Error propagation: condition and stability

The condition of a function $f(x)$ measures the sensitivity of $f(x)$ to small changes in the variable x:

$$C = \left| \frac{E_{rel}(f(x))}{E_{rel}(x)} \right|$$

where $E_{rel}(f(x))$ is the relative error of $f(x)$ for a relative error of $E_{rel}(x)$ in the variable x.

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Then, [1]

$$f(x_T) - f(x_A) \simeq f'(x_t)(x_T - x_A) \rightarrow E_{rel}(f(x)) \simeq \frac{f'(x_T)}{f(x_T)}(x_T - x_A).$$

Therefore

$$C \simeq \left| x_T \frac{f'(x_T)}{f(x_T)} \right| \tag{1}$$

[1] Apply the following theorem: if $g(x)$ continuous in $[a, b]$ and differentiable in (a, b) then $\exists c \in (a, b) \, : \, f(b) - f(a) = f'(c)(b - a)$

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We will use the expression given in (1) as the definition of condition for functions of real variable. We will define the condition numbers as

$$C(x) = \left| x \frac{f'(x)}{f(x)} \right|$$

Given x, if $0 < C(x) < 1$ the problem will be well conditioned; if $C(x) > 1$ the problem will be ill conditioned. If $C(x) = 1$, the relative error is maintained.

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Example

$f(x) = \sqrt{x}$ is well conditioned: $C(x) = 1/2$.

$f(x) = x^2 - 1$ is ill-conditioned for $x \simeq 1$:

$$C(x) = \left| \frac{2x^2}{x^2 - 1} \right|$$

The concept of condition can be extended to more general situations. Consider, for instance, a classical problem which is the study of the condition of recurrence relations:

The Bessel functions $J_n(x)$ satisfy the recurrence relation $J_{n+1}(x) = -J_{n-1}(x) + \frac{2n}{x} J_n(x)$. The computation of J_n from J_0 and J_1, is ill-conditioned (an small perturbation in the initial data has catastrophic consecuences on the final value J_n).

Error propagation: condition and stability

CONDITION and STABILITY are related concepts but not equal.
Condition does not depends on round-off errors; the stability of an algorithm depends on the condition of the function to be computed.

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Example

Given the function $f(x) = \sqrt{x+1} - \sqrt{x}$, the condition number is:

$$C(x) = \left| x \frac{f'(x)}{f(x)} \right| = \frac{x}{2\sqrt{x}\sqrt{x+1}}$$

and we see that $C(x) < 1/2$ for $x > 0$; then, the function is well conditioned.

However, the following algorithm

1. Input: x
2. $y = x + 1$
3. $f_1 = \sqrt{x+1}$
4. $f_2 = \sqrt{x}$
5. $f = f_1 - f_2$

is unstable for x large (see step 5).

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Example

We can get a stable algorithm by considering:

$$f(x) = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$