Linear Multistep Methods for I.V. Problems

The linear multistep methods can be written in the general form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \,, \tag{1}$$

where k is called the step number and without loss of generality we let $\alpha_k = 1$. Explicit methods are characterised by $\beta_k = 0$ and implicit methods have $\beta_k \neq 0$.

The Adams Family

This familiy of methods is derived from the identity

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))dt\,. \qquad (2)$$

With a view to using previously computed values of y_n, we replace $f(t, y)$ by the polynomial of degree $k - 1$ passing through the k points

$$(t_{n-k+1}, f_{n-k+1}),\ \dots, (t_{n-1}, f_{n-1}), (t_n, f_n)\,.$$

Using a constant step size, this polynomial can be written (in the Newton backward difference form) as follows

$$P_{k-1}(r) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \dots + \frac{r(r+1)\dots(r+k-2)}{(k-1)!}\nabla^{k-1} f_n\,.$$

donde $t = t_n + rh$, $r \in [-(k-1), 0]$.

Then, the Adams-Bashforth method of k steps has the form

$$y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} P_{k-1}(t)dt\,.$$

and by integrating the polynomial

$$y_{n+1} - y_n = \int_0^1 P_{k-1}(r)hdr = h\sum_{i=0}^{k-1}\gamma_i \nabla^i f_n = \sum_{i=0}^{k-1}\gamma_i \Delta^i f_{n-i} \qquad (3)$$

where $\gamma_i = (-1)^i \int_0^1 \begin{pmatrix} -r \\ i \end{pmatrix} dr$

$$\gamma_0 = \int_0^1 dr = 1,\;\; \gamma_1 = \int_0^1 rdr = \frac{1}{2},\;\; \gamma_2 = \int_0^1 \frac{r(r+1)}{2}dr = \frac{5}{2}\,,$$

$$\gamma_3 = \int_0^1 \frac{r(r+1)(r+2)}{6}dr = \frac{3}{8}\,,\;\ldots$$

The first Adams-Bashforth methods are then:

$$
\begin{aligned}
y_{n+1} &= y_n + hf_n && \text{(AB1), Euler Method} \\
y_{n+1} &= y_n + \frac{h}{2}(3f_n - f_{n-1}) && \text{(AB2)} \\
y_{n+1} &= y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}) && \text{(AB3)} \\
y_{n+1} &= y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}) && \text{(AB4)}
\end{aligned} \tag{4}
$$

and so on.

Linear Multistep Methods for I.V. Problems

The Adams-Bashforth methods are **explicit methods**. **Implicit Adams methods** are derived by integrating the order k polynomial passing through the points

$$(t_{n+1}, f_{n+1}),\ (t_n, f_n),\ (t_{n-1}, f_{n-1})\dots, (t_{n-k+1}, f_{n-k+1})\,.$$

Note that we are considering the additional data point (t_{n+1}, f_{n+1}). The corresponding polynomial is

$$P_k(r) = f_{n+1} + r\nabla f_{n+1} + \frac{r(r+1)}{2!}\nabla^2 f_{n+1} + \dots + \frac{r(r+1)\dots(r+k-1)}{k!}\nabla^k f_{n+1}\,.$$

where $t = t_{n+1} + rh$, $r \in [-(k-1), 0]$.

Linear Multistep Methods for I.V. Problems

Then, for the implicit methods we will have

$$y_{n+1} - y_n = \int_{-1}^{0} P_k(r) h dr = h \sum_{i=0}^{k} \gamma_i' \nabla^i f_{n+1} = h \sum_{i=0}^{k} \gamma_i' \Delta^i f_{n+1-i} \, ,$$

where

$$\gamma_0' = \int_{-1}^{0} dr = 1, \;\; \gamma_1' = \int_{-1}^{0} r dr = -\frac{1}{2}, \;\; \gamma_2' = \int_{-1}^{0} \frac{r(r+1)}{2} dr = -\frac{1}{12} \, ,$$

$$\gamma_3' = \int_{-1}^{0} \frac{r(r+1)(r+2)}{6} dr = -\frac{1}{24} \, , \;\; \dots$$

Linear Multistep Methods for I.V. Problems

The implicit Adams methods are called Adams-Moulton methods (AM). The first AM methods are:

$$\begin{aligned}
y_{n+1} &= y_n + hf_{n+1} = y_n + hf(t_{n+1} y_{n+1}) && \text{(AM0), implicit Euler method} \\
y_{n+1} &= y_n + \frac{h}{2}(f_{n+1} + f_n) && \text{(AM1), trapezoidal method} \\
y_{n+1} &= y_n + \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1}) && \text{(AM2)} \\
y_{n+1} &= y_n + \frac{h}{24}(9f_{n+1} - 19f_n - 5f_{n-1} + f_{n-2}) && \text{(AM3)}
\end{aligned} \tag{5}$$

Note that, as typical for any implicit method, we have to solve a nonlinear equation.

Consider an Adams-Moulton scheme with 2 steps (AM2), order 3. We start from the initial values y_0, y_1 (the value of y_1 can be obtained, for instance, by using a Taylor method of order 4) and we have to compute y_2. In order to compute this value we have to solve the, in general, nonlinear equation

$$y_2 = y_1 + \frac{h}{12}(5f(t_2, y_2) + 8f(t_1, y_1) - f(t_0, y_0))$$

So far we have solved this kind of equations by using, for instance, a fixed point method. As we already know, the fixed point method is a good choice if the initial value is near enough to the solution of the equation and h is small enough.

Predictor-Corrector Methods for I.V. Problems

In general, at each step $n+1$ we have to solve the nonlinear equation

$$y_{n+1} = G(y_{n+1}) = y_n - h\Phi(y_{n+1})\,.$$

In order to apply a fixed point method, a first estimation of the root of the equation would be very convenient. This estimation can be obtained by using, for instance, an Adams-Bashforth method.

Predictor-Corrector Methods for I.V. Problems

In this way, we will have the following **predictor-corrector scheme**: Starting form h and the initial values $(y_0, ..., y_{k-1})$, the following scheme is repeated at each step n (the values $f_n, f_{n-1}, ..., f_{n-k+1}$ are obtained in previous iterations):

P: Predict by using AB. For instance, by using AB2 and using the known values f_n, f_{n-1},

$$y_{n+1}^{(0)} = y_n + \frac{h}{2}(3f_n - f_{n-1})$$

.

E: Evaluate $f(t_{n+1}, y_{n+1}^{(0)})$

C: Correct by using the AM formula. For example, for AM2 we have:

$$y_{n+1} = y_n + \frac{h}{12}(5f(t_{n+1}, y_{n+1}^{(0)}) + 8f_n - f_{n-1})$$

E: Evaluate $f_{n+1} = f(t_{n+1}, y_{n+1})$ in order to be apply in subsequent steps.

Predictor-Corrector Methods for I.V. Problems

This algorithm is denoted simbolically by **PECE**. Another possibilities are: **PECECE** (2 fixed point iterations) or **PE(CE)**^{m} (m fixed point iterations).

Example

Consider the Forward Euler method as predictor and the AM of 2nd order as corrector.
The corresponding PECE algorithm reads: $y_n \to y_{n+1}^{(0)} \to y_{n+1}$, where

$$\text{Predictor: } y_{n+1}^{(0)} = y_n + hf(t_n, y_n)$$

$$\text{Corrector: } y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(0)})\right].$$

This is a RK Method of 2nd order!

Example

Consider the ODE

$$y'(x) = -y(x) + 2\cos x, \quad y(0) = 1.$$

(1) Give the 2nd order Adams-Moulton iteration formula.

(2) Consider the Forward Euler Method as predictor and the Adams-Moulton scheme of 2nd order as corrector. Give the corresponding iteration formula.

(3) Use the previous algorithm to evaluate y_1 for $h = 0.1$.

Sol:

∘ The 2nd order Adams-Moulton scheme reads

$$\begin{aligned} y_{k+1} &= y_k + \tfrac{h}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_{k+1})\right] \\ &= y_k + \frac{h}{2}\left[2\cos x_k - y_k + 2\cos x_{k+1} - y_{k+1}\right] \end{aligned}$$

with $y_0 = y(0) = 1$. Therefore, the iteration formula is:

$$y_{k+1} = \frac{1 - h/2}{1 + h/2} y_k + \frac{h}{1 + h/2}\left[\cos x_k + \cos(x_k + h)\right],$$

$$k = 0, 1, 2, \ldots$$

Sol (Cont.):

∘ The Predictor-Corrector formulas are

$$
\begin{aligned}
y_{k+1}^{(0)} &= y_k + f(x_k, y_k)h, \\
y_{k+1} &= y_k + \frac{h}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(0)})\right].
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
y_{k+1} &= y_k + \frac{h}{2}[2\cos x_k - y_k + \\
& \quad 2\cos x_{k+1} - h(2\cos x_k - y_k) - y_k] \\
&= \left[1 - h + \frac{h^2}{2}\right] y_k + \\
& \quad h(\cos x_k + \cos(x_k + h)) - h^2 \cos x_k.
\end{aligned}
$$

Sol (Cont.):

∘ By using the previous formulas, we obtain

$$y_1 \approx 1.0945.$$

Th exact solution at the point $x = x_1$ is $y(x_1) = (\cos x_1 + \sin x_1)|_{x=0.01} \approx 1.0948$. The approximate solution x_1 by using the Adams-Moulton method is $y_1|_{AM} \approx 1.0948$. Then, in this case, $y_1|_{AM}$ is more accurate than the corresponding solution of the PC scheme; however, the PC approximation is better than the approximation obtained with a single use of a predictor method.

Boundary Value Problems

Prototype example:
Solve

$$y'' = f(t, y, y') \text{ en } a \leq t \leq b$$

with the boundary conditions

$$y(a) = \alpha\,,\; y(b) = \beta\,.$$

Boundary conditions on the derivatives of the solution or mixed boundary conditions can be also considered $y(a) + \gamma y'(a) = \alpha\,,$ ...