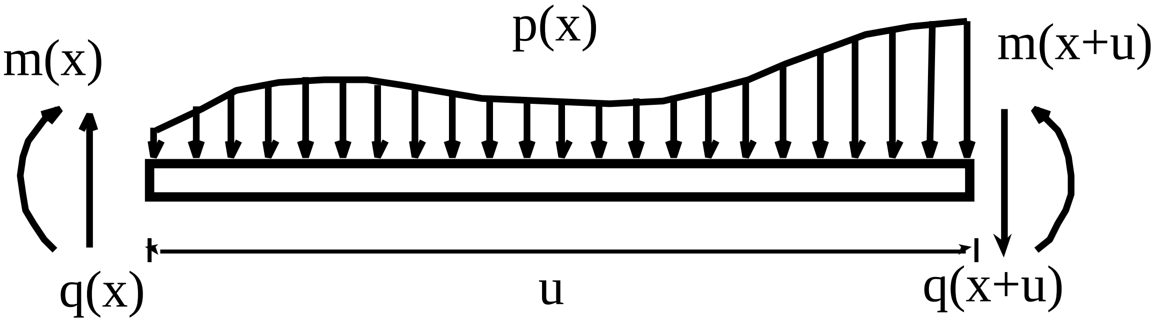
$$\boxed{\begin{array}{l} q'(x) = p(x) \\ m'(x) = q(x) \\ w'(x) = \dfrac{m(x)}{EI} \\ z'(x) = w(x), \end{array}} \tag{9}$$

which is the usual mathematical model in terms of differential equations.

NEW APPROACH

Functional equations

In the new approach, the equilibrium is analyzed for discrete pieces.



The equilibrium of vertical forces leads to

$$\boxed{q(x+u) = q(x) + A(x,u),} \qquad (10)$$

where

$$A(x,u) = \int_x^{x+u} p(s)ds. \qquad (11)$$

The equilibrium of moments gives

$$\boxed{m(x+u) = m(x) + uq(x) + B(x,u),} \qquad (12)$$

where

$$B(x,u) = \int_x^{x+u} (x+u-s)p(s)ds. \qquad (13)$$

THE BEAM EQUATION

Using Equation (7) we get

$$\begin{aligned} w(x+u) &= w(x) + \frac{1}{EI}\int_x^{x+u} m(s)ds \\ &= w(x) + \frac{1}{EI}\left[m(x)u + q(x)\frac{u^2}{2} + C(x,u)\right]. \end{aligned} \tag{14}$$

In addition we have

$$\begin{aligned} z(x+u) &= z(x) + \int_x^{x+u} w(s)ds \\ &= z(x) + w(x)u + \frac{1}{EI}\left[m(x)\frac{u^2}{2} + q(x)\frac{u^3}{6} + D(x,u)\right]. \end{aligned} \tag{15}$$

Thus, we get the system of functional equations

$$\begin{aligned} q(x+u) &= q(x) + A(x,u) \\ m(x+u) &= m(x) + uq(x) + B(x,u) \\ w(x+u) &= w(x) + \frac{1}{EI}\left[m(x)u + q(x)\frac{u^2}{2} + C(x,u)\right] \\ z(x+u) &= z(x) + w(x)u + \frac{1}{EI}\left[m(x)\frac{u^2}{2} + q(x)\frac{u^3}{6} + D(x,u)\right] \end{aligned} \tag{16}$$

where

$$\begin{aligned} A(x,u) &= \int_x^{x+u} p(s)ds \\ B(x,u) &= \int_x^{x+u} (x+u-s)p(s)ds \\ C(x,u) &= \int_x^{x+u} B(x,s-x)ds \\ D(x,u) &= \int_x^{x+u} C(x,s-x)ds \end{aligned} \tag{17}$$

Equation (16) is equivalent to the system of functional equations (9).

Theorem 1 If the equation

$$f(x+y) = f(x) + f(y) \quad ; \quad x, y \in \mathbf{R} \tag{18}$$

is satisfied for all real x, y, and if the function $f(x)$ is (a) continuous at a point, or (b) nonnegative for small x, or (c) bounded in an interval or (d) integrable or (e) measurable, then

$$f(x) = cx \quad , \quad x \in \mathbf{R} \tag{19}$$

where c is an arbitrary constant.

Theorem 2 The most general solutions, which are continuous-at-a-point, of the functional equation

$$f(xy) = f(x) + f(y) \quad x, y \in \mathbf{T} \tag{20}$$

are

$$f(x) = \begin{cases} c\log(x) & if \quad \mathbf{T} = \mathbf{R}_{++} \\ c\log(|x|) & if \quad \mathbf{T} = \mathbf{R} - \{0\} \\ 0 & if \quad \mathbf{T} = \mathbf{R} \end{cases} \tag{21}$$

Theorem 3 *The most general solutions of the functional equation*

$$f(x+y) = f(x)f(y) \ ; \ x, y \in \mathbf{R} \ or \ x, y \in \mathbf{R}_{++} \quad (22)$$

which are continuous-at-a-point are

$$f(x) = \exp(cx) \ and \ f(x) = 0. \quad (23)$$

Theorem 4 *The most general solutions, which are continuous-at-a-point, of the functional equation*

$$f(xy) = f(x)f(y) \quad x, y \in \mathbf{T} \quad (24)$$

are

$$\left.\begin{array}{l} f(x) = 1 \\ f(x) = \begin{cases} |x|^c & x \neq 0 \\ 0 & x = 0 \end{cases} \\ f(x) = \begin{cases} |x|^c sgn(x) & x \neq 0 \\ 0 & x = 0 \end{cases} \end{array}\right\} \ if \quad \mathbf{T} = \mathbf{R}$$

$$\left.\begin{array}{l} f(x) = |x|^c \\ f(x) = |x|^c sgn(x) \end{array}\right\} \ if \quad \mathbf{T} = \mathbf{R} - \{0\} \quad (25)$$

$$f(x) = x^c \ \ if \quad \mathbf{T} = \mathbf{R}_{++}$$

where c is an arbitrary real number, together with

$$f(x) = 0 \ ; \ f(x) = \begin{cases} 0 & |x| \neq 1 \\ x & |x| = 1 \end{cases} \ ; \ f(x) = \begin{cases} 0 & |x| \neq 1 \\ 1 & |x| = 1 \end{cases}$$

which are common to the three domains.

Theorem 5 *(Pexider's main equation)The most general system of solutions of*

$$f(x+y) = g(x)+h(y) \ ; \ x, y \in \mathbf{R} \ or \ [a, b] \ with \ a, b \in \mathbf{R} \quad (26)$$

with f: (a) continuous at a point, or (b) non-negative for small x, or (c) bounded in an interval, is

$$f(x) = Ax+B+C \ ; \ g(x) = Ax+B \ ; \ h(x) = Ax+C \quad (27)$$

where A, B and C are arbitrary constants.

Theorem 6 *The most general system of solutions of*

$$f(xy) = g(x) + h(y); x, y \in \mathbf{R} \ or \ \mathbf{R}_{++} \ or \ \mathbf{R} - \{0\} \quad (28)$$

with f continuous at a point is

$$\left.\begin{array}{l} f(x) = A\log(BCx) \\ g(x) = A\log(Bx) \\ h(x) = A\log(Cx) \end{array}\right\} ; x, y \in \mathbf{R}_{++}$$

$$\left.\begin{array}{l} f(x) = A\log(BC\,|x|) \\ g(x) = A\log(B\,|x|) \\ h(x) = A\log(C\,|x|) \end{array}\right\} \ x, y \in \mathbf{R} - \{0\} \quad (29)$$

$$f(x) = A + B; g(x) = A; h(x) = B; \quad if \ x, y \in \mathbf{R} \ or \ \mathbf{R} - \{0\} \ or \ \mathbf{R}_{++}$$

Theorem 7 *The most general system of solutions of*

$$f(x+y) = g(x)h(y); x, y \in \mathbf{R} \tag{30}$$

with f continuous at a point is

$$f(x) = ABe^{Cx}; g(x) = Ae^{Cx}; h(x) = Be^{Cx} \tag{31}$$

where A, B and C are arbitrary non-zero constants, together with the trivial solutions

$$\begin{array}{l} f(x) = g(x) = 0; h(x)\ arbitrary \\ f(x) = h(x) = 0; g(x)\ arbitrary. \end{array} \tag{32}$$

Theorem 8 *The most general system of solutions of*

$$f(xy) = g(x)h(y); x, y \in \mathbf{R}\ or\ \mathbf{R}_{++}\ or\ \mathbf{R} - \{0\} \tag{33}$$

with f continuous at a point is

$$\begin{array}{l} f(x) = AB; g(x) = A; h(x) = B \\ if\ x, y \in \mathbf{R}\ or\ \mathbf{R} - \{0\}\ or\ \mathbf{R}_{++} \end{array} \tag{34}$$

$$\left.\begin{array}{l} f(x) = ABx^C \\ g(x) = Ax^C \\ h(x) = Bx^C \end{array}\right\} \quad if\ \ x, y \in \mathbf{R}_{++} \tag{35}$$

$$\begin{array}{l} f(x) = g(x) = 0; h(x)\ arbitrary \\ f(x) = h(x) = 0; g(x)\ arbitrary. \end{array} \tag{36}$$

$$\left.\begin{array}{l} f(x) = AB\,|x|^C \\ g(x) = A\,|x|^C \\ h(x) = B\,|x|^C \end{array}\right\} \; or \; \left.\begin{array}{l} f(x) = AB\,|x|^C\, sgn(x) \\ g(x) = A\,|x|^C\, sgn(x) \\ h(x) = B\,|x|^C\, sgn(x) \end{array}\right\}$$
$$if \;\; x, y \in \mathbf{R} - \{0\},$$

$$\left.\begin{array}{l} f(x) = \begin{cases} AB\,|x|^C & x \neq 0 \\ 0 & x = 0 \end{cases} \\ g(x) = \begin{cases} A\,|x|^C & x \neq 0 \\ 0 & x = 0 \end{cases} \\ h(x) = \begin{cases} B\,|x|^C & x \neq 0 \\ 0 & x = 0 \end{cases} \end{array}\right\} \; or$$

$$\left.\begin{array}{l} f(x) = \begin{cases} AB\,|x|^C\, sgn(x) & x \neq 0 \\ 0 & x = 0 \end{cases} \\ g(x) = \begin{cases} A\,|x|^C\, sgn(x) & x \neq 0 \\ 0 & x = 0 \end{cases} \\ h(x) = \begin{cases} B\,|x|^C\, sgn(x) & x \neq 0 \\ 0 & x = 0 \end{cases} \end{array}\right\} \; if \;\; x, y \in \mathbf{R}, \tag{37}$$

where A, B and C are arbitrary constants.

Theorem 9 The general continuous solution of the translation equation

$$\begin{array}{l} F[F(x,u),v] = F(x,u+v) \\ x, F(x,u) \in (a,b)\ ;\ u,v \in (-\infty,\infty) \end{array} \quad (38)$$

is

$$F(x,y) = f[f^{-1}(x) + y] \quad (39)$$

where f is an arbitrary function which is continuous and strictly monotonic in $(-\infty,\infty)$, if one of the following conditions holds:

- (a) $F(x,u)$ is strictly monotonic for each value of x with respect to u and for uncountably many values of u with respect to x.
- (b) $F(x,u)$ is continuous for each value of u with respect to x and for $x = x_0$ with respect to u and nonconstant for every fixed value of x.

The translation equation has many interesting applications. Let us define the n-th iterate $F(x,n) = g_n(x)$:

$$\begin{cases} g_0(x) = x \\ g_n(x) = g[g_{n-1}(x)] \;\; for\; n > 0 \end{cases}$$

Then $F(x,n)$ satisfies translation equation:

$$g_n[g_m(x)] = g_{m+n}(x) \Leftrightarrow$$
$$F[F(x,m),n] = F(x,m+n)$$

Thus:

$$F(x,y) = f[f^{-1}(x) + y] \;;\; F(x,1) = g(x) \quad (40)$$

which implies

$$g[f(x)] = f(x+1) \quad (41)$$

Thus, the problem of finding the n-th iterate can be solved by solving the equivalent functional equation $g[f(x)] = f(x+1)$ or $f^{-1}[g(x)] = f^{-1}(x) + 1$, which is a particular case of the Abel equation.

The most common methods for solving functional equations are:

1. Replacing variables by given values.
2. Transforming one or several variables.
3. Transforming one or several functions.
4. Using a more general functional equation.
5. Treating variables as constants.
6. Inductive methods.
7. Iterative methods.
8. Separation of variables.
9. Analytical techniques (differentiation, integration, etc.).
10. Mixed methods.

REPLACING VARIABLES BY GIVEN VALUES

Example 1 (The homogeneous functions) The general solution of the functional equation

$$\boxed{f(yx) = y^k f(x) \quad ; \quad x, y \in \mathbf{R}_+,} \tag{42}$$

where f is real and k is constant, is

$$\boxed{f(x) = cx^k} \tag{43}$$

where c is an arbitrary constant. ■

Proof: Letting $x = 1$ in (42) we get $f(y) = cy^k$, where $c = f(1)$. ■

Example 2 (Sincov's equation). The general solution of the functional equation

$$\boxed{f(x, y) + f(y, z) = f(x, z)} \tag{44}$$

is

$$f(x, y) = g(y) - g(x) \tag{45}$$

where g is an arbitrary function. ■

Proof: Letting $z = 0$ in (44) and $g(x) = f(x, 0)$, we get (45), which satisfies (44). ■

Example 3 The general solution of the functional equation

$$\boxed{G(x+z, y+z) = G(x, y) + z} \tag{46}$$

is

$$\boxed{G(x, y) = x + g(y - x)} \tag{47}$$

where g is an arbitrary function. ■

Proof: Letting $z = -x$ in (46) and $g(x) = G(0, x)$, we get (47). ■

Example 4 If the functional equation

$$\boxed{f(x+y) = f(x) + f(y) + K,} \tag{48}$$

where K is a real constant, is satisfied for every pair of real numbers x and y and if the function $f(x)$ is (a) continuous in at least one point, or (b) bounded by K for small values of x, or (c) bounded in a given interval, then

$$\boxed{f(x) = cx - K,} \tag{49}$$

where c is an arbitrary constant. ■

Proof: Letting

$$f(x) = g(x) - K, \tag{50}$$

the functional equation (48) transforms to

$$g(x+y) = g(x) + g(y)$$

which is the Cauchy functional equation, with solution $g(x) = cx$. Replacing this in (50) we can check that (48) holds. ■

USING A MORE GENERAL FUNCTIONAL EQUATION

Example 5 The general solution of the functional equation

$$F[G(x,y), G(u,v)] = K[x+u, y+v] \qquad (51)$$

can be obtained from the solution

$$\begin{array}{ll} F(x,y) = k[f(x)+g(y)] & , \; G(x,y) = f^{-1}[p(x)+q(y)], \\ K(x,y) = k[l(x)+m(y)] & , \; H(x,y) = g^{-1}[r(x)+s(y)], \\ M(x,y) = l^{-1}[p(x)+r(y)] & , N(x,y) = m^{-1}[q(x)+s(y)], \end{array}$$

of the functional equation

$$\boxed{F[G(x,y), H(u,v)] = K[M(x,u), N(y,v)]} \qquad (52)$$

taking into account that

$H(x,y) = G(x,y); M(x,u) = x+u; N(y,v) = y+v.$ ■

Example 6 The two functional equations

$$\begin{array}{l} F(x+y, u+v) = K(M(x,u), N(y,v)) \\ F(F(x,y), z) = F(x, F(y,z)), \end{array}$$

and

$$F[G(x,y), H(u,v)] = K[M(x,u), N(y,v)].$$

can be solved as particular cases of the above functional equation (52). ■

Example 7 The continuous (with respect to its first argument) general solution of the functional equation

$$\boxed{f(x+y,z)=f(x,z)f(y,z) \quad ; \quad x,y,z\in \mathbf{R}} \qquad (53)$$

is

$$\boxed{f(x,z)=\exp c(z)x} \qquad (54)$$

where c is an arbitrary function. ■

Proof: For each value z, (53) is Cauchy II, the solution of which is (54) (the constant depends on z). ■

Example 8 The general solution of

$$\boxed{f(ux,y,z)=u^k f(x,y,z)} \qquad (55)$$

is

$$\boxed{f(x,y,z)=x^k c(y,z)} \qquad (56)$$

where c is an arbitrary constant. ■

Since the general solution of $f(ux) = u^k f(x)$ is $f(x) = cx^k$, and for each fixed y and z, Equation (55) is of this form, then (56) holds.

Example 9 The general solution of the functional equation

$$\boxed{f^{-1}(g(x)+h(y)) = \exp(x) \quad x,y \in \mathbf{R}} \tag{57}$$

is

$$\boxed{g(x) = f(\exp(x)) - c \quad ; \quad h(y) = c} \tag{58}$$

where f is an invertible arbitrary function and c is an arbitrary constant. ∎

Proof: Equation (57) can be written as

$$g(x) + h(y) = f(\exp(x)) \quad \forall x, y \in \mathbf{R}$$

and then we get

$$-g(x) + f(\exp(x)) = h(y) = c,$$

from which we finally obtain (58). ∎

Example 10 To solve the equation

$$f(x,y) + f(y,z) = f(x,z), \tag{59}$$

we differentiate with respect to x, y and z, independently, and we get

$$\begin{cases} f_1'(x,y) = f_1'(x,z) \\ f_2'(x,y) + f_1'(y,z) = 0 \\ f_2'(y,z) = f_2'(x,z), \end{cases} \tag{60}$$

where the subindices refer to partial derivatives with respect to the indicated arguments. From (60), we obtain

$$\begin{array}{l} f_1'(x,y) = s'(x) \Rightarrow f(x,y) = s(x) + g(y) \\ f_2'(x,y) = -f_1'(y,z) \Rightarrow g'(y) = -s'(y) \\ \Rightarrow g(y) = -s(y) + k \end{array} \tag{61}$$

and we get $f(x,y) = s(x) - s(y) + k$, but substitution into (59) leads to $k = 0$. Thus the general differentiable solution of (59) becomes

$$f(x,y) = s(x) - s(y) \tag{62}$$