Solving Ordinary Differential equations

Our starting point is the following ODE:

$$\left.\begin{array}{rcl} y'(t) & = & f(t, y(t)), \;\; a \leq t \leq b \\ y(a) & = & \eta. \end{array}\right\} \tag{1}$$

Why do we need numerical methods for solving such an "innocuous" differential equation? Because problems of this form are often difficult, if not impossible, to solve analitically. This is specially the case when the equation is nonlinear.

Solving Ordinary Differential equations

Although we will concentrate on first-order scalar equations, the numerical methods are easily extended to cover first-order systems of equations. Furthermore, it is easy to convert higher order scalar initial value problems to first-order systems. Therefore, we are not restricted to first order equations. In fact, the methods can be used as part of numerical algorithms to find solutions of evolutionary **partial** differential equations. Additionally, we will consider also methods for solving **Boundary Value Problems**.

Solving Ordinary Differential equations

Outline of Study of ODE's:

1. Single-Step Methods for I.V. Problems
   - a. Euler.
   - b. Trapezoidal.
   - c. Taylor methods.
   - d. General Runge-Kutta
   - e. Adaptive step-size control.
2. Stiff ODEs
3. Multi-Step Methods for I.V. Problems.
4. Boundary Value Problems
   1. Shooting Method
   2. Finite Difference Method
   3. Finite element Method

Discretisation

The central idea behind numerical methods is that of discretisation. That is we partition the continuous interval $[a, b]$ by a discrete set of $N + 1$ points:

$$a = t_0 < t_1 < t_2 < ... < t_{N-1} < t_N = b.$$

The parameters

$$h_n = t_{n+1} - t_n\,, \quad n = 0, 1, ..., N - 1 \tag{2}$$

are called the step-sizes. We will be often be interested in using an equally spaced partition where

$$h_n = h = \frac{(b - a)}{N}, \;\; n = 0, 1, ..., N - 1.$$

Single-Step Methods for I.V. Problems

We will let y_n denote the numerical approximation to the exact solution $y(t_n)$. A numerical solution of (1) consists of a set of discrete approximations $\{y_n\}_{n=0}^{N}$. A numerical method is a difference equation involving a number of consecutive approximations

$$y_j, \quad j = 0, 1, ..., k$$

from which we sequentially compute the sequence

$$y_{k+n}, \quad n = 1, 2, ..., N.$$

Single-Step Methods for I.V. Problems

The derivation of a number of numerical methods begins by integrating (1) between t_n and t_{n+1}. This gives

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \frac{dy}{dt}\, dt &= \int_{t_n}^{t_{n+1}} f(t,y)\, dt \\ \Rightarrow y(t_{n+1}) - y(t_n) &= \int_{t_n}^{t_{n+1}} f(t,y)\, dt. \end{aligned}$$

Now, if we make the approximation

$$f(t,y) \approx f(t_n, y(t_n)), \quad t \in (t_n, t_{n+1})$$

then

$$y(t_{n+1}) - y(t_n) \approx \int_{t_n}^{t_{n+1}} f(t_n, y(t_n))\, dt = (t_{n+1} - t_n) f(t_n, y(t_n)).$$

Therefore

$$y(t_{n+1}) \approx y(t_n) + (t_{n+1} - t_n) f(t_n, y(t_n)).$$

This suggest the numerical method

$$y_{n+1} = y_n + (t_{n+1} - t_n)f(t_n, y(t_n)), \quad n = 0, 1, ..., N-1 \qquad (3)$$

which is called the forward or explicit Euler method. Note that from the initial condition $y_0 = \eta$ we can explicitly calculate y_1 by applying (3). This in turn allows us to calculate y_2 and then y_3 and so on.

A geometrical interpretation of the forward Euler method is that instead of following the possibly curved solution trajectory passing through (t_n, y_n), the method actually follows a straight line trajectory which has slope $f(t_n, y_n)$.

The forward Euler method is, of course, an approximate method which will only be exact in the trivial case of a linear solution in t. But we hope that the method will be closer to the exact solution as the step-size h is taken smaller. This is a neccesary condition for any reasonable numerical method.

Definition

We will say that a numerical method is convergent when for all IVP (1) with solution sufficiently differentiable, the following condition applies

$$\lim_{h\to 0}\left(\max_{1\le n\le N} ||y_n - y(t_n)||\right) = 0.$$

being $y_0 = y(t_0)$.
We will say that the order of convergence of the method is p, if as h is taken smaller (ie, with N large enough), then

$$\left(\max_{1\le n\le N} ||y_n - y(t_n)||\right) = \mathcal{O}(h^p)\,,\ \ Nh = \text{constante}$$

ie, if there $\exists C \ge 0$ such that $\max_{1\le n\le N} ||y_n - y(t_n)|| \le C|h|^p$ for N large enough.

Single-Step Methods for I.V. Problems

Theorem

For all IVP (1) with f continuous and satisfying a Lipschitz condition on D, the forward Euler method is convergent and its order of convergence is 1. The error of the Euler method can be bound as follows

$$||y(t_n) - y_n|| \leq \frac{C}{2L}(e^{(t_n - a)L} - 1)h\,,\ 0 \leq n \leq N$$

being $y_0 = y(t_0)$*,* $C = \max_{x\in[a,b]} ||y''(x)||$ *and L a Lipschitz constant.*

Single-Step Methods for I.V. Problems

An obvious question is whether we can easily improve upon the forward Euler method. Remembering that Euler method replaces $f(t, y(t))$ by the slope $f(t_n, y_n)$, it seems likely that an improved approximation would be the average of the slopes at t_n and t_{n+1}. That is

$$y(t_{n+1}) - y(t_n) \approx (t_{n+1} - t_n)\frac{1}{2}\left(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\right) .$$

This suggest the following numerical method:

$$y_{n+1} = y_n + \frac{h_n}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right) . \tag{4}$$

This method is called the trapezoidal method and differs from the forward Euler method in an important way:

At the $(n+1)$st step we have to solve the (generally no linear) equation

$$g(y_{n+1}) \equiv y_{n+1} - y_n - \frac{h_n}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right) = 0 \tag{5}$$

to determine y_{n+1}. Therefore y_{n+1} is defined implicitly and for that reason the trapezoidal method is an example of an implicit method. The forward Euler method, on the other hand, is an example of an explicit method.

Single-Step Methods for I.V. Problems

For the trapezoidal method, the following convergence theorem can be stablished

Theorem

For all IVP (1) satisfying a Lipschitz condition, the trapezoidal method is convergent and for $hL < 2$ (being L a Lipschitz constant) the error can be bound by

$$|e_n| \leq \frac{Ch^2}{L} \exp\left(\frac{L(t_n - a)}{1 - \frac{hL}{2}}\right),$$

where $|y'''| < C$.
Therefore, the trapezoidal method is convergent with order 2.