On the connectedness of finite distance graphs

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Abstract. We describe a polynomial-time algorithm for deciding whether a given distance graph with a finite number of vertices is connected.

Keywords: Distance graphs, circulant graphs, connectedness.

1 Introduction

An *integer distance graph* has a countably infinite set of vertices, labelled by the integer numbers, being two of them linked by an edge when their distance lies in a fixed parameter set. These graphs were defined in [5] and several authors have studied their chromatic number [3, 10]. In this article we deal with the finite analogue of those graphs, which we simply refer to as *distance graphs*, and study in which cases they are connected. These graphs generalize the so-called *circulant graphs*, which are the Cayley graphs of cyclic groups. Whereas the latter have been widely studied [1, 2, 14], specially because of their interest in networks and computer science, the research carried out on distance graphs deals mainly with Hamiltonian properties [8, 13].

Throughout this article, we assume $0 \notin \mathbb{N}$ and consider undirected graphs. We use the letter n for the number of vertices of a graph and the notation $\overline{x}$ for the residue class modulo n of an integer x. As a graph with only one vertex is trivially connected, we are not interested in the case $n = 1$. In order to avoid possible ambiguities, let us say that two vertices in a graph are *connected* when there is a path joining them and reserve the word *linked* for adjacent nodes.

Denifinition 1 *Let $n \geq 2$ be an integer and $S = \{s_1, \ldots, s_r\} \subset \mathbb{N}$. The* circulant graph $C_n(S) = C_n(s_1, \ldots, s_r)$ *is the graph with vertex set $\mathbb{Z}/n\mathbb{Z}$ and edge set $\{(\bar{x}, \bar{x}+\bar{s}) \mid \bar{x} \in \mathbb{Z}/n\mathbb{Z},\ s \in S\}$. The* distance graph $D_n(S) = D_n(s_1, \ldots, s_r)$ *is the graph with vertex set $\{0, \ldots, n-1\}$ and edge set $\{(x, x+s) \mid 0 \leq x < x+s < n,\ s \in S\}$.*

Therefore, a distance graph is a subgraph of the associated circulant, obtained by keeping only those edges $(\bar{x}, \bar{x}+\bar{s}) \in (\mathbb{Z}/n\mathbb{Z})^2$ where the sum $x + s \in \mathbb{Z}$ does not exceed the maximum vertex $n-1$ (see Figure 1). As is stated in [4], the class of circulant graphs coincides with the class of regular distance graphs. We refer to the elements of the set S as *distances* or *jumps*.

Note that an (undirected) graph is a distance graph if and only if its adjacency matrix is a symmetric Toeplitz matrix, hence the term *Toeplitz graph* used by some authors. It is a direct consequence of Bézout's identity that a circulant graph $C_n(s_1, \ldots, s_r)$ (and the associated directed graph) is connected if and only if $\gcd(n, s_1, \ldots, s_r) = 1$. In our case, it is easy to see that $\gcd(s_1, \ldots, s_r) = 1$ is a necessary condition for $D_n(s_1, \ldots, s_r)$ to be connected, whereas the example depicted in Figure 1 shows that it is not sufficient.

The following observation is immediate from [4, Proposition 4] and will lead to a simple method for deciding whether a distance graph is connected or not.

Lemma 2 *Let S be a finite subset of $\mathbb{N}$. If $\gcd(S) = 1$, there exists a (unique) integer $N(S) \geq 2$ such that, for $n \geq 2$:*

$$D_n(S) \text{ is connected} \iff n \geq N(S).$$

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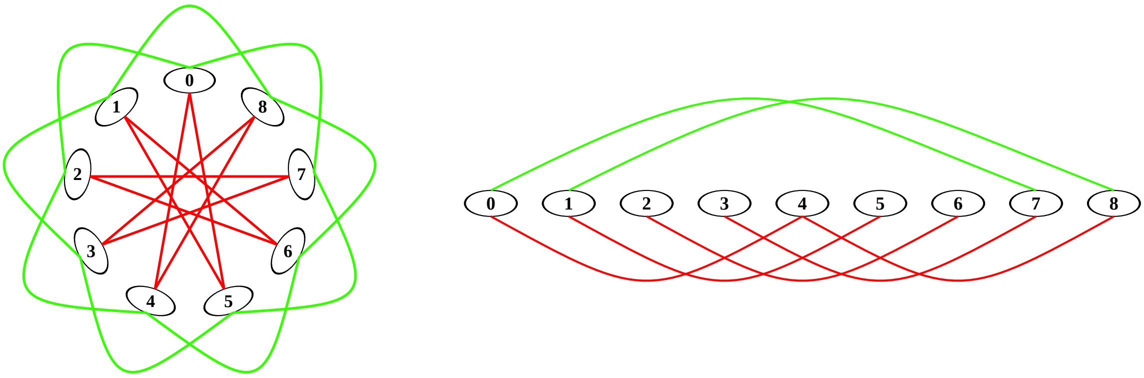

Figure 1: $C_9(4,7)$ and $D_9(4,7)$

As a consequence of this result, computing $N(S)$ is enough for testing the connectedness of a distance graph. In the case $1 \in S$, every graph $D_n(S)$ is connected and $N(S) = 2$, therefore. The simple formula $N(s_1, s_2) = s_1 + s_2 - 1$, valid for graphs of two distances distinct to 1, has been proved in [8] and in Section 2 we present an alternative proof. The computation of $N(S)$ has been proposed [4, Conjecture 2] as an NP-hard problem. Nevertheless, we develop (Section 3) a simple iterative algorithm that computes in polynomial time that index for an arbitrary finite set S. Finally, in Section 4 we present some additional results improving that algorithm.

2 Two distances case

In their study of the directed variant of circulant graphs with two jumps, Wong and Coppersmith [14] made use of *minimum distance diagrams* attaining an "L" shape. They identified paths starting from node 0 with pairs of integers and associated to every path its destination node. These diagrams and their extensions to higher dimensions (graphs with more than two jumps) have aroused interest both from the pure combinatorial point of view and because of its applications in optimal network design [6, 7, 9, 12]. Let us reproduce an analogue construction in the case of a two-jump (and undirected) circulant graph $C_n(s_1, s_2)$.

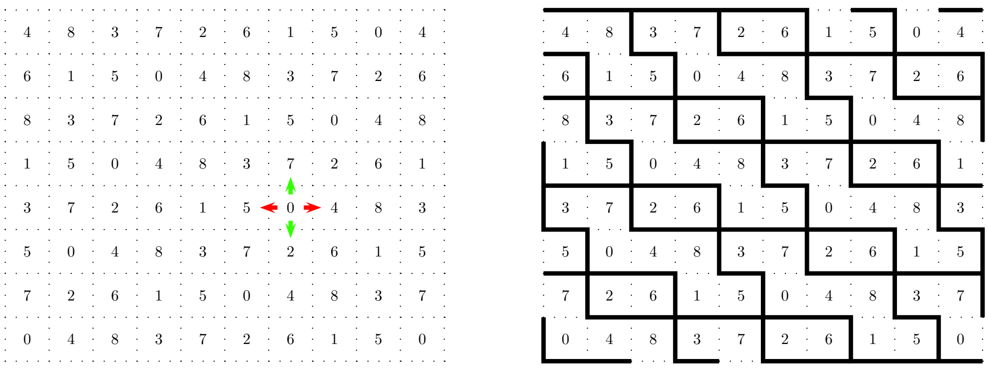

Figure 2: Diagrams associated to $C_9(4,7)$ and $D_9(4,7)$

We map each pair $(i,j) \in \mathbb{Z}^2$ to the vertex $\overline{is_1 + js_2} \in \mathbb{Z}/n\mathbb{Z}$, i.e. the node reached from node 0 by any path composed of i edges of the type $(x, x+s_1)$ and j edges of the other type (see Figure 2.I). We can adapt this diagram to the corresponding distance graph. In the circulant case, the cells adjacent to a given one show the nodes adjacent to its label. For instance, in the picture node 0 is linked to 2,4,5, and 7. In the corresponding distance graph, however, not every pair of adjacent cells is linked. A horizontal move is allowed when the left label is smaller than the right one; and a vertical move, when the label increases upwards. We have marked in Figure 2.II those lines which cannot be crossed by moving through edges of the distance graph. Therefore, the grid becomes tiled by copies of the graph's connected components. Indeed, we only get those nodes in $\mathbb{Z}\langle\gcd(S)\rangle$; but we focus on the case $\gcd(S) = 1$, since otherwise the graph is always disconnected. With these diagrams in mind, we are ready to prove next result (see [8, Corollary 6] and [4, Theorem 5] for alternative proofs).

Proposition 3 *Let $S = \{s_1, s_2\} \subset \mathbb{N}$ be such that $\gcd(s_1, s_2) = 1$. Then,*

$$N(S) = \begin{cases} s_1 + s_2 - 1, & \text{if } \min(S) \geq 2; \\ 2, & \text{otherwise.} \end{cases}$$

Proof. The tiles of the previous discussion are composed of several *horizontal stripes*. For instance, the bigger component of Figure 2.II involves the stripes $\{1, 5\}$, $\{0, 4, 8\}$, and $\{3, 7\}$ and the other one consists of the stripe $\{2, 6\}$. In the general case $D_n(s_1, s_2)$ we may assume $s_1 < s_2$. In order to make the presentation simpler, let us assume also $s_1 \leq n$. Note that if $2 \leq n < s_1$, the graph is disconnected and $n < s_1 + s_2 - 1$. We have then s_1 horizontal stripes, starting by $0, \ldots, s_1 - 1$, because these are the nodes not linked to the left. The stripe starting by the node i is joined to the stripe above in the diagram if and only if $i + s_2 < n$, so there are $\max(0, \min(s_1, n - s_2))$ of them and $\min(s_1, \max(0, s_1 + s_2 - n))$ "ceiling" stripes. Note that a tile may have a single ceiling stripe or none of them.

Let us prove that a tile with no ceiling stripe, i.e. an infinite tile, contains all the graph's nodes, and there is no other connected component therefore. Assume that $x < n - 1$ is the label of a tile's cell (i, j). We will show that the label $x + 1$ also appears in the tile. The condition $\gcd(s_1, s_2) = 1$ implies the existence of integers λ, μ such that $x + 1 = x + \lambda s_1 + \mu s_2$. Then, the cell $(i + \lambda, j + \mu)$ is labelled by $x + 1$. Note that, within a row, the cells (u, v) that belong to the same tile are characterized by the quotient of $us_1 + vs_2$ modulo n. As the quotient of $(i + \lambda)s_1 + (j + \mu)s_2 = is_1 + js_2 + 1$ modulo n is the same as that of $is_1 + js_2$, and the considered tile has cells in the row $j + \mu$ (in every row indeed), cell $(i + \lambda, j + \mu)$ must be in the same tile as (i, j). This argument can be easily extended to prove that an infinite tiling covers the whole set of vertices.

Finally, if there are at least two ceiling stripes, i.e. $\min(s_1 + s_2 - n, s_1) \geq 2$, there must be at least two connected components. On the other hand, if there is one or no ceilings ($\min(s_1 + s_2 - n, s_1) \leq 1$), the graph is connected. ∎

3 General algorithm

For sets of parameters S with more than two distances, we propose a simple algorithm that computes the threshold $N(S)$ of the required number of vertices for the distance graph to be connected. We use the following two technical results:

Lemma 4 *Let S be a finite subset of $\mathbb{N}$ such that $\gcd(S) = 1$. Let $a \geq 2$, $\lambda \geq 1$ be integers such that $\min(S) = a$ and $\min(S \backslash \{a\}) \geq \lambda a$. Then, $N(S) \geq (\lambda + 1)a$.*

Proof. Consider the distance graph $D_{(\lambda+1)a-1}(S)$ and the subset of vertices with remainder $a - 1$ modulo a, i.e. $C := \{a - 1, 2a - 1, \ldots, \lambda a - 1\}$. There is no edge involving these vertices but those of the type $(x, x + a)$; for, if $u \in C$ and $b \in S \backslash \{a\}$:

$$u + b \geq a - 1 + \lambda a = (\lambda + 1)a - 1, \quad u - b \leq \lambda a - 1 - \lambda a = -1.$$

Therefore, C (more exactly, the subgraph induced in this set of vertices) is a connected component, and $D_{(\lambda+1)a-1}(S)$ is disconnected. ∎

Next result, key of the iterative algorithm we propose, leads to a relation between the index $N(S)$ of a set S and the index of a modified set of distances; where every element of S, except its minimum, has been reduced by a suitable multiple of that minimum. We use the notation $S - \mu = \{s - \mu \mid s \in S\}$, for $\mu \in \mathbb{N}$.

Lemma 5 *Let S be a finite subset of $\mathbb{N}$ such that $\#S \geq 2$ and $a := \min(S) \geq 2$. Let n, λ be positive integers such that $n \geq (\lambda+1)a$ and $\min(S \backslash \{a\}) \geq \lambda a$. Then, $D_n(S)$ is connected if and only if $D_{n-\lambda a}(\{a\} \cup (S\backslash\{a\}) - \lambda a)$ is.*

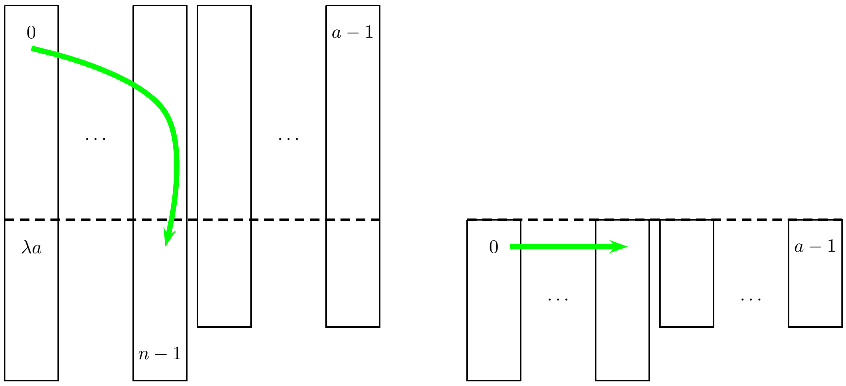

Figure 3: $D_n(S)$ and $D_{n-\lambda a}(\{a\} \cup (S\backslash\{a\}) - \lambda a)$

Proof. In both graphs considered in this lemma, a is an element of the distance set. Therefore, two nodes with the same remainder modulo a are obviously connected and each column of Figure 3 is contained in a connected component of the corresponding graph. Let π be a path in a distance graph $D_m(T)$; i.e. a list of integers (nodes) $\pi = (x_1, \ldots, x_{l+1})$ such that $0 \leq x_i < m$, for $1 \leq i \leq l+1$ and $x_{i+1} - x_i \in T \cup (-T)$, for $1 \leq i \leq l$. In this proof we use the alternative notation $\pi = [x_1; x_2 - x_1, \ldots, x_{l+1} - x_l]$ and $\text{end}(\pi) = x_{l+1}$. The hypothesis $n \geq (\lambda+1)a$ guarantees that $\{0, \ldots, a-1\}$ is a subset of the set of vertices of $D_{n-\lambda a}(\{a\} \cup (S\backslash\{a\}) - \lambda a)$.
$\Leftarrow$: Let i, j be two vertices in $\{0, \ldots, n-1\}$, and let $\tilde{\imath}$ and $\tilde{\jmath}$ be their respective remainders modulo a. Let us prove that $\lambda a + \tilde{\imath}$ and $\lambda a + \tilde{\jmath}$ are connected in $D_n(S)$. We consider the vertices $\tilde{\imath}$ and $\tilde{\jmath}$ in $D_{n-\lambda a}(\{a\} \cup (S\backslash\{a\}) - \lambda a)$, connected through path $\pi = [\tilde{\imath}; s_1, \ldots, s_l]$. We build a path $\pi' = [\lambda a + \tilde{\imath}; f(s_1), \ldots, f(s_l)]$ in $D_n(S)$, substituting a list $f(s_i)$ for each edge s_i in π as follows:

$$\pm a \overset{f}{\mapsto} \pm a; \quad b - \lambda a \overset{f}{\mapsto} [\overbrace{-a, \ldots, -a}^{\lambda}, b]; \quad \lambda a - b \overset{f}{\mapsto} [-b, \overbrace{a, \ldots, a}^{\lambda}], \text{ for } b \in S\backslash\{a\}.$$

Note that the resulting π' is a path in $D_n(S)$: the end node of every subpath $[\lambda a + \tilde{\imath}; f(s_1), \ldots, f(s_i)]$ is $e_i = \lambda a + \text{end}([\tilde{\imath}; s_1, \ldots, s_i]) < \lambda a + (n - \lambda a)$. We can concatenate the list $(\overbrace{-a, \ldots, -a}^{\lambda})$ (in the second case), for $e_{i-1} \geq \lambda a$. In the third case, $\text{end}([\tilde{\imath}; s_1, \ldots, s_{i-1}]) \geq b - \lambda a$, so $e_{i-1} \geq b$.
$\Rightarrow$: We just need to prove that every pair (u, v) of vertices in $\{0, \ldots, a-1\}$ is connected in the second graph. Let $\pi = [u + \lambda a; s_1, \ldots, s_l]$ be a path in $D_n(S)$ that connects $u + \lambda a$ with $v + \lambda a$. We build a path π', replacing each edge s_i of π by a list $f(s_i)$ such that $\text{end}([u; f(s_1), \ldots, f(s_i)]) \in \{0, \ldots, a-1\}$ and $\text{end}([u; f(s_1), \ldots, f(s_i)]) \equiv \text{end}([u + \lambda a; s_1, \ldots, s_i]) \bmod a$.

$$\pm a \mapsto \emptyset; \quad b \mapsto [b - \lambda a, \overbrace{-a, \ldots, -a}^{k_1}]; \quad -b \mapsto [\overbrace{a, \ldots, a}^{k_2}, \lambda a - b], \text{ for } b \in S\backslash\{a\}.$$

Note that for the above specification to be correct it is enough to define

$$k_1 = \left\lfloor \frac{\text{end}([u; f(s_1), \ldots, f(s_{i-1})]) + b}{a} \right\rfloor - \lambda, \quad k_2 = \left\lceil \frac{b - \text{end}([u; f(s_1), \ldots, f(s_{i-1})])}{a} \right\rceil - \lambda.$$

Note that if the hypothesis $n \geq (\lambda+1)a$ is suppressed, the theorem is no longer true, but the "$\Rightarrow$" part remains valid. ■

Using this reduction, we propose the Euclidean-like Algorithm 1 to compute the threshold number of nodes $N(S)$. Note that in Step 7, the length of S may decrease in one unit, for we consider $0 \notin \mathbb{N} \supset S$. This happens when the second element of S is a multiple of the first one.

Algorithm 1:

Input: S, finite subset of $\mathbb{N}$ with $\gcd(S) = 1$

Output: $N(S)$

1 $a \leftarrow \min(S)$

2 $M \leftarrow 0,\ D \leftarrow 0$

3 **while** $a \neq 1$ **do**

4 $\quad \lambda \leftarrow \mathrm{quo}(\min(S \backslash \{a\}), a)$

5 $\quad M \leftarrow \max(M, D + (\lambda + 1)a)$

6 $\quad D \leftarrow D + \lambda a$

7 $\quad S \leftarrow \{a\} \cup (S \backslash \{a\}) - \lambda a$ `/* Removing the 0 entry, if appears */`

8 $\quad a \leftarrow \min(S)$

9 **end**

10 **return** $\max(M, D + 2)$

Proposition 6 *Algorithm 1 returns the value $N(S)$ in time polynomial on* $(\log(\max(S)), \mathrm{len}(S))$.

Proof. Firstly, we prove that if the algorithm terminates, the output is correct. If $a \geq 2$, Lemma 4 gives $N(S) \geq (\lambda + 1)a$. Set $N' = N\big(\{a\} \cup (S \backslash \{a\}) - \lambda a\big)$ and let us prove

$$N(S) = \max((\lambda + 1)a, \lambda a + N'). \tag{1}$$

- If $\lambda a + N' \leq (\lambda + 1)a$, the graph $D_a(\{a\} \cup (S \backslash \{a\}) - \lambda a)$ is connected and, by Lemma 5, so is $D_{(\lambda+1)a}(S)$.
- If $\lambda a + N' > (\lambda + 1)a$; as the graph $D_{N'}(\{a\} \cup (S \backslash \{a\}) - \lambda a)$ is connected and $D_{N'-1}(\{a\} \cup (S \backslash \{a\}) - \lambda a)$ is not, Lemma 5 gives: $D_{N'+\lambda a}(S)$ connected, $D_{N'+\lambda a-1}(S)$ disconnected.

Let S_i, M_i, and D_i be the values of the variables S, M, and D before the ith iteration of loop 3-9. Using (1), we can recursively prove that the expected output $N(S_1)$ equals $\max(M_i, D_i + N(S_i))$. Finally, if $a = 1$, the answer $N(1, \ldots) = 2$ is correct.

Now, let us estimate the number of required loops in a similar vein to Lamé's [11] well-known proof. We consider the succession of pairs $\big(a_i = \min(S_i), b_i = \min(S_i \backslash \{a_i\})\big)$. We have, if $b_i \notin \mathbb{Z}\langle a_i \rangle$: $0 < a_{i+1} < b_{i+1} \leq a_i$. Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_n + f_{n+1}$, for $n \geq 0$. We have the following two possibilities:

- If $f_r < a_i < b_i \leq f_{r+1}$, we get $a_{i+1} = b_i - \lambda_i a_i < f_{r+1} - f_r = f_{r-1}$.
- If $f_r < a_i \leq f_{r+1} \leq f_s < b_i \leq f_{s+1}$ but $a_{i+1} > f_r$, it follows: $a_{i+2} = b_{i+1} - \lambda_{i+1} a_{i+1} < f_{r+1} - f_r = f_{r-1}$.

On the other hand, when $b_i \in \mathbb{Z}\langle a_i \rangle$, the length of S decreases and we get $a_{i+1} \leq a_i$. Therefore, if

$$\min(S) \leq f_r, \tag{2}$$

the number of iterations is upper bounded by $\mathrm{len}(S) + r - 4$. Writing $\varphi = (1 + \sqrt{5})/2$, we have $f_r > \varphi^{r-2}$. Therefore, (2) holds for $r = \lceil \log_\varphi(\min(S)) \rceil + 2$ and the algorithm ends after at most $\lceil \log_\varphi(\min(S)) \rceil + \mathrm{len}(S) - 2$ loops. Finally, note that the length of the variables evaluated in the algorithm remains polynomial on $\max(S)$ and the number of iterations. ■

The presentation above is intended to prove that the connectedness problem is polynomially solvable, but not to provide an optimal procedure. For instance, in the two distances case ($\#S = 2$) Algorithm 1 follows the classic Euclidean algorithm instead of evaluating the formula provided in Section 2. On the other hand, we will show in next section (Proposition 10) that the variables used in the algorithm keep below the bound $\max(S) + \min(S) - 1$.

4 Bounds for the threshold $N(s)$

In this section we delve more into the behaviour of the parameter $N(S)$, giving lower and upper bounds related to the formula $s_1 + s_2 - 1$ of Proposition 3. This leads to an improvement of Algorithm 1. For graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with the same vertex set, we write $G_1 \leq G_2$ when the partition of V composed by the connected components of G_1 is finer than that of G_2. We also use the notation $G_1 \equiv G_2$ when the associated partitions coincide and write $G_1 \cup G_2$ for the graph $(V, E_1 \cup E_2)$. Note that in the distance graphs case we have: $D_n(S_1) \cup D_n(S_2) = D_n(S_1 \cup S_2)$. We use the following properties:

Lemma 7 *Let G_1, G_2, and G_3 be graphs with the same vertex set. We have:*

- $G_1 \subseteq G_2 \Rightarrow G_1 \leq G_2$
- $G_1 \leq G_3$ *and* $G_2 \leq G_3 \Rightarrow G_1 \cup G_2 \leq G_3$

Lemma 8 *Let S be a finite subset of $\mathbb{N}$ such that $\#S \geq 2$, $\gcd(S) = 1$, and $d = \gcd(S \backslash \{s_r\}) \geq 2$, where $s_r = \max(S)$. Then, $N(S) \geq s_r + d - 1$.*

Proof. For any integer $n \geq 2$, we have $D_n(S\backslash\{s_r\}) \leq D_n(d) \subseteq D_n(d, s_r)$, so $D_n(S) \leq D_n(d, s_r)$, using Lemma 7. Now, $D_{s_r+d-2}(d, s_r)$ is disconnected by Proposition 3. ∎

Next result generalizes Lemma 2 when the greatest common divisor of the set of distances S is bigger than one.

Lemma 9 *Let S be a finite subset of $\mathbb{N}$ and $n \in \mathbb{N}$ such that $d = \gcd(S) \leq n/2$. The graph $D_n(S)$ has exactly d connected components (i.e. $D_n(S) \equiv D_n(d)$) if and only if $n \geq d\, N(S/d)$.*

Proof. As $D_n(S) \leq D_n(d)$, the set $\left\{i, i+d, \ldots, i + \left\lfloor \frac{n-i-1}{d} \right\rfloor d\right\}$ contains the connected component of i in $D_n(S)$. Let $\lambda \in \mathbb{N}$ such that $\lambda \geq 2$ and $i + (\lambda - 1)d < n$. The subgraph that $D_n(S)$ induces on vertices $\{i, i+d, \ldots, i+(\lambda-1)d\}$ is isomorphic to $D_\lambda(S/d)$. Then, it is connected when $\lambda \geq N(S/d)$. We have:

$$\left\lfloor \frac{n-i-1}{d} \right\rfloor + 1 \geq N(S/d), \forall i = 0, \ldots, d-1 \iff \left\lfloor \frac{n-d}{d} \right\rfloor + 1 \geq N(S/d).$$

∎

Next result, stated also in [8, Corollary 4], provides a bound for the threshold $N(S)$. We will show in Corollary 12 how to build instances of arbitrary length attaining the proposed bound.

Proposition 10 *Let $S = \{s_1, \ldots, s_r\} \subset \mathbb{N}$ such that $s_1 < \cdots < s_r$ and let $t \in \{2, \ldots, r\}$ such that $\gcd(s_1, \ldots, s_t) = 1$. Then, $N(S) \leq s_t + s_1 - 1$.*

Proof. We use induction on the parameter t. If $t = 2$, Proposition 3 gives the result. If the result is valid up to t, consider the sequence $s_1 < \cdots < s_{t+1}$ with $\gcd(s_1, \ldots, s_t) = d$ and $\gcd(s_1, \ldots, s_{t+1}) = 1$. We have $d\,N(s_1/d, \ldots, s_t/d) \leq s_t + s_1 - d < s_{t+1} + s_1 - 1$ and by Lemma 9, $D_{s_{t+1}+s_1-1}(s_1, \ldots, s_t) \equiv D_{s_{t+1}+s_1-1}(d)$. Now, $D_{s_{t+1}+s_1-1}(d, s_{t+1}) \leq D_{s_{t+1}+s_1-1}(s_1, \ldots, s_{t+1})$ by Lemma 7 and this graph is connected by Proposition 3. ∎

Proposition 11 *Let S be a finite subset of $\mathbb{N}$ such that $\#S \geq 2$, $\gcd(S) = 1$, and $d = \gcd(S\backslash\{s_r\}) \geq 2$, where $s_r = \max(S)$. If $s_r \geq d\, N((S\backslash\{s_r\})/d) - d + 1$, we have $N(S) = s_r + d - 1$.*

Proof. Let $n = s_r + d - 1$. By Lemma 8, we just need to prove that $D_n(S)$ is connected. From the hypotheses, we have $n \geq d\, N((S\backslash\{s_r\})/d)$, so $D_n(s_1, \ldots, s_{r-1})$ is under the hypotheses of Lemma 9 and $D_n(s_1, \ldots, s_{r-1}) \equiv D_n(d)$. It follows that $D_n(d, s_r) \leq D_n(S)$ and the former is connected by Proposition 3. ∎

Using Proposition 10, we derive an alternative stop criterion (Step 3) for Algorithm 1:

Corollary 12 *Let $S = \{s_1, \ldots, s_r\} \subset \mathbb{N}$ such that $r \geq 2$, $\gcd(s_1, \ldots, s_r) = 1 < d := \gcd(s_1, \ldots, s_{r-1}) \leq s_1 < \cdots < s_r$. If $s_r \geq s_{r-1} + s_1 - 2d + 1$, we have $N(S) = s_r + d - 1$.*

Removing the hypothesis $d = 1$, we still have $N(S) \leq s_r + d - 1$ but may get smaller connected distance graphs, involving only jumps in $S \backslash \{s_r\}$. In this case, the distance s_r is not relevant for the computation of $N(S)$:

Proposition 13 *Let $S = \{s_1, \ldots, s_r\} \subset \mathbb{N}$ such that $r \geq 2$, $\gcd(s_1, \ldots, s_{r-1}) = 1 \leq s_1 < \cdots < s_r$. If $s_r \geq s_{r-1} + s_1 - 1$, we have $N(S) = N(S \backslash \{s_r\})$.*

Proof. The case $r = 2$ is trivially proved. If $r \geq 3$, Proposition 10 gives $N(S) \leq s_{r-1} + s_1 - 1 \leq s_r$. Therefore, the smallest connected graph with distance set S does not include any edge of the type $(x, x + s_r)$. ■

Tying all these pieces together, we present Algorithm 2, another version of the procedure for computing the threshold $N(S)$. Note that at Step 6, we have $\#S \geq 2$ because $\gcd(S) = 1$ and $\min(S) > 1$.

Algorithm 2:

```
   Input: S, finite subset of ℕ with gcd(S) = 1 and #S ≥ 2
   Output: N(S)
1  M ← 0, D ← 0
2  a ← min(S)
3  while a ≠ 1 do
4      z ← max(S)
5      K ← z + a − 1
6      y ← max(S\{z})
7      d ← gcd(S\{z})
8      if M ≥ D + K then
9          return M
10     else if z ≥ y + a − 2d + 1 then
11         if d > 1 then
12             return max(M, D + z + d − 1)
13         else
14             S ← S\{z}
15         end
16     else
17         λ ← quo(min(S\{a}), a)
18         M ← max(M, D + (λ + 1)a)
19         D ← D + λa
20         S ← {a} ∪ (S\{a}) − λa          /* Removing the 0 entry, if appears */
21     end
22     a ← min(S)
23 end
24 return max(M, D + 2)
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References

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