Boundary Value Problems

Prototype example:
Solve

$$y'' = f(t, y, y') \text{ en } a \leq t \leq b$$

with the boundary conditions

$$y(a) = \alpha\,,\; y(b) = \beta\,.$$

Boundary conditions on the derivatives of the solution or mixed boundary conditions can be also considered $y(a) + \gamma y'(a) = \alpha\,,$ ...

Shooting Methods:

1. Linear Shooting Methods:

Consider the linear ODE

$$\begin{aligned} & y'' = p(t)y' + q(t)y + r(t) \\ & y(a) = \alpha\,,\ y(b) = \beta\,. \end{aligned}$$

These problems can be transform into two IVPs as the following theorem shows:

Boundary Value Problems

Theorem

Let $u(t)$ the solution of the IVP

$$
\begin{aligned}
&u'' = pu' + qu + r\\
&u(a) = \alpha\,,\ u'(a) = 0
\end{aligned}
$$

and $v(t)$ the solution of the IVP

$$
\begin{aligned}
&v'' = pv' + qv\\
&v(a) = 0\,,\ v'(a) = 1
\end{aligned}
$$

then, if $v(b) \neq 0$ ($v(t) \neq 0$), $y(t) = u(t) + cv(t)$ with $c = \dfrac{\beta - u(b)}{v(b)}$ is the solution of the problem

$$
\begin{aligned}
&y'' = py' + qy + r\\
&y(a) = \alpha\,,\ y(b) = \beta\,.
\end{aligned}
$$

Therefore, the resulting algorithm for solving linear 2nd order BVPs will be:

- Solve the inhomogeneous equation with $u(a) = \alpha$, $u'(a) = 0$.
- Solve the homogeneous equation with $v(a) = 0$, $v'(a) = 1$.
- Compute

$$y(t) = u(t) + \frac{\beta - u(b)}{v(b)} v(t) \,.$$

```
% The function linear implements the shooting methods for linear problems
function [t,y,C]=linear(fun1,fun2,xi,xf,a0,b0,eps)
%The function fehlb implements a Runge-Kutta-Fehlberg scheme
[t1,y1]=fehlb(fun1,xi,xf,[a0 0]',eps);
l1=length(t1);
[t2,y2]=fehlb(fun2,xi,xf,[0 1]',eps);
l2=length(t2);
C=(b0-y1(2,l1))/y2(2,l2);
[t,y]=fehlb(fun1,xi,xf,[a0 C]',eps);
```

2. Nonlinear Shooting Methods:
Consider the (in general nonlinear) ODE

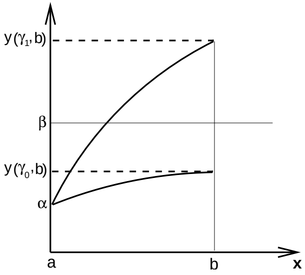
$$\begin{aligned} y'' &= f(x, y, y') \\ y(a) &= \alpha\,, y(b) = \beta \end{aligned}$$

The nonlinear shooting method consists in finding the solution (for different values of the parameter γ_k) of the IVP

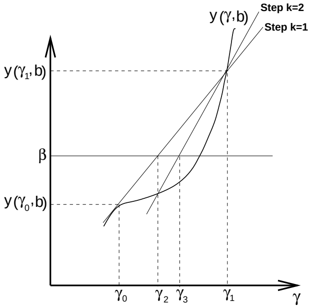
$$\begin{aligned} y'' &= f(x, y, y') \\ y(a) &= \alpha\,, y'(a) = \gamma_k \end{aligned}$$

The algorithm will stop when a γ_k value is found for which $y(b) = \beta$. We will start from two values of the parameter γ (γ_0, γ_1) and then apply the secant method in order to estimate the rest of values for this parameter.

Shooting method: graphical representation (1)



Shooting method: graphical representation (2)



Algorithm: Nonlinear shooting method

Input: $\gamma_0, \gamma_1, \epsilon$ (tolerance)

Solve the IVP:

$$\begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha \ , \ y'(a) = \gamma_0 \end{cases} \quad \rightarrow y(\gamma_0; x)$$

$err = \epsilon + 1 \ ; \ k = 0$

Do while $err > \epsilon$

$k = k + 1$
Solve the IVP:

$$\begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha \ , \ y'(a) = \gamma_k \end{cases} \quad \rightarrow y(\gamma_k; x)$$

$$\gamma_{k+1} = \gamma_k - \frac{\gamma_k - \gamma_{k-1}}{y(\gamma_k; b) - y(\gamma_{k-1}; b)} (y(\gamma_k; b) - \beta)$$

$$y(x) = y(\gamma_k; x)$$

Finite Difference Method

The shooting method is inefficient for higher-order equations with several boundary conditions.

Finite Difference method has the advantage of being direct (not iterative) for linear problems, but requires the solution of simultaneous algebraic equations.

The scheme for the finite difference method consists of the following steps:

1. Consider a partition of the domain in n interior discrete points.
2. Write a finite divided difference expression for the ODE at each interior point.
3. Use the known values of y at $x = x_0$ and $x = x_f$.
4. Set-up the n-linear equations with n-unknowns. Realizing that the system is banded and often symmetric, solve with most efficient method.

Merits of Different Numerical Methods for ODE Boundary Value Problems:

Shooting method

1. Conceptually simple and easy.
2. Inefficient for higher-order systems w/ many boundary conditions.
3. May not converge for nonlinear problems.
4. Can blow up for bad guess of initial conditions.

Finite Difference method

1. Stable
2. Direct (not iterative) for linear problems.
3. Requires solution of simultaneous algebraic eqns.
4. More complex.