

# On the Probability Distribution of Data at Points in Real Complete Intersection Varieties.<sup>1</sup>

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## Abstract

We show several estimates on the probability distribution of some data at points in real complete intersection varieties: norms of real affine solutions, condition number of real solution of real systems of multivariate polynomial equations and convergence radius of Newton's operator for under-determined system of multi-variate polynomial equations.

*Key words:* Newton's operator, condition number, real complete intersection varieties, polynomial equations solver.

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## 1 Introduction.

These pages are concerned with the probability distribution of data at points in real complete intersection algebraic varieties. The ultimate goal of this study will be the design and analysis of efficient algorithms for real solving of multi-variate real polynomial equations.

In several recent works ([2], [5], [6]) probabilistic algorithms in average polynomial time that compute complex solutions of complex polynomial equations have been introduced. Unfortunately, the same algorithmic scheme does not apply for real solving. In fact, linear homotopy outside the discriminant variety does not apply to real solving of zero-dimensional equations.

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In conclusion, we feel that the study of real solving of multi-variate polynomial equations must be studied re-initiating the program established by M. Shub and S. Smale in his seminal of papers on Bézout's Theorem ([33–37]). These pages are a modest attempt to re-initiate this study from the most elementary aspects: study of the probability distribution of three data at points in real complete intersection varieties:

- (1) Norms of real affine zeros of real complete intersections.
- (2) Condition number at real zeros of real system of multi-variate polynomial equations.
- (3) Convergence radius of Newton's operator at zeros of complete intersection real algebraic varieties.

In order to be precise on stating our main outcomes we introduce a few notations.

Let  $n, d \in \mathbb{N}$  be two positive integer numbers. We denote by  $H_d^{\mathbb{R}}$  the real vector space of all homogeneous polynomials  $f \in \mathbb{R}[X_0, \dots, X_n]$  of degree  $d$ . For every positive integer number  $m \in \mathbb{N}$ ,  $m \leq n$ , and for every degree list  $(d) := (d_1, \dots, d_m) \in \mathbb{N}^m$ , we denote by  $\mathcal{H}_{(d)}^{\mathbb{R}}$  the real vector space given as the product

$$\mathcal{H}_{(d)}^{\mathbb{R}} := H_{d_1}^{\mathbb{R}} \times \dots \times H_{d_m}^{\mathbb{R}}.$$

Note that  $\mathcal{H}_{(d)}^{\mathbb{R}}$  is a real vector space of dimension  $N := \sum_{i=1}^m \binom{d_i+n}{n}$ .

For every list  $f := (f_1, \dots, f_m) \in \mathcal{H}_{(d)}^{\mathbb{R}}$  of homogeneous polynomials we denote by  $V_{\mathbb{P}}(f) \subseteq \mathbb{P}_n(\mathbb{R})$  the algebraic variety of its common zeros in the real projective space  $\mathbb{P}_n(\mathbb{R})$ .

We assume that  $\mathcal{H}_{(d)}^{\mathbb{R}}$  is endowed with the Bombieri-Weyl-Kostlan metric that we denote by  $\langle \cdot, \cdot \rangle_{\Delta}$  (cf. Section 2.1 below for details). This metric induces a Riemannian structure in  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  and a volume form that we denote by  $d\nu_{\Delta}$ . Moreover, the total volume of  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  is finite and normalizing the metric  $\langle \cdot, \cdot \rangle_{\Delta}$  also induces a probability distribution on  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  that we will denote by  $\text{Prob}_{\Delta}[\cdot]$ .

Note that there is a subset of measure zero  $\Sigma$  of  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  (i.e.  $\text{Prob}_{\Delta}[\Sigma] = 0$ ) such that for every system  $f = (f_1, \dots, f_m) \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) \setminus \Sigma$ , the real projective variety  $V_{\mathbb{P}}(f)$  is either empty or a Riemannian sub-manifold of  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  of codimension  $m$ . We assume that  $V_{\mathbb{P}}(f)$  is endowed with the Riemannian structure as sub-manifold of  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$ .

A relevant point estimate in polynomial equation solving is the norm of the affine solutions of the given system. This has been used in [6] to design efficient algorithms to compute complex affine solutions of complex systems. Here we study its real version.

For every system  $f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$ , we denote by  $V_{\mathbb{A}}(f) \subseteq \mathbb{R}^n$  the real algebraic set of its real affine zeros. Namely, let  $\varphi_0 : \mathbb{R}^n \longrightarrow \mathbb{P}_n(\mathbb{R})$  be the affine chart given by the following equality:

$$\varphi_0(x_1, \dots, x_n) := (1 : x_1 : \dots : x_n), \forall (x_1, \dots, x_n) \in \mathbb{R}^n,$$

where  $(1 : x_1 : \dots : x_n)$  denote homogeneous coordinates. Thus,  $V_{\mathbb{A}}(f) := \varphi_0^{-1}(V_{\mathbb{P}}(f))$ . We also assume that  $V_{\mathbb{A}}(f)$  is endowed with the pullback probability distribution defined from the Riemannian probability defined on  $V_{\mathbb{P}}(f)$ .

We denote by  $\mathcal{R}_{(d)} \subseteq \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  the subset of all system  $f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  such that  $V_{\mathbb{P}}(f) \neq \emptyset$ . Finally, for every positive real number  $r, 0 < r < 1$ , we denote by  $N_{av}^r(f)$  the expected norm of the affine points in  $V_{\mathbb{A}}(f)$ . Namely,  $N_{av}^r := E_{V_{\mathbb{A}}(f)}[\|\cdot\|^r]$ . We also prove the following statement:

**Theorem 1** *With the same notations as above, for every positive real number  $r, 0 < r < 1$  the following equality holds:*

$$E_{\Delta}[N_{av}^r(f)] = \beta\left(\frac{n+r}{2}, \frac{1-r}{2}\right) \frac{\nu_{\Delta}[\mathcal{R}_{(d)}]}{\vartheta_N}.$$

where  $\beta$  is the standard hyper-geometric function and  $\vartheta_k = \nu_k[\mathbb{P}_k(\mathbb{R})]$  denotes the volume of the real projective space of dimension  $k$ . In particular, if all the degrees  $d_1, \dots, d_m$  in the list  $(d) = (d_1, \dots, d_m)$  are odd integer numbers, the following equality holds:

$$E_{\Delta}[N_{av}^r(f)] = \beta\left(\frac{n+r}{2}, \frac{1-r}{2}\right).$$

From this statement we immediately obtain the following consequences.

**Corollary 2** *With the same notations as in the Theorem 1 above, if  $n = m$ ,  $n \geq 3$  and for every positive real number  $\varepsilon < 1$ , the following inequality holds:*

$$\text{Prob}_{\Delta}\left[\max_{\zeta \in V_{\mathbb{A}}(f)} \|\zeta\| \geq \varepsilon^{-1}\right] \leq \mathcal{D}^{1/2} \varepsilon^{1/2} \Gamma\left(\frac{1}{4}\right),$$

where  $\Gamma$  is the Euler's  $\Gamma$ -function and  $\mathcal{D}$  denotes the Bézout number, namely  $\mathcal{D} = \prod_{i=1}^m d_i$ .

**Corollary 3** *With the same notations as in the Theorem 1 above and for every positive real number  $\varepsilon < 1$  the following inequality holds:*

$$\text{Prob}_{\Delta}\left[\dim(V_{\mathbb{A}}(f) \cap B_{\mathbb{R}^n}(0, \varepsilon^{-2})) < n - m\right] \leq \varepsilon \beta\left(\frac{n}{2} + \frac{1}{4}, \frac{1}{4}\right) \frac{\nu_{\Delta}[\mathcal{R}_{(d)}]}{\vartheta_N},$$

where  $\dim$  is the dimension as semi-algebraic set and  $B_{\mathbb{R}^n}(0, \varepsilon^{-2})$  is the closed ball of center 0 and radius  $\varepsilon^{-2}$  in  $\mathbb{R}^n$ . In particular, if all the degrees  $d_1, \dots, d_m$

in the list  $(d) = (d_1, \dots, d_m)$  are odd integer numbers,  $n \geq 3$ , for every positive real number  $\varepsilon < 1$ , the following inequality holds:

$$\text{Prob}_\Delta \left[ \dim(V_{\mathbb{A}}(f) \cap B_{\mathbb{R}^n}(0, \varepsilon^{-2})) < n - m \right] \leq \varepsilon \Gamma \left( \frac{1}{4} \right).$$

Note that this last statement means that with high probability  $B_{\mathbb{R}^n}(0, \varepsilon^{-2})$  contains a portion in  $V_{\mathbb{A}}(f)$  which is a Zariski dense in some of its irreducible components.

The second ingredient in the design of algorithms for polynomial equation solving is the normalized condition number  $\mu_{norm}^m$  (cf. Subsection 2.4) introduced by M. Shub and S. Smale in [33] (see also references there in), and then generalized to undetermined system of polynomials by J. Dégot and J. Dedieu in [18,15]. The complex sparse case had been studied by M. Rojas and G. Malajovic in [29]. Let the reader observe that  $\mu_{norm}^m$  is not a “conic” condition number as established in [9].

Let the reader recall that the condition number has been show to be a central tool in the design and analysis of algorithms for polynomial equation solving. For instance, Dégot proved in [18] that  $\mu_{norm}$  describe the stability of the algorithm, it also determines the convergence of Newton’s operator (cf. [16,4]) and, finally, it estimates the complexity of certain polynomial equation solvers (as in [37,5,6]). Thus, the knowledge of its probability distribution is essential also to understand the behaviour of such algorithms.

For every positive real number  $\varepsilon > 0$  we define the function  $Vol_\varepsilon : \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) \longrightarrow \mathbb{R} \cup \{+\infty\}$  in the following term. For every  $f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) \setminus \Sigma$  we define

$$Vol_\varepsilon^m(f) := \nu_{n-m} \left[ \zeta \in V_{\mathbb{P}}(f) : \mu_{norm}^m(f, \zeta) > \varepsilon^{-1} \right],$$

where  $\mu_{norm}^m(f, \zeta)$  is the normalized condition number of  $f$  at  $\zeta \in V_{\mathbb{P}}(f)$  and  $\nu_{n-m}[\cdot]$  denotes the  $(n - m)$ -th dimensional Hausdorff measure in  $V_{\mathbb{P}}(f)$ .

**Theorem 4** *With these notations, assume  $2 \leq m$ . Then, there are two universal constants  $c, C \in \mathbb{R}$ ,  $0.245 \leq c \leq 2$  and  $5.013 \leq C \leq 6.414$  such that for every positive real number  $\varepsilon > 0$  satisfying  $\varepsilon \leq \frac{1}{n-m+2}$ , the following inequalities hold:*

$$\begin{aligned} \frac{4}{\sqrt{\pi}} \left( \frac{\mathcal{D}}{(n+1)m-1} \right)^{1/2} \left( \frac{(n+1)c\varepsilon}{n-m+2} \right)^{n-m+2} &\leq \\ &\leq \mathbb{E}_\Delta [Vol_\varepsilon^m(f)] \leq \\ &\leq 2 \left( \frac{N\mathcal{D}}{\pi} \right)^{1/2} \left( \frac{(n+1)^{3/2}C\varepsilon}{n-m+2} \right)^{n-m+2} \end{aligned}$$

Let the reader compare these estimates with the ones in Theorem 3.7 of [9] page 21.

For zero-dimensional varieties (case  $m = n$ ) we also define  $\mu_{worst}(f)$  as the maximum of the condition numbers of its zeros in  $V_{\mathbb{P}}(f)$ . We also prove

**Corollary 5** *With the same notations as above, provided that  $n = m$ , for every positive number  $\varepsilon, 0 < \varepsilon \leq 1/2$ , the following inequalities holds:*

$$\text{Prob}_{\Delta} \left[ f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) : \mu_{worst}(f) \geq \varepsilon^{-1} \right] \leq \left( \frac{N\mathcal{D}}{\pi} \right)^{1/2} \frac{(n+1)^3 (C\varepsilon)^2}{2}$$

$$E_{\Delta} [\mu_{worst}(f)] \leq 2\sqrt{2} \left( \frac{N\mathcal{D}}{\pi} \right)^{1/4} (n+1)^{3/2} C$$

Now, we consider the average condition number  $\mu_{av,\varepsilon}^m(f)$  on the variety  $V_{\mathbb{P}}(f)$ . Namely, for every positive real number  $\varepsilon > 0$ , we define

$$\mu_{av,\varepsilon}^m(f) = \frac{Vol_{\varepsilon}(f)}{\nu_{n-m}[V_{\mathbb{P}}(f)]} = \frac{1}{\nu_{n-m}[V_{\mathbb{P}}(f)]} \int_{V_{\mathbb{P}}(f)} \chi_{\varepsilon} dV_{\mathbb{P}}(f),$$

provided that  $\nu_{n-m}[V_{\mathbb{P}}(f)] > 0$  and where  $\chi_{\varepsilon} : V_{\mathbb{P}}(f) \rightarrow \{0, 1\}$  is the characteristic function of the set  $A_{\varepsilon}(f) = \{\zeta \in V_{\mathbb{P}}(f) : \mu_{norm}^m(f, \zeta) > \varepsilon^{-1}\}$ .

The following statement essentially follows from Theorem 4.

**Corollary 6** *With the same notations as above, provided that  $(d) = (d_1, \dots, d_m)$  are all odd integer numbers, for every positive real number  $\varepsilon > 0$  satisfying  $\varepsilon \leq \frac{1}{n-m+2}$ , the following inequalities hold:*

$$\begin{aligned} \frac{4}{\sqrt{\pi}} \frac{1}{\mathcal{D}^{1/2} \vartheta_{n-m}} \left( \frac{1}{(n+1)m-1} \right)^{1/2} \left( \frac{(n+1)c\varepsilon}{n-m+2} \right)^{n-m+2} &\leq \\ &\leq E_{\Delta} [Vol_{\varepsilon}^m(f)] \leq \\ &\leq \frac{2}{\sqrt{\pi}} \frac{(N\mathcal{D})^{1/2}}{\vartheta_{n-m}} \left( \frac{(n+1)^{3/2} C\varepsilon}{n-m+2} \right)^{n-m+2} \end{aligned}$$

where  $0.245 \leq c \leq 2$  and  $5.013 \leq C \leq 6.414$ .

These probability estimates are helpful to design algorithmic procedures to compute points in the solution variety  $V_{\mathbb{A}}(f)$  (as in [13]). They are also helpful to have sharp probability estimates on the behaviour of the convergence radius of Newton's method in either affine or projective context (as done in [4] for the complex case). In order to illustrate this second application we just discuss the probability distribution of the convergence radius of projective Newton's method in the under-determined real case.

Thus, we assume  $m \leq n$ ,  $(d) = (d_1, \dots, d_m)$  a degree list and  $f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  a system of multivariate homogeneous polynomials with real coefficients. For every projective point  $x \in \mathbb{P}_n(\mathbb{R})$  we define the Newton operator in positive dimension in the following terms:

$$N_f(x) := \pi \left( z - Df(z)^\dagger f(z) \right),$$

where  $\pi : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}_n(\mathbb{R})$  is the canonical projection,  $Df(z)^\dagger$  is the Moore-Penrose pseudo-inverse and  $z \in \mathbb{R}^{n+1} \setminus \{0\}$  is any point such that  $\pi(z) = x \in \mathbb{P}_n(\mathbb{R})$ . Note that if  $Df(z) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m$  is surjective

$$Df(z)^\dagger = \left( Df(z)|_{\ker(f)^\perp} \right)^{-1}.$$

In the case  $m = n$ , this is the standard Newton's operator in the projective case.

For every  $f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  such that  $V_{\mathbb{P}}(f)$  is a smooth real algebraic variety of codimension  $m$ , and for every  $\zeta \in V_{\mathbb{P}}(f)$  there is an open ball  $B_{\mathbb{P}}(\zeta, r)$  (with respect to the projective distance  $d_{\mathbb{P}}$ ) of positive radius  $r > 0$ , such that for all  $x \in B(\zeta, r)$  the following property holds:

$$d_{\mathbb{P}}(N_f^k(x), V_{\mathbb{P}}(f)) \leq \frac{a}{2^{2^k-1}} d_{\mathbb{P}}(x, \zeta), \quad (1)$$

where  $a$  is any universal constant independent of  $f$  and  $\zeta$  (we shall use  $a = 2^{3/2}3$ , but this may be suppressed),  $N_f^k(x)$  is the  $k$ -th iterate of projective Newton's method applied to  $x$ . This sentence easily follows from Theorem 130 in [16]. All points in  $\mathbb{P}_n(\mathbb{R})$  satisfying property (1) are called *approximate zeros of  $f$* . Thus, we may consider the best of such radius  $r$  by defining

$$R(f, \zeta) := \sup\{r \in \mathbb{R}^+ : \forall x \in B_{\mathbb{P}}(\zeta, r), x \text{ is an approximate zero of } f\}.$$

The following statement follows from Theorem 4 above (see also Proposition 25 below).

**Corollary 7** *With the same notations of above, for every positive real number  $0 < \varepsilon < \min\{1/3, \frac{1}{n-m+2}\}$  the following inequality hold:*

$$E_{\Delta} [\nu_{n-m} [\zeta \in V_{\mathbb{P}}(f) : R(f, \zeta) < \varepsilon]] \leq 2 \left( \frac{N\mathcal{D}}{\pi} \right)^{1/2} \left( \frac{(n+1)^{3/2}C}{n-m+2} \right)^{n-m+2} \left( \frac{2u_0}{d^{3/2}} \varepsilon \right)^{n-m+2},$$

where  $5.013 \leq C \leq 6.414$  and  $u_0 \approx 0.05992$  is an universal constant.

Moreover, in the zero-dimensional case we may even obtain a tubular neighborhood of the variety of solutions. With the same notations as above, let us

define

$$R_{\text{worst}} := \min\{R(f, \zeta) : \zeta \in V_{\mathbb{P}}(f)\}.$$

Note that this quantity measure the worst radius of convergence around the roots of a system of equations.

**Corollary 8** *With the same notations of above, provided that  $n = m$ , we have that the expected radius of convergence is at least*

$$E_{\Delta} [R_{\text{worst}}] \geq \frac{2u_0\pi^{1/4}}{6u_0\pi^{1/4} + 2\sqrt{2}d^{3/2}(N\mathcal{D})^{1/4}(n+1)^{3/2}C},$$

where  $d := \max\{d_1, \dots, d_m\}$  is the maximum degree,  $5.013 \leq C \leq 6.414$  and  $u_0 \approx 0.05992$  is an universal constant.

The manuscript is structured as follows. In Section 2 we state the basic facts and essential notations used in subsequent sections. Sections 3 and 4 are devoted to establish the main technical objects to be used in the proof of Corollaries 2 and 3. Then, Section 5 is devoted to prove Corollaries 5, 6, 7 and 8.

## 2 Notations and Basic Results.

### 2.1 Norms.

In this subsection we introduce the Bombieri-Weyl-Kostlan metric in  $\mathcal{H}_{(d)}^{\mathbb{R}}$  (see [27,8,15,18,29,33,34] for further bibliographical references). As in the previous section, for every positive integer number  $l \in \mathbb{N}$ , let  $H_l^{\mathbb{R}} \subseteq \mathbb{R}[X_0, \dots, X_n]$  be the vector space of all homogeneous polynomials of degree  $l$  with coefficients in  $\mathbb{R}$ . The monomial basis of  $H_l^{\mathbb{R}}$  can be identified with the set of multi-indices

$$\mathbb{N}_l^{n+1} := \{\mu = (\mu_0, \mu_1, \dots, \mu_n) \in \mathbb{N}^{n+1} : |\mu| := \mu_0 + \dots + \mu_n = l\}.$$

As in standard elimination theory we can choose a monomial order  $\leq_l$  in  $\mathbb{N}_l^{n+1}$  (see [12,7,25] and the references therein for an introduction to monomial orders, Gröbner bases and Computational Commutative Algebra). Any monomial order in  $\mathbb{N}_l^{n+1}$  allows us to see the elements of  $H_l^{\mathbb{R}}$  as vectors given by their coordinates (with respect to this monomial order). This is called in the literature of effective algebraic geometry as “dense encoding of polynomials”. For every  $\mu \in \mathbb{N}_l^{n+1}$ , we define the multinomial coefficient

$$\binom{l}{\mu} := \frac{l!}{\mu_0! \cdots \mu_n!}.$$

We define the matrix  $\Delta_l \in \mathcal{M}_{N_l}(\mathbb{R})$  with  $N_l := \binom{l+n}{n}$ , associated with  $H_l^{\mathbb{R}}$ , as the diagonal matrix whose  $\mu$ -th entry (with respect to the monomial order  $\leq_l$ ) at the diagonal is  $\binom{l}{\mu}^{-1/2}$ . Namely,  $\Delta_l$  is the diagonal matrix given by the following identity:

$$\Delta_l := \bigoplus_{\mu \in \mathbb{N}_l^{n+1}} \left( \binom{l}{\mu}^{-1/2} \right).$$

Let  $\langle \cdot, \cdot \rangle_l : H_l^{\mathbb{R}} \times H_l^{\mathbb{R}} \rightarrow \mathbb{R}$  be the canonical Euclidean product on  $H_l^{\mathbb{R}}$  and let  $(d) := (d_1, \dots, d_m) \in \mathbb{N}^m$  be a list of positive degrees. We also have the canonical Euclidean product on  $\mathcal{H}_{(d)}^{\mathbb{R}}$  given by the following identity:

$$\langle f, g \rangle := \sum_{i=1}^m \langle f_i, g_i \rangle_{d_i} \in \mathbb{R},$$

where  $f = (f_1, \dots, f_m), g = (g_1, \dots, g_m) \in \mathcal{H}_{(d)}^{\mathbb{R}}$ . We finally denote by  $\langle \cdot, \cdot \rangle_{\Delta}$  the Euclidean product over  $\mathcal{H}_{(d)}^{\mathbb{R}}$  defined by the respective matrices  $\Delta_{d_i}$  and given by the following identity:

$$\langle f, g \rangle_{\Delta} := \sum_{i=1}^m \langle \Delta_{d_i} f_i, \Delta_{d_i} g_i \rangle_{d_i} = \sum_{i=1}^m \langle f_i, g_i \rangle_{\Delta_{d_i}}$$

where  $f := (f_1, \dots, f_m), g := (g_1, \dots, g_m) \in \mathcal{H}_{(d)}^{\mathbb{R}}$ . We denote by  $\Delta$  the following matrix,

$$\Delta := \bigoplus_{i=1}^m \Delta_{d_i} \in \mathcal{M}_N(\mathbb{R}).$$

The bi-linear map  $\langle \cdot, \cdot \rangle_{\Delta}$  is invariant under the action of the orthogonal group. The reader may follow details in [8].

We consider the following group action

$$\begin{aligned} \mathcal{O}_{n+1}(\mathbb{R}) \times \mathcal{H}_{(d)}^{\mathbb{R}} &\longrightarrow \mathcal{H}_{(d)}^{\mathbb{R}} \\ (U, f) &\longmapsto f \circ U, \end{aligned}$$

where “ $\circ$ ” denotes the composition and  $\mathcal{O}_{n+1}(\mathbb{R})$  is the orthogonal group. The action naturally extend to the projective space  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$ . The following statement holds (cf. [8]):

**Proposition 9 (Orthogonal Invariance)** *With the same notations as above,  $f, g \in \mathcal{H}_{(d)}^{\mathbb{R}}$  and  $U \in \mathcal{O}_{n+1}(\mathbb{R})$  the following holds*

$$\langle f \circ U, g \circ U \rangle_{\Delta} = \langle f, g \rangle_{\Delta}.$$

The euclidean product  $\langle \cdot, \cdot \rangle_{\Delta}$  defined by the Kostlan matrix induces a Riemannian structure in the projective space  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$ . For every point  $f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$

we denote by  $f^\perp$  the orthogonal complement with respect to  $\langle \cdot, \cdot \rangle_\Delta$ . Note that the tangent space  $T_f \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  can be identified with the orthogonal complement  $f^\perp$  with the inner product inherited from that of  $\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$ . Moreover the affine chart at a point  $f \in \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  is the mapping  $\varphi_f : f^\perp \longrightarrow \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R}) \setminus f^\perp$  given by the following equality:

$$\varphi_f(g) = \pi(\tilde{f} + \tilde{g}),$$

where  $\pi : \mathcal{H}_{(d)}^\mathbb{R} \setminus \{0\} \longrightarrow \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  is the canonical projection and  $\tilde{f}, \tilde{g} \in \mathcal{H}_{(d)}^\mathbb{R}$  are respectively represents of  $f$  and  $g$  of norm 1. Moreover, the tangent mapping  $T_0 \varphi_f : f^\perp \longrightarrow T_f \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  is a linear isometry.

As for the space of solution  $\mathbb{P}_n(\mathbb{R})$ , we assume it is endowed with the standard canonical Riemannian structure.

Given any pair  $(f, x) \in \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R}) \times \mathbb{P}_n(\mathbb{R})$ , we denote by  $T_x f := (d_x f)|_{x^\perp}$  the restriction of the tangent mapping  $d_x f$  to the tangent space  $x^\perp$ , where  $f, x$  are any fixed affine representations such that  $\|f\|_\Delta = \|x\|_2 = 1$ . Sometimes we identify  $T_x f$  and the jacobian matrix in any orthonormal basis of  $x^\perp$ . In the case that  $x = e_0 := (1 : 0 : \dots : 0)$ , we identify

$$T_{e_0} f \equiv \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(e_0) & \dots & \frac{\partial f^1}{\partial x_n}(e_0) \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x_1}(e_0) & \dots & \frac{\partial f^m}{\partial x_n}(e_0) \end{pmatrix},$$

for any fixed representation  $f \in \mathcal{H}_{(d)}^\mathbb{R}$ ,  $\|f\|_\Delta = 1$ .

## 2.2 Incidence variety, Discriminant Variety and a Triangle of Projections.

We first define the real incidence variety  $V \subseteq \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R}) \times \mathbb{P}_n(\mathbb{R})$  as the multi-homogeneous variety given by the following identity:

$$V := \{(f, x) \in \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R}) \times \mathbb{P}_n(\mathbb{R}) : f(x) = 0\}.$$

The following statement essentially follows the same arguments as in Proposition 1, Chapter 10 of [8].

**Theorem 10** *The multi-homogeneous variety  $V \subseteq \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R}) \times \mathbb{P}_n(\mathbb{R})$  is a compact, connected, real Riemannian manifold of real dimension*

$$\dim V = N + n - m.$$

Moreover, for each point  $(f, \zeta) \in V$ , the tangent space  $T_{(f, \zeta)}V$  is given by the following identity:

$$T_{(f, \zeta)}V = \{(h, w) \in T_f\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) \times T_{\zeta}\mathbb{P}_n(\mathbb{R}) : h(\zeta) + d_{\zeta}f \cdot w^t = 0\},$$

where  $w^t$  is the transpose of the vector  $w \in \langle x \rangle^{\perp} \subseteq \mathbb{R}^{n+1}$ .

Let  $\pi_1$  and  $\pi_2$  be the canonical projections from  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) \times \mathbb{P}_n(\mathbb{R})$  respectively onto  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  and  $\mathbb{P}_n(\mathbb{R})$ . We denote by  $\pi_1$  and  $\pi_2$  its restrictions to  $V$  and we have the following triangular diagram:

$$\begin{array}{ccc} & V & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) & & \mathbb{P}_n(\mathbb{R}). \end{array}$$

We consider the following isometric action on the real incidence variety.

$$\begin{aligned} \mathcal{O}_{n+1}(\mathbb{R}) \times V &\longrightarrow V \\ (U, (f, x)) &\longmapsto (f \circ U^{-1}, Ux). \end{aligned}$$

**Proposition 11** *The projection  $\pi_2$  is an onto submersion at any  $(f, \zeta) \in V$ . Namely, the classes of critical points and critical values of  $\pi_2$  are empty.*

*The fibers of  $\pi_2$  at any point is a real projective linear subvariety given by the following identity*

$$V_x := \pi_2^{-1}(x) = \{f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) : f(x) = 0\}.$$

*The real of codimension of  $V_x$  in  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  is  $m$ , and hence,*

$$\dim V_x = N - m.$$

*Moreover, for every orthogonal matrix  $U \in \mathcal{O}_{n+1}(\mathbb{R})$ , given  $x, y \in \mathbb{P}_n(\mathbb{R})$  such that  $Ux = y$ , the following is a isometry between the fibers  $V_x$  and  $V_y$*

$$\begin{aligned} V_x &\longrightarrow V_y \\ f &\longrightarrow f \cdot U^{-1}. \end{aligned}$$

The mapping  $\pi_1$  is not onto. We denote by  $\mathcal{R}_{(d)} \subseteq \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  the image of  $\pi_1$ . Namely,  $\mathcal{R}_{(d)} := \pi_1(V)$ . The following statement follows from elementary Elimination Theory over the reals and explain some of the properties of  $\mathcal{R}_{(d)}$ .

**Proposition 12** *The set  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) \setminus \mathcal{R}_{(d)}$  is a set of measure zero if and only if the degrees  $d_1, \dots, d_m$  are odd positive integers numbers.*

We denote by  $\Sigma' \subseteq V$  the set of critical points of the projection  $\pi_1$  and by  $\Sigma$  set of critical values of  $\pi_1$  that belong to  $\mathcal{R}_{(d)}$ . The following equality also characterizes the regular points of  $\pi_1$  in  $V$ .

$$V \setminus \Sigma' = \{(f, x) \in V : T_x f \text{ has rank } m\}.$$

The following properties hold.

**Proposition 13** (1)  $\pi_1 : V \setminus \Sigma' \longrightarrow \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) \setminus \Sigma$  is a submersion at any point  $(f, x) \in V \setminus \Sigma'$ .

(2) The real dimension of  $\Sigma'$  and  $\Sigma$  satisfy the following equalities

$$\dim \Sigma' \leq N + n - m - 1, \quad \dim \Sigma \leq N - 1.$$

(3) The open set  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) \setminus \Sigma$  is connected provided that  $m < n$ .

(4) For every  $f \in \mathcal{R}_{(d)} \setminus \Sigma$  the fiber  $\pi_1^{-1}(f)$  is a real smooth projective variety of dimension  $n - m$ .

### 2.3 Some General Integral Formulae.

Now we fix some notations and recall some relevant integral formulae that we use in the sequel. For every  $d$ -dimensional Riemannian manifold  $M$  and for every measurable subset  $A \subseteq M$  we usually denote by  $\nu_d[A]$  its  $d$ -dimensional Hausdorff measure.

Let  $\mathbb{P}_k(\mathbb{R})$  be the  $k$ -dimensional real projective space. We will denote by  $d\nu_{\mathbb{P}}$  the canonical volume form associated to the Riemannian structure of  $\mathbb{P}_k(\mathbb{R})$ . For every subset  $A \subseteq \mathbb{P}_k(\mathbb{R})$  we also denote by  $\nu_k[A]$  its  $k$ -dimensional volume. We finally denote by  $\vartheta_k$  the volume of  $\mathbb{P}_k(\mathbb{R})$ . Namely,

$$\vartheta_k := \nu_k[\mathbb{P}_k(\mathbb{R})] = \frac{\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)}.$$

For a linear space  $\mathbb{R}^{k+1}$  and a positive real number  $t > 0$  we denote by  $\mathbb{S}_k^t \subseteq \mathbb{R}^{k+1}$  the sphere of radius  $t$  centered at 0. Observe that the volume of  $\mathbb{S}_k^t$  as sub-manifold of  $\mathbb{R}^{k+1}$  is equal to:

$$\nu[\mathbb{S}_k^t] := 2t^k \frac{\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)} = 2t^k \vartheta_k.$$

Note that we denote by  $\mathbb{S}_k$  the sphere of radius  $t = 1$ .

Let  $\Delta$  be the diagonal matrix used in subsection 2.1 (see [8, pg. 236] for further bibliographical references). From now on, we call  $\Delta$  the Kostlan matrix associated to the degree list  $(d)$ . We denote by  $\|\cdot\|_\Delta$  the norm on  $\mathcal{H}_{(d)}^\mathbb{R}$  defined by  $\langle \cdot, \cdot \rangle_\Delta$ . As in [34] (cf. also [8]), we consider the Riemannian structure in  $\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  induced by the Euclidean product  $\langle \cdot, \cdot \rangle_\Delta$ . We denote by  $d\nu_\Delta$  the volume form on  $\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  associated to this Riemannian structure. As  $\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  is compact of finite volume, we also have a probability distribution defined on  $\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  in the following terms: For every integrable function  $f : \mathbb{P}_n(\mathbb{R}) \longrightarrow \mathbb{R}$  we define its expectation  $E_\Delta[f]$  by the following equality:

$$E_\Delta[f] := \frac{1}{\vartheta_N} \int_{\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})} f d\nu_\Delta.$$

Let the reader observe that  $\nu_\Delta[\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})] = \vartheta_N$ .

For every measurable subset  $A \subseteq \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$ , we define its probability as the expectation of its characteristic function  $\chi_A : \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R}) \longrightarrow \{0, 1\}$ . Namely,

$$\text{Prob}_\Delta[A] := \frac{\nu_\Delta[A]}{\vartheta_N} = \frac{1}{\vartheta_N} \int_{\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})} \chi_A d\nu_\Delta.$$

For every subset  $A$  of  $\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  we denote as  $\tilde{A} \subseteq \mathcal{H}_{(d)}^\mathbb{R}$  the affine cone associated to  $A$ . Namely,

$$\tilde{A} := \pi^{-1}(A) \cup \{0\},$$

where  $\pi : \mathcal{H}_{(d)}^\mathbb{R} \setminus \{0\} \longrightarrow \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  is the canonical projection onto the projective space.

From Proposition 1, Chapter 11 of [8], the probability distribution defined by  $d\nu_\Delta$  on  $\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  agrees with the Gaussian distribution in  $\mathcal{H}_{(d)}^\mathbb{R}$  associated to the norm  $\|\cdot\|_\Delta$ . Namely, for every measurable subset  $A \subseteq \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})$  we have

$$\text{Prob}_\Delta[A] = \frac{1}{(2\pi)^{N/2}} \int_{\mathcal{H}_{(d)}^\mathbb{R}} \chi_{\tilde{A}} e^{-\frac{\|f\|_\Delta^2}{2}} d\mathcal{H}_{(d)}^\mathbb{R}$$

where  $\tilde{A} \subseteq \mathcal{H}_{(d)}^\mathbb{R}$  is the affine cone over  $A$  and  $\chi_{\tilde{A}} : \mathcal{H}_{(d)}^\mathbb{R} \longrightarrow \{0, 1\}$  is the characteristic function of  $\tilde{A}$ .

We will make extensive use of the so called Co-area Formula. A general formulation is known as Federer's Co-area Formula (cf. [24]). Another general formulation may be seen in [26]. For our purposes we use the smooth version used in [34] (cf. also [8]).

**Definition 14** Let  $X$  and  $Y$  be Riemannian manifolds, and let  $F : X \rightarrow Y$  be a  $C^1$  surjective map. Let  $k := \dim(Y)$  be the real dimension of  $Y$ . For every regular point  $x \in X$  (i.e.  $F$  is a submersion at  $x$ ), let  $v_1^x, \dots, v_k^x$  be an orthonormal basis of  $\ker(d_x F)^\perp$ . Then, we define the Normal Jacobian of  $F$  at  $x$ ,  $\text{NJ}_x F$ , as the volume in  $T_{F(x)}Y$  of the parallelepiped spanned by  $d_x F(v_1^x), \dots, d_x F(v_k^x)$ . In the case that  $d_x F$  is not surjective, we define  $\text{NJ}_x F := 0$ .

The following Proposition immediately follows from the definition.

**Proposition 15** Let  $X, Y$  be two Riemannian manifolds, and let  $F : X \rightarrow Y$  be a  $C^1$  map. Let  $x_1, x_2 \in X$  be two points. Assume that there exist isometries  $\varphi_X : X \rightarrow X$  and  $\varphi_Y : Y \rightarrow Y$  such that  $\varphi_X(x_1) = x_2$ , and

$$F \circ \varphi_X = \varphi_Y \circ F.$$

Then, the following equality holds:

$$\text{NJ}_{x_1} F = \text{NJ}_{x_2} F.$$

Moreover, if there exists an inverse  $G : Y \rightarrow X$ , then

$$\text{NJ}_x F = \frac{1}{\text{NJ}_{F(x)} G}.$$

**Theorem 16 (Co-area Formula)** Let  $X, Y$  be two Riemannian manifolds of respective dimensions  $k_1 \geq k_2$ . Let  $F : X \rightarrow Y$  be a  $C^\infty$  mapping. Assume that  $F$  is a submersion almost everywhere in  $X$  and let  $\psi : X \rightarrow \mathbb{R}$  be an integrable mapping. Then, the following equality holds:

$$\int_X \psi \, dX = \int_{y \in Y} \int_{x \in F^{-1}(y)} \psi(x) \frac{1}{\text{NJ}_x F} \, dF^{-1}(y) dY,$$

where  $\text{NJ}_x F$  is the normal jacobian of  $F$  at  $x$ .

We now state a translations to the real case of a proposition due to Howard (cf. [26]). Note that, in the following, the volume of  $\mathcal{O}_{n+1}(\mathbb{R})$  is normalized in such a way that the volume of  $\mathcal{O}_{n+1}(\mathbb{R})$  is equal to 1.

**Theorem 17** Let  $V \subseteq \mathbb{P}_n(\mathbb{R})$  be a smooth real equidimensional projective algebraic variety of dimension  $m$ . Then, we have:

$$\nu_m[V] = \vartheta_m \int_{U \in \mathcal{O}_{n+1}(\mathbb{R})} \sharp(V \cap UL) d\mathcal{O}_{n+1}(\mathbb{R}),$$

where  $L \subseteq \mathbb{P}_n(\mathbb{R})$  is any fixed linear projective subspace of  $\mathbb{P}_n(\mathbb{R})$  of co-dimension  $m$ .

**Corollary 18** Let  $f \in \mathcal{H}_{(d)}^{\mathbb{R}}$  be a system  $f = (f_1, \dots, f_m)$  of real homogeneous polynomial equations. Assume that  $(d) = (d_1, \dots, d_m)$  is a vector of

odd positive integers and assume that  $V_{\mathbb{P}}(f) = \pi^{-1}(f) \subseteq \mathbb{P}_n(\mathbb{R})$  is smooth of co-dimension  $m$ . Then, the following inequality holds:

$$\vartheta_{n-m} \leq \nu_{n-m} [V_{\mathbb{P}}(f)] \leq \mathcal{D}\vartheta_{n-m}.$$

**PROOF.** We make use of Theorem 17 above. Let  $L \subseteq \mathbb{P}_n(\mathbb{R})$  be a linear subspace of codimension  $n - m$ . Then, there is a matrix  $A \in \mathcal{M}_{(n+1) \times m}(\mathbb{R})$  of rank  $m$  such that

$$L = \left\{ (x_0, \dots, x_n) \in \mathbb{P}_n(\mathbb{R}) : A \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = 0 \right\}.$$

For every orthogonal matrix  $U \in \mathcal{O}_{n+1}(\mathbb{R})$ , we have

$$UL = \left\{ (x_0, \dots, x_n) \in \mathbb{P}_n(\mathbb{R}) : AU^{-1} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = 0 \in \mathbb{R}^m \right\}.$$

For every  $f = (f_1, \dots, f_m) \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  there is a non-empty Zariski open subset  $R_f \subseteq \mathcal{O}_{n+1}(\mathbb{R})$  such that the following holds for every  $U \in R_f$ :

- The variety

$$W_U := \left\{ (x_0, \dots, x_n) \in \mathbb{P}_n(\mathbb{C}) : f_i(x) = 0, AU^{-1} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = 0 \in \mathbb{C}^m \right\},$$

is a zero-dimensional (and hence finite) complex algebraic subvariety of the projective space  $\mathbb{P}_n(\mathbb{C})$ .

- The cardinality of  $W_U$  equals the Bézout number of the system. Namely

$$\#W_U = d_1 \cdot \dots \cdot d_m \cdot 1 \cdot \dots \cdot 1 = \mathcal{D}.$$

Now, observe that  $W_U \cap \mathbb{P}_n(\mathbb{R}) = V_{\mathbb{P}}(f) \cap UL$  and, hence, we conclude

$$\#(V_{\mathbb{P}}(f) \cap UL) \leq \#W_U = \mathcal{D}, \quad \forall U \in R_f.$$

On the other hand, as  $\mathcal{D}$  is odd, for all  $U \in R_f$ ,  $W_U \cap \mathbb{P}_n(\mathbb{R}) \neq \emptyset$  and hence, we also conclude

$$1 \leq \#(W_U \cap \mathbb{P}_n(\mathbb{R})) = \#(V_{\mathbb{P}}(f) \cap UL), \quad \forall U \in R_f.$$

As  $R_f$  is Zariski dense in  $\mathcal{O}_{n+1}(\mathbb{R})$  and Zariski open, using Theorem 17, we conclude

$$\begin{aligned} \vartheta_{n-m} &\leq \vartheta_{n-m} \int_{R_f} \#(V_{\mathbb{P}}(f) \cap UL) d\mathcal{O}_{n+1}(\mathbb{R}) = \\ &= \vartheta_{n-m} \int_{\mathcal{O}_{n+1}(\mathbb{R})} \#(V_{\mathbb{P}}(f) \cap UL) d\mathcal{O}_{n+1}(\mathbb{R}) \leq \mathcal{D}\vartheta_{n-m} \end{aligned}$$

and the claim follows.

The following classical statements will be used in forthcoming pages.

**Theorem 19 (Chebysheff-Markov's inequality)** *Let  $(\Omega, \Sigma, \mu)$  be a measure space with  $\nu[\Omega] < \infty$ ,  $f : \Omega \rightarrow \mathbb{R}$  a measurable extended real-valued function, and  $\varepsilon > 0$  then:*

$$\text{Prob}_{\Omega} [|f| \geq \varepsilon] \leq \frac{\mathbb{E}_{\Omega} [|f|]}{\varepsilon}$$

**Theorem 20 (Jensen Inequality)** *Let  $(\Omega, \Sigma, \mu)$  be a measure space with  $\nu[\Omega] < \infty$ ,  $X$  an integrable real-valued random variable then*

$$\mathbb{E}_{\Omega} \left[ \frac{1}{X} \right] \geq \frac{1}{\mathbb{E}_{\Omega} [X]}.$$

**Theorem 21** *Let  $X$  be a positive real-valued random variable such that for every positive real number  $t > 1$ ,*

$$\text{Prob} [X > t] \leq ct^{-\alpha},$$

*where  $c > 1, \alpha > 1$  are some positive constants. Then, the following inequality holds:*

$$\mathbb{E} [X] \leq c^{1/\alpha} \frac{\alpha}{\alpha - 1}.$$

**PROOF.** A proof can be followed in [4].

#### 2.4 Some Basic Facts about the Condition Number.

The condition number in the numerical study of matrices was introduced by A. Turing in [39]. Further developments of the notion were done by [30,40]. Several authors discussed results on the probability distribution of condition number of real and complex matrices. We may cite the seminal work by [38,21,22]. Other discussion on the condition number of complex matrices were done

by [32,19,3]. In the real case the works [1,9,11,14,17] also contain relevant material.

We use the following version of the condition number for rectangular matrices. We consider the projective function:

$$\mu : \mathbb{P}(\mathcal{M}_{m \times n}(\mathbb{R})) \longrightarrow \mathbb{R}^+ \cup \{\infty\},$$

where for every matrix  $A \in \mathbb{P}(\mathcal{M}_{m \times n}(\mathbb{R}))$  we define

$$\mu(A) := \|A\|_F \|A^\dagger\|_2,$$

where  $\|A\|_F = (\text{Tr}(A^t A))^{1/2}$  is the usual Frobenius norm,  $A^\dagger$  is the Moore-Penrose pseudo-inverse and  $\|A^\dagger\|_2$  is the norm of  $A^\dagger$  as linear operator.

Note that  $\mu(A) = +\infty$  if and only if  $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$  is not a surjective matrix. This condition number satisfies Eckart and Young's Condition Number Theorem (cf. [20] and also [8,3]).

The following statement immediately follows from the main outcome of [11].

**Corollary 22** *Let  $m, n \in \mathbb{N}$  two positive integer numbers. Assume  $2 \leq m \leq n$  and let  $\varepsilon \leq \frac{1}{n-m+1}$  be a positive real number. Then the following inequalities holds:*

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \left( \frac{nc\varepsilon}{n-m+1} \right)^{n-m+1} &\leq \\ &\leq \text{Prob}_{\mathbb{P}(\mathcal{M}_{m \times n}(\mathbb{R}))} [\mu \geq \varepsilon^{-1}] \leq \\ &\leq \frac{1}{\sqrt{2\pi}} \left( \frac{n^{3/2}C\varepsilon}{n-m+1} \right)^{n-m+1}. \end{aligned}$$

where  $\text{Prob}_{\mathbb{P}(\mathcal{M}_{m \times n}(\mathbb{R}))}$  is the probability distribution associated to the canonical Riemannian structure in  $\mathbb{P}(\mathcal{M}_{m \times n}(\mathbb{R})) = \mathbb{P}_{nm-1}(\mathbb{R})$  and where  $0.245 \leq c \leq 2$  and  $5.013 \leq C \leq 6.414$ .

Now we recall some basic facts about the condition number of systems of polynomial equations as introduced by [33] (cf. also [8] and [16]).

Along this subsection we fix a positive integer number  $m, 1 \leq m \leq n$ , and a degree list  $(d) = (d_1, \dots, d_m)$ .

**Definition 23** *Let  $(f, \zeta) \in V$  be a point in the incidence variety. We define the normalized condition number of  $f$  at  $\zeta$  as:*

$$\mu_{norm}^m(f, \zeta) := \|f\|_\Delta \|(T_\zeta f)^\dagger \Delta \left( (d)^{1/2} \right)\|,$$

where  $\|\cdot\|_\Delta$  stands for the Bombieri-Weyl-Kostlan norm,  $\|\cdot\|$  is the norm as linear operator,  $(T_x f)^\dagger$  is the Moore-Penrose pseudo-inverse of  $T_x f : T_x \mathbb{P}_n(\mathbb{R}) \rightarrow T_0 \mathbb{R}^m = \mathbb{R}^m$  and,  $\Delta((d)^{1/2}) \in \mathcal{M}_m(\mathbb{R})$  is the real matrix given by:

$$\Delta((d)^{1/2}) := \begin{pmatrix} d_1^{1/2} & & \\ & \ddots & \\ & & d_m^{1/2} \end{pmatrix}.$$

Let the reader observe that in the case  $m = n$ , the condition number  $\mu_{norm}^n(f, \zeta)$  equals the usual condition number  $\mu_{norm}(f, \zeta)$  as defined in [33].

Note that the condition number  $\mu_{norm}^m(f, \zeta) = +\infty$  if and only if  $\zeta \in V_{\mathbb{P}}(f)$  is a critical point of the mapping  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m$ .

The normalized condition number is invariant under the action of the orthogonal group  $\mathcal{O}_{n+1}(\mathbb{R})$  over the incidence variety  $V$ . Namely, for every orthogonal matrix  $U \in \mathcal{O}_{n+1}(\mathbb{R})$  and for every point  $(f, \zeta) \in V$  at the incidence variety we have:

$$\mu_{norm}^m(f, \zeta) = \mu_{norm}^m(f \circ U^{-1}, U\zeta).$$

The next statement essentially follows from the same arguments of [34] (cf. also [4]).

**Theorem 24** *For every pair  $(f, \zeta) \in V$  the following equality holds:*

$$\mu_{norm}^m(f, \zeta) = \frac{1}{d_{\mathbb{P}}((f, \zeta), \Sigma \cap V_\zeta)},$$

where  $d_{\mathbb{P}}(\cdot, \cdot)$  is the projective distance in  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  induced by the Bombieri-Weyl-Kostlan metric.

This condition number  $\mu_{norm}^m$  satisfies the following relevant property that follows from similar arguments that those of Theorems 130 and 137 of [16], Lemma 12 page 269 of [8] and [33].

**Proposition 25** *There is a universal constant  $u_0$  (approximately equal to 0.05992) that satisfies the following property:*

Let  $f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  be a system of multivariate polynomial equations,  $\zeta \in V_{\mathbb{P}}(f)$  and  $x \in \mathbb{P}_n(\mathbb{R})$ . Assume that

$$d_{\mathbb{P}}(x, \zeta) \leq \min \left\{ \frac{2u_0}{d^{3/2} \mu_{norm}^m(f, \zeta)}, \frac{1}{3} \right\}.$$

Then, there is a projective zero  $M_{f(x)} \in V_{\mathbb{P}}(f)$  such that the following holds:

$$d_{\mathbb{P}}(N_f^k(x), M_{f(x)}) \leq \frac{2^{3/2}3}{2^{2k-1}} d_{\mathbb{P}}(x, \zeta).$$

In particular, we conclude that

$$\min \left\{ \frac{2u_0}{d^{3/2} \mu_{norm}^m(f, \zeta)}, \frac{1}{3} \right\} \leq R(f, \zeta),$$

where  $R(f, \zeta)$  is the quantity defined at the introduction.

### 3 Some Basic Integral Identities.

In this section we state some basic integral identities related to the real incidence variety.

Here we follow the notation of Section 2 above. Let  $e_0 = (1 : 0 : \dots : 0) \in \mathbb{P}_n(\mathbb{R})$  be a fixed projective point in the real projective space. According to Section 2 we denote by  $V_{e_0} = \pi_2^{-1}(\{e_0\})$  the fiber of  $\pi_2$  over  $e_0$ .

**Proposition 26** *Let  $f \in V_{e_0}$  be a system such that  $\text{rank}(T_{e_0}f) = m$ . With the previous notations, we have*

$$\frac{\text{NJ}_{(f, e_0)} \pi_1}{\text{NJ}_{(f, e_0)} \pi_2} = \det((T_{e_0}f)(T_{e_0}f)^t).$$

**PROOF.** A proof of this equality can be followed in [8] and for a more general version in [4].

**Proposition 27** *Let  $\Phi : V \rightarrow \mathbb{R}$  be a positive integrable function. Assume that  $\Phi$  is orthogonally invariant by the action of the orthogonal group  $\mathcal{O}_{n+1}(\mathbb{R})$  on  $V$ . Then, the following equality holds:*

$$\begin{aligned} \int_{\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})} \int_{V_{\mathbb{P}}(f)} \Phi(f, \zeta) dV_{\mathbb{P}}(f) d\nu_{\Delta} &= \\ &= \vartheta_n \int_{V_{e_0}} \Phi(f, e_0) \left| \det \left( (T_{e_0}f)(T_{e_0}f)^t \right) \right|^{1/2} dV_{e_0} \end{aligned}$$

**PROOF.** This equality follows from the standard arguments using the Co-

area Formula (16 above) applied to the following diagram

$$\begin{array}{ccc} & V & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) & & \mathbb{P}_n(\mathbb{R}). \end{array}$$

See [34,8,3] for details.

As in [34] we can decompose  $V_{e_0}$  as the orthogonal sum of two vector subspaces:

$$V_{e_0} = V'_{e_0} \perp L_{e_0},$$

where  $L_{e_0}$  is the orthogonal complement of  $V'_{e_0}$  and  $V'_{e_0}$  is the vector subspace of  $V_{e_0}$  given by the following identities:

$$V'_{e_0} := \{f \in V_{e_0} : T_{e_0}f = D_{e_0}f|_{(e_0)^\perp} \equiv 0\}.$$

Note that  $V'_{e_0}$  is the subspace of all system  $f = (f_1, \dots, f_m)$  where the order of  $f_i$  at  $e_0$  is at least 2.

We may also describe the subspace  $L_{e_0}$  in the following terms. A system  $f = (f_1, \dots, f_m) \in L_{e_0}$  if and only if there are linear mappings  $l_1, \dots, l_m : \mathbb{R}^n \rightarrow \mathbb{R}$  such that for every positive integer  $i, 1 \leq i \leq m$  the following equality holds:

$$f_i = x_0^{d_i-1} l_i.$$

We may then identify  $\mathcal{M}_{m \times n}(\mathbb{R})$  with  $L_{e_0}$  by the following mapping: To every matrix  $M \in \mathcal{M}_{m \times n}(\mathbb{R})$  we associate the system  $\varphi(M) := (f_1, \dots, f_m) \in L_{e_0}$  given by the following identity:

$$\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} := \begin{pmatrix} x_0^{d_1-1} & & \\ & \ddots & \\ & & x_0^{d_m-1} \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

The following proposition is a link between the condition number of system of equations and the condition number of matrices.

**Proposition 28** *With the previous notations the following mapping is an isometry*

$$\begin{aligned} \rho : B^1(L_{e_0}) &\longrightarrow B^1(\mathcal{M}_{m \times n}(\mathbb{R})) \\ f &\longmapsto \Delta(d_i^{1/2}) T_{e_0} f, \end{aligned}$$

where  $B^1(L_{e_0})$  is the closed ball of radius 1 in  $L_{e_0}$  with respect to the norm induced by  $\langle \cdot, \cdot \rangle_\Delta$  and  $B^1(\mathcal{M}_{m \times n}(\mathbb{R}))$  is the closed ball of radius 1 in  $\mathcal{M}_{m \times n}(\mathbb{R})$  with respect to the canonical Frobenius norm. Moreover, for every  $f \in L_{e_0}$ ,  $\mu_{norm}^m(f, e_0) = \mu(\rho(f))$

We now state another proposition that we use in the sequel.

**Proposition 29** *With the same notations as above, assume that the integrable function  $\Phi : V \rightarrow \mathbb{R}$  satisfies for every  $f \in V_{e_0}$  that:*

$$\Phi(f, e_0) = \Phi(T_{e_0}f, e_0) = \Phi(g, e_0),$$

where  $g \in L_{e_0}$  is the component of  $f$  in the orthogonal decomposition  $V_{e_0} = V'_{e_0} \perp L_{e_0}$ . Then the following equality holds:

$$\int_{\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})} \int_{V_{\mathbb{P}(f)}} \Phi(f, z) dV_{\mathbb{P}(f)} d\nu_\Delta = 2\vartheta_n \vartheta_{N-(n+1)m-1} I(\Phi, m, n, (d)),$$

where  $N = \dim(\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}))$  and

$$\begin{aligned} I(\Phi, m, n, (d)) &= \\ &= \int_{B^1(L_{e_0})} \Phi(f, e_0) \left| \det \left( (T_{e_0}f)(T_{e_0}f)^t \right) \right|^{1/2} \left( 1 - \|f\|_\Delta^2 \right)^{\frac{N-(n+1)m-2}{2}} dB^1, \end{aligned}$$

where  $B^1(L_{e_0})$  stands for the unit ball with respect to the Bombieri-Weyl-Kostlan metric in  $L_{e_0}$ .

**PROOF.** Let  $\mathbb{S}$  be the unit sphere in  $\tilde{V}_{e_0}$  with respect to the Bombieri-Weyl-Kostlan metric. Now, let  $p : \mathbb{S} \rightarrow B^1(L_{e_0})$  be the canonical projection onto the unit ball of  $L_{e_0}$  with respect to the Bombieri-Kostlan metric.

As in Example 9 and Lemma 2 of Chapter 11 of [8] we have:

- For every point  $g \in B^1(L_{e_0})$  the fiber  $p^{-1}(g)$  can be identified with a  $(\dim(V_{e_0}) - \dim(L_{e_0}) - 1)$ -sphere in  $V_{e_0}$  of radius  $(1 - \|g\|_\Delta^2)^{1/2}$ . Namely,  $p^{-1}(g) \equiv (N - (n+1)m - 1)$ -sphere of radius  $(1 - \|g\|_\Delta^2)^{1/2}$  in  $V_{e_0}$ .
- The normal jacobian of  $p$  at any  $f \in \mathbb{S}$  satisfies:

$$\text{NJ}_f p = \left( 1 - \|p(f)\|_\Delta^2 \right)^{1/2}.$$

Thus, we conclude:

$$\begin{aligned}
& \int_{\mathbb{S}} \Phi(f, e_0) \left| \det \left( (T_{e_0} f)(T_{e_0} f)^t \right) \right|^{1/2} d\mathbb{S} = \\
& = \int_{B^1(L_{e_0})} \int_{p^{-1}(g)} \Phi(g, e_0) \left| \det \left( (T_{e_0} g)(T_{e_0} g)^t \right) \right|^{1/2} (1 - \|g\|_{\Delta}^2)^{-1/2} dp^{-1}(g) dB^1 = \\
& = \int_{B^1(L_{e_0})} \Phi(g, e_0) \left| \det \left( (T_{e_0} g)(T_{e_0} g)^t \right) \right|^{1/2} (1 - \|g\|_{\Delta}^2)^{-1/2} \nu [p^{-1}(g)] dB^1
\end{aligned}$$

Now, as  $p^{-1}(g)$  is a real  $(N - (n + 1)m - 1)$ -sphere of radius  $(1 - \|g\|_{\Delta}^2)^{1/2}$ , then we have:

$$\nu [p^{-1}(g)] = 2(1 - \|g\|_{\Delta}^2)^{\frac{N-(n+1)m-1}{2}} \vartheta_{N-(n+1)m-1}.$$

Thus, we conclude:

$$\int_{V_{e_0}} \Phi(f, e_0) \left| \det \left( (T_{e_0} f)(T_{e_0} f)^t \right) \right|^{1/2} dV_{e_0} = 2\vartheta_{N-(n+1)m-1} I(\Phi, m, n, (d)),$$

where

$$\begin{aligned}
I(\Phi, m, n, (d)) & = \\
& = \int_{B^1(L_{e_0})} \Phi(g, e_0) \left| \det \left( (T_{e_0} g)(T_{e_0} g)^t \right) \right|^{1/2} (1 - \|g\|_{\Delta}^2)^{\frac{N-(n+1)m-2}{2}} dB^1
\end{aligned}$$

**Proposition 30** *With the same notations as above, assume that the integrable function  $\Phi : V \rightarrow \mathbb{R}$  satisfies for every  $f \in V_{e_0}$  that:*

$$\Phi(f, e_0) = \|g\|_{\Delta}^{-1} \Phi(g, e_0),$$

where  $g \in L_{e_0}$  is the component of  $f$  in the orthogonal decomposition  $V_{e_0} = V'_{e_0} \perp L_{e_0}$ . Then the following equality holds:

$$\int_{\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})} \int_{V_{\mathbb{P}(f)}} \Phi(f, z) dV_{\mathbb{P}(f)} d\nu_{\Delta} = 2\vartheta_n \vartheta_{N-(n+1)m-1} J(\Phi, m, n, (d)),$$

where  $N = \dim(\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}))$  and

$$\begin{aligned}
J(\Phi, m, n, (d)) & = \\
& = \int_{B^1(L_{e_0})} \frac{\Phi(g, e_0)}{\|g\|_{\Delta}} \left| \det \left( (T_{e_0} f)(T_{e_0} f)^t \right) \right|^{1/2} (1 - \|f\|_{\Delta}^2)^{\frac{N-(n+1)m-2}{2}} dB^1,
\end{aligned}$$

where  $B^1(L_{e_0})$  stands for the unit ball with respect to the Bombieri-Weyl-Kostlan metric in  $L_{e_0}$ .

**PROOF.** The proof is similar to the previous proposition.

The following corollary immediately follows.

**Corollary 31 (Ko00,Pr06,Bu06)** *With the same notations as above*

$$E_{\Delta} [\nu_{n-m} [V_{\mathbb{P}}(f)]] = \mathcal{D}^{1/2} \vartheta_{n-m}.$$

Now, we state another useful identity.

**Lemma 32** *With the same notations as above, the following identity holds:*

$$\int_{(\mathcal{R}_{(d)})_{e_0}} \frac{1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} \frac{\text{NJ}_{(f,e_0)} \pi_1}{\text{NJ}_{(f,e_0)} \pi_2} dV_{e_0} = \frac{\nu_{\Delta} [\mathcal{R}_{(d)}]}{\vartheta_n},$$

where  $\mathcal{R}_{(d)} := \pi_1(V) = \{f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) : V_{\mathbb{P}}(f) \neq \emptyset\}$  and  $(\mathcal{R}_{(d)})_{e_0} = \pi_1(V_{e_0})$ .

**PROOF.** First of all, the following equality holds

$$I := \int_{\mathcal{R}_{(d)}} \frac{1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} \left( \int_{V_{\mathbb{P}}(f)} dV_{\mathbb{P}}(f) \right) d\nu_{\Delta} = \int_{\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})} \chi_{\mathcal{R}_{(d)}} d\nu_{\Delta} = \nu_{\Delta} [\mathcal{R}_{(d)}].$$

On the other hand, noting that  $V_{\mathbb{P}}(f) = \pi_1^{-1}(f)$  and applying the Co-area Formula we have

$$I = \int_V \frac{\text{NJ}_{(f,x)} \pi_1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} dV = \int_{x \in \mathbb{P}_n(\mathbb{R})} \int_{\pi_2^{-1}(x)} \frac{1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} \frac{\text{NJ}_{(f,x)} \pi_1}{\text{NJ}_{(f,x)} \pi_2} d\pi_2^{-1}(x) d\nu_{\mathbb{P}}.$$

As  $\nu_{n-m} [V_{\mathbb{P}}(f)]$  is invariant under the action of the unitary group  $\mathcal{O}_{n+1}(\mathbb{R})$ , we then have

$$I = \vartheta_n \int_{V_{e_0}} \frac{1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} \frac{\text{NJ}_{(f,x)} \pi_1}{\text{NJ}_{(f,x)} \pi_2} dV_{e_0}.$$

**Remark 33** *Note that if  $(d) = (d_1, \dots, d_m)$  is a vector of odd integers numbers,  $\nu_{\Delta} [\mathcal{R}_{(d)}] = \nu_{\Delta} [\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})]$  (see Proposition 12 above) and the statement of the previous lemma becomes*

$$\int_{V_{e_0}} \frac{1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} \frac{\text{NJ}_{(f,x)} \pi_1}{\text{NJ}_{(f,x)} \pi_2} dV_{e_0} = \frac{\vartheta_N}{\vartheta_n}.$$

#### 4 Average Norm of Real Complete Intersection.

In this section we prove Corollaries 2 and 3, but first we need to prove the more technical Theorem 1. In order to make this proof we introduce some notations and lemmas.

#### 4.1 Technical Lemmas.

First we introduce the notations of the standard atlas by affine charts of the real projective space  $\mathbb{P}_n(\mathbb{R})$ . For every positive integer  $i$ ,  $0 \leq i \leq n$ , we define the open and dense subset  $\mathbb{A}_i(\mathbb{R}) \subset \mathbb{P}_n(\mathbb{R})$  given as the set of projective points whose  $i$ -th homogeneous coordinate is not zero. Namely, we define:

$$\mathbb{A}_i(\mathbb{R}) := \{x = (x_0 : \dots : x_n) \in \mathbb{P}_n(\mathbb{R}) : x_i \neq 0\}.$$

The complement of  $\mathbb{A}_i(\mathbb{R})$  in  $\mathbb{P}_n(\mathbb{R})$  is the hyperplane  $H_i = \mathbb{P}_n(\mathbb{R}) \setminus \mathbb{A}_i(\mathbb{R})$ . For every  $i$ ,  $0 \leq i \leq n$  we have the differentiable canonical mapping

$$\varphi_i : \mathbb{R}^n \longrightarrow \mathbb{A}_i(\mathbb{R}) \subset \mathbb{P}_n(\mathbb{R}).$$

given by the following identity. For every  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$\varphi_i(x) := (x_1 : \dots : x_{i-1} : 1 : x_i : \dots, x_n).$$

We denote by  $e_i \in \mathbb{A}_i(\mathbb{R})$  the projective point given by  $e_i = \varphi_i(0) \in \mathbb{A}_i(\mathbb{R})$  where  $0 \in \mathbb{R}^n$  is the origin. Note that for every  $i$ ,  $0 \leq i \leq n$ ,  $\varphi_i$  is an isometry at  $0 \in \mathbb{R}^n$ . In particular  $\text{NJ}_0 \varphi_i = 1$  for every  $i$ ,  $0 \leq i \leq n$ .

The following lemma translates to real coordinates Lemma 21 of [3].

**Lemma 34** *With the same notation as above, the following equality holds  $\forall x \in \mathbb{R}^n$  and  $\forall i, 0 \leq i \leq n$  the following property hold:*

$$\text{NJ}_x \varphi_i = \frac{1}{(1 + \|x\|^2)^{\frac{n+1}{2}}},$$

where  $\text{NJ}_x \varphi_i$  is the normal jacobian of  $\varphi_i$  at  $x \in \mathbb{R}^n$ .

**PROOF.** Proving this equality for  $i = 0$  suffices. In order to simplify notations we write  $\varphi$  instead of  $\varphi_0$  and  $e = e_0 = (1 : 0 \dots : 0)$ . As we already observed  $d_0 \varphi : T_0 \mathbb{R}^n \longrightarrow T_e \mathbb{P}_n(\mathbb{R})$  is a linear isomorphism and an isometry. Then,  $\text{NJ}_0 \varphi = 1$ .

On the other hand, let  $\mathcal{O}_{n+1}(\mathbb{R}) \subseteq \mathcal{M}_{n+1}(\mathbb{R})$  be the group of orthogonal matrices. As in previous notation let  $\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}_n(\mathbb{R})$  be the canonical projection. For every orthogonal matrix  $U \in \mathcal{O}_{n+1}(\mathbb{R})$  we will denote by  $U$  the affine isometry  $U : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$  given by:

$$U(x) := Ux^t \text{ for every } x \in \mathbb{R}^{n+1},$$

where  $x^t$  means transpose of  $x$ . We denote by  $\tilde{U}$  the unique isometry  $\tilde{U} : \mathbb{P}_n(\mathbb{R}) \longrightarrow \mathbb{P}_n(\mathbb{R})$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xrightarrow{U} & \mathbb{R}^{n+1} \setminus \{0\} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{P}_n(\mathbb{R}) & \xrightarrow{\tilde{U}} & \mathbb{P}_n(\mathbb{R}) \end{array}$$

For every  $x \in \mathbb{R}^n$  there is an orthogonal matrix  $U \in \mathcal{O}_{n+1}(\mathbb{R})$  such that  $\tilde{U}(\varphi(x)) = e = (1 : 0 : \dots : 0)$ . Note that we may choose the matrix  $U$  such that:

$$U \begin{pmatrix} 1 \\ x^t \end{pmatrix} = \begin{pmatrix} (1 + \|x\|^2)^{1/2} \\ 0 \end{pmatrix}.$$

Note that  $\tilde{U}$  is an isometry at any projective point  $z \in \mathbb{P}_n(\mathbb{R})$ . Hence the Normal Jacobian satisfies  $\text{NJ}_z \tilde{U} = 1, \forall z \in \mathbb{P}_n(\mathbb{R})$ .

Let us now define the differentiable mapping  $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$  given as:

$$\phi := \varphi^{-1} \circ \tilde{U} \circ \varphi.$$

Note that  $\phi(x) = 0$  and  $\varphi \circ \phi = U \circ \varphi$ . Then, the Chain Rule supplies that the following equality holds:

$$\text{NJ}_0 \varphi \text{NJ}_x \phi = \text{NJ}_{\varphi(x)} \tilde{U} \text{NJ}_x \varphi.$$

As  $\varphi$  is an isometry at the origin  $0 \in \mathbb{R}^n$  and  $\tilde{U}$  is an isometry at any projective point, we immediately conclude:

$$\text{NJ}_x \phi = \text{NJ}_x \varphi.$$

Moreover, let  $g : \mathbb{R}^n \longrightarrow \mathbb{R}$  the mapping given by:

$$g(z) := \langle U_0, (1, z) \rangle, \forall z \in \mathbb{R}^n,$$

where  $\langle \cdot, \cdot \rangle$  stands for the standard euclidean product in  $\mathbb{R}^{n+1}$ . Now observe that:

$$(g\phi)(z) = (\langle U_1, (1, z) \rangle, \dots, \langle U_n, (1, z) \rangle),$$

for all  $z \in \mathbb{R}^n$ . Since  $\phi(x) = 0$  we immediately conclude that for every  $v \in T_x \mathbb{R}^n$ ,

$$g(x) d_x \phi(v) = (\langle U_1, (1, v) \rangle, \dots, \langle U_n, (1, v) \rangle).$$

Thus we conclude:

$$d_x \phi(v) = \frac{1}{(1 + \|x\|^2)^{1/2}} (\langle U_1, (1, v) \rangle, \dots, \langle U_n, (1, v) \rangle),$$

for all  $v \in T_x \mathbb{R}^n$

Now we prove the following claim:

For every tangent vector  $v \in T_x \mathbb{R}^n$  such that  $\langle x, v \rangle_{\mathbb{R}^n} = 0$  and for any other tangent vector  $w \in T_x \mathbb{R}^n$  the following equality holds:

$$\langle d_x \phi(v), d_x \phi(w) \rangle_{T_0 \mathbb{R}^n} = \frac{1}{1 + \|x\|_2^2} \langle v, w \rangle_{\mathbb{R}^n}.$$

In order to prove this claim note that for all  $v, w \in T_x \mathbb{R}^n$ :

$$\langle d_x \phi(v), d_x \phi(w) \rangle_{T_0 \mathbb{R}^n} = \frac{1}{1 + \|x\|_2^2} \sum_{i=1}^n \langle U_i, (0, v) \rangle \langle U_i, (0, w) \rangle.$$

Since  $\left\langle U \begin{pmatrix} 1 \\ v^t \end{pmatrix}, U \begin{pmatrix} 1 \\ w^t \end{pmatrix} \right\rangle_{\mathbb{R}^{n+1}} = \langle v, w \rangle_{\mathbb{R}^n}$  we immediately conclude:

$$\langle d_x \phi(v), d_x \phi(w) \rangle_{T_0 \mathbb{R}^n} = \frac{1}{1 + \|x\|_2^2} [\langle u, v \rangle_{\mathbb{R}^n} - \langle U_0, (0, v) \rangle \langle U_0, (0, w) \rangle].$$

Now if  $\langle x, v \rangle_{\mathbb{R}^n} = 0$  we have  $\langle U_0, (0, v) \rangle = 0$  and the claim follow.

Let us now consider an orthonormal basis  $\{b_i, \dots, b_n\}$  of  $T_x \mathbb{R}^n$  such that  $b_n = \frac{x}{\|x\|}$ . Then, for every  $i, 1 \leq i \leq n-1$  and for every  $j, 1 \leq j \leq n$  we have:

$$\langle d_x \phi(v), d_x \phi(w) \rangle_{T_0 \mathbb{R}^n} = \frac{1}{1 + \|x\|_2^2} \langle b_i, b_j \rangle_{T_x \mathbb{R}^n} = \begin{cases} \frac{1}{1 + \|x\|_2^2} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

As for  $i = n$  we have:

$$\langle d_x \phi(v), d_x \phi(w) \rangle_{T_0 \mathbb{R}^n} = \frac{1}{1 + \|x\|_2^2} \left[ 1 - \frac{1}{\|x\|_2^2} \langle U_0, (0, x) \rangle^2 \right].$$

Now observe that:

$$\langle U_0, (0, x) \rangle^2 = \frac{\|x\|_2^4}{1 + \|x\|_2^2},$$

then, we conclude:

$$\langle d_x \phi(v), d_x \phi(w) \rangle_{T_0 \mathbb{R}^n} = \frac{1}{1 + \|x\|_2^2} \left[ 1 - \frac{\|x\|_2^2}{1 + \|x\|_2^2} \right] = \frac{1}{(1 + \|x\|_2^2)^2}.$$

Finally,

$$\text{NJ}_x \varphi \text{NJ}_x \phi = |\det W(\phi, x)|^{1/2} = \left( \frac{1}{1 + \|x\|_2^2} \right)^{\frac{n+1}{2}},$$

where  $W(\phi, x)$  is the symmetric matrix given by:

$$W(\phi, x) = (\langle d_x \phi(b_i), d_x \phi(b_j) \rangle)_{1 \leq i, j \leq n}.$$

Then, the statement follows.

**Lemma 35** *For every positive real number  $r, 0 < r < 1$  and for every  $i, 0 \leq i \leq n$  the following equality holds:*

$$\frac{1}{\vartheta_n} \int_{\mathbb{P}_n(\mathbb{R})} \|\varphi_i^{-1}(x)\|^r d\nu_{\mathbb{P}} = \beta\left(\frac{n+r}{2}, \frac{1-r}{2}\right).$$

**PROOF.** As in the proof of the previous Lemma, the case  $i = 0$  suffices. Once again, we simplify notations by writing  $\varphi := \varphi_0$ . Since  $\mathbb{P}_n(\mathbb{R}) \setminus \mathbb{A}_0(\mathbb{R})$  is a zero measure hyperplane, we are allowed to use the Co-area Formula (Theorem 16 above) to conclude:

$$I(r) := \int_{\mathbb{P}_n(\mathbb{R})} \|\varphi^{-1}(x)\|^r d\nu_{\mathbb{P}} = \int_{\mathbb{R}^n} \int_{x \in \varphi^{-1}(z)} \frac{\|x\|^r}{\text{NJ}_x \varphi} d\varphi^{-1}(z) d\mathbb{R}^n.$$

From Lemma 34 we conclude:

$$I(r) = \int_{\mathbb{R}^n} \frac{\|x\|^r}{(1 + \|x\|^2)^{\frac{n+1}{2}}} d\mathbb{R}^n.$$

Now we use spherical coordinates to compute this integral. We have the differentiable mapping:

$$\begin{aligned} \rho : (0, +\infty) \times \mathbb{S}_{n-1} &\longrightarrow \mathbb{R}^n \setminus \{0\} \\ (t, v) &\longmapsto tv \end{aligned},$$

whose inverse is given by:

$$\begin{aligned} \rho^{-1} : \mathbb{R}^n \setminus \{0\} &\longrightarrow (0, +\infty) \times \mathbb{S}_{n-1} \\ x &\longmapsto \left(\|x\|, \frac{x}{\|x\|}\right). \end{aligned}$$

The Normal Jacobians of  $\rho$  and  $\rho^{-1}$  satisfy:

$$\begin{aligned} \text{NJ}_x \rho^{-1} &= \frac{1}{\|x\|^{n-1}} \quad \forall x \in \mathbb{R}^n \setminus \{0\} \\ \text{NJ}_x \rho &= t^{n-1} \quad \forall (t, v) \in (0, +\infty) \times \mathbb{S}_{n-1}. \end{aligned}$$

Thus, using the Co-area Formula again we conclude:

$$I(r) = \int_{(0, +\infty) \times \mathbb{S}_{n-1}} \int_{\rho(t, v)} \frac{\|x\|^r}{(1 + \|x\|^2)^{\frac{n+1}{2}} \text{NJ}_x \rho^{-1}} d\rho(t, v) d\mathbb{R} \times \mathbb{S}.$$

This yields:

$$I(r) = \nu[\mathbb{S}_{n-1}] \int_0^{+\infty} \frac{t^{n+r-1}}{(1+t^2)^{\frac{n+1}{2}}} dt.$$

Now using:

$$\beta(x, y) = 2 \int_0^\infty \frac{t^{2x-1}}{(1+t^2)^{x+y}} dt,$$

we obtain:

$$I(r) = \frac{\nu[\mathbb{S}_{n-1}]}{2} \beta\left(\frac{n+r}{2}, \frac{1-r}{2}\right).$$

The claim follows noting that:

$$\vartheta_n = \frac{\nu[\mathbb{S}_{n-1}]}{2}.$$

#### 4.2 Proof of Theorem 1.

We have that

$$\vartheta_N \mathbf{E}_\Delta [N_{av}^r(f)] = \int_{\mathcal{R}_{(d)}} \frac{1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} \left( \int_{V_{\mathbb{P}}(f)} \|\varphi_0^{-1}(x)\|^r dV_{\mathbb{P}}(f) \right) d\nu_\Delta$$

As  $V_{\mathbb{P}}(f) = \pi_1^{-1}(f)$ , applying the Co-area Formula we conclude

$$\begin{aligned} \vartheta_N \mathbf{E}_\Delta [N_{av}^r(f)] &= \int_V \frac{1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} \|\varphi_0^{-1}(x)\|^r \mathbf{NJ}_{(f,x)} \pi_1 dV = \\ &= \int_{\mathbb{P}_n(\mathbb{R})} \int_{(\mathcal{R}_{(d)})_x} \frac{1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} \|\varphi_0^{-1}(x)\|^r \frac{\mathbf{NJ}_{(f,x)} \pi_1}{\mathbf{NJ}_{(f,x)} \pi_2} dV_x d\nu_n, \end{aligned}$$

where  $(\mathcal{R}_{(d)})_x = \mathcal{R}_{(d)} \cap \pi_2^{-1}(x) = V_x$ . Now, we observe that the following integral does not depend upon  $x$

$$\int_{V_x} \frac{1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} \frac{\mathbf{NJ}_{(f,x)} \pi_1}{\mathbf{NJ}_{(f,x)} \pi_2} dV_x.$$

Thus, we conclude the following equality

$$\vartheta_N \mathbf{E}_\Delta [N_{av}^r(f)] = \left( \int_{\mathbb{P}_n(\mathbb{R})} \|\varphi_0^{-1}(x)\|^r d\nu_n \right) \left( \int_{V_{e_0}} \frac{1}{\nu_{n-m} [V_{\mathbb{P}}(f)]} \frac{\mathbf{NJ}_{(f,e_0)} \pi_1}{\mathbf{NJ}_{(f,e_0)} \pi_2} dV_{e_0} \right).$$

Now, applying Lemma 32 and Lemma 35, we conclude

$$\vartheta_N \mathbf{E}_\Delta [N_{av}^r(f)] = \beta\left(\frac{n+r}{2}, \frac{1-r}{2}\right) \nu_\Delta [\mathcal{R}_{(d)}],$$

as wanted.  $\square$

**Corollary 36** *With the same notations as above*

$$\text{Prob}_\Delta \left[ N_{av}^r(f) \geq \varepsilon^{-1} \right] \leq \frac{\nu_\Delta \left[ \mathcal{R}_{(d)} \right]}{\vartheta_N} \beta \left( \frac{n+r}{2}, \frac{1-r}{2} \right) \varepsilon.$$

*In particular, if all the degrees  $d_1, \dots, d_m$  in the list  $(d) = (d_1, \dots, d_m)$  are odd integer numbers, the following equality holds:*

$$\text{Prob}_\Delta \left[ N_{av}^r(f) \geq \varepsilon^{-1} \right] \leq \beta \left( \frac{n+r}{2}, \frac{1-r}{2} \right) \varepsilon.$$

**PROOF.** Apply Markov's Inequality (19 above) to the previous theorem.

### 4.3 Proof of Corollary 3

Note that for every positive real number  $\varepsilon > 0$ , the following holds almost every where in  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$

$$\begin{aligned} \dim(V_{\mathbb{A}}(f) \cap B(0, \varepsilon^{-2})) < n - m \quad \text{if and only if} \\ V_{\mathbb{A}}(f) \cap B(0, \varepsilon^{-2}) \text{ has zero measure in } V_{\mathbb{A}}(f). \end{aligned}$$

If  $V_{\mathbb{A}}(f) \cap B(0, \varepsilon^{-2})$  has zero measure in  $V_{\mathbb{A}}(f)$ , then, the following holds almost everywhere in  $V_{\mathbb{A}}(f)$ :

$$\forall x \in V_{\mathbb{A}}(f), \|x\| > \varepsilon^{-2}.$$

Equivalently, the following also holds almost every where in  $V_{\mathbb{A}}(f)$ :

$$\forall x \in V_{\mathbb{A}}(f), \|x\|^{1/2} > \varepsilon^{-1},$$

and hence  $N_{av}^{1/2}(f) \geq \varepsilon^{-1}$ . Thus, we have

$$\begin{aligned} \text{Prob}_\Delta \left[ f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) : \dim(V_{\mathbb{A}}(f) \cap B(0, \varepsilon^{-1})) < n - m \right] &\leq \\ &\leq \text{Prob}_\Delta \left[ f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) : N_{av}^{1/2} \geq \varepsilon^{-1} \right] \end{aligned}$$

Applying Markov's Inequality (cf. Theorem 19), we then conclude

$$\text{Prob}_\Delta \left[ f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}}) : \dim(V_{\mathbb{A}}(f) \cap B(0, \varepsilon^{-1})) < n - m \right] \leq \varepsilon E_\Delta \left[ N_{av}^{1/2}(f) \right],$$

and the inequality follows from Theorem 1 above.  $\square$

#### 4.4 Proof of Corollary 2.

In order to prove Corollary 2 first, we need to prove the following more technical theorem.

**Theorem 37** *For every positive real number  $r \in \mathbb{R}, 0 < r < 1$  the following inequality holds:*

$$\mathbb{E}_\Delta \left[ \int_{V_{\mathbb{A}}(f)} \|z\|^r dV_{\mathbb{A}}(f) \right] = \mathcal{D}^{1/2} \vartheta_{n-m} \beta \left( \frac{n+r}{2}, \frac{1-r}{2} \right).$$

**PROOF.** For every  $f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  let  $V_{\mathbb{A}}(f) \subseteq \mathbb{C}^n$  be the affine algebraic set of all complex solution of the system of polynomial equations defined by  $f$ . Namely,

$$V_{\mathbb{A}'}(f) := \{x \in \mathbb{C}^n : f_i(x) = 0, 1 \leq i \leq m\},$$

where  $f = (f_1 \dots, f_m)$ . Let  $U_\infty \subseteq \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  be the class of all list  $f \in \mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$  such that  $V_{\mathbb{A}'}(f)$  is a complex algebraic variety of dimension lower than  $n - m$ . Standard result of Elimination Theory implies that  $U_\infty$  is included in a proper algebraic subvariety of  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$ . In particular, the following equality holds in  $\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})$ :

$$\nu_N [U_\infty] = 0.$$

Now, for every  $\alpha \in \{0, 1\}$  we consider the following integral:

$$I^\alpha(r) = \int_{\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})} \int_{V_{\mathbb{P}}(f)} \|\varphi_0^{-1}(x)\|^{r\alpha} dV_{\mathbb{P}}(f) d\nu_\Delta.$$

Moreover, for every  $f \in \mathcal{H}_{(d)}^{\mathbb{R}} \setminus U_\infty$  the following equality holds:

$$\mathbb{E}_{\mathcal{H}_{(d)}^{\mathbb{R}}} \left[ \int_{V_{\mathbb{A}}(f)} \|z\|^r dV_{\mathbb{A}}(f) \right] = \frac{I^1(r)}{\vartheta_N}. \quad (2)$$

We then focus our proof on the calculation of  $I^\alpha(r)$ . Using the Co-area Formula (Theorem 16 above) we have:

$$I^\alpha(r) = \int_{\mathbb{P}_n(\mathbb{R})} \int_{V_x} \|\varphi_0^{-1}(x)\|^{r\alpha} \frac{\text{NJ}_{(f,x)} \pi_1}{\text{NJ}_{(f,x)} \pi_2} dV_x d\nu_{\mathbb{P}}.$$

It is clear that

$$\frac{\text{NJ}_{(f,x)} \pi_1}{\text{NJ}_{(f,x)} \pi_2} dV_x,$$

is independent of  $x$ , so we conclude

$$I^\alpha(r) = \int_{\mathbb{P}_n(\mathbb{R})} \|\varphi_0^{-1}(x)\|^{r\alpha} d\nu_{\mathbb{P}} \int_{V_{e_0}} \frac{\text{NJ}_{(f,e_0)} \pi_1}{\text{NJ}_{(f,e_0)} \pi_2} dV_{e_0}.$$

Now we consider the case  $\alpha = 0$ . Then, from Theorem 31:

$$I^0(r) = \int_{\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})} \int_{V_{\mathbb{P}}(f)} dV_{\mathbb{P}}(f) d\nu_{\Delta} = \mathcal{D}^{1/2} \vartheta_{n-m} \vartheta_N,$$

on the other hand:

$$I^0(r) = \vartheta_n \int_{V_{e_0}} \frac{\text{NJ}_{(f,e_0)} \pi_1}{\text{NJ}_{(f,e_0)} \pi_2} dV_{e_0}.$$

We thus conclude:

$$\int_{V_{e_0}} \frac{\text{NJ}_{(f,e_0)} \pi_1}{\text{NJ}_{(f,e_0)} \pi_2} dV_{e_0} = \mathcal{D}^{1/2} \frac{\vartheta_{n-m} \vartheta_N}{\vartheta_n}. \quad (3)$$

As for the case  $\alpha = 1$ , using Equation (3) and Theorem 35 we have:

$$I^1(r) = \mathcal{D}^{1/2} \vartheta_{n-m} \vartheta_N \beta \left( \frac{n+r}{2}, \frac{1-r}{2} \right).$$

Finally, using equality (2) above, we conclude:

$$\mathbb{E}_{\Delta} \left[ \int_{V_{\mathbb{A}}(f)} \|z\|^r dV_{\mathbb{A}}(f) \right] = \mathcal{D}^{1/2} \vartheta_{n-m} \beta \left( \frac{n+r}{2}, \frac{1-r}{2} \right),$$

and the statement is proved.

Now we can prove Corollary 2. In fact, we prove the following more technical corollary:

**Corollary 38** *With the same notations as in the Theorem 1 above, if  $n = m$ ,  $n \geq 3$  and for every positive real number  $r, 0 < r < 1$  and  $\varepsilon < 1$ , the following inequality holds:*

$$\text{Prob}_{\Delta} \left[ \max_{x \in V_{\mathbb{A}}(f)} \|x\| \geq \varepsilon^{-1} \right] \leq \mathcal{D}^{1/2} \varepsilon^r \Gamma \left( \frac{1-r}{2} \right),$$

where  $\Gamma$  is the Euler's  $\Gamma$ -function and  $\mathcal{D}$  denotes the Bézout number, namely  $\mathcal{D} = \prod_{i=1}^m d_i$ .

**PROOF.**

Trivially we have the following equality for  $0 < r < 1$ :

$$\text{Prob}_{\Delta} \left[ \max_{x \in V_{\mathbb{A}}(f)} \|x\| \geq \varepsilon^{-1} \right] = \text{Prob}_{\Delta} \left[ \max_{x \in V_{\mathbb{A}}(f)} \|x\|^r \geq \varepsilon^{-r} \right].$$

Also the following inequality is trivial:

$$\text{Prob}_\Delta \left[ \max_{x \in V_{\mathbb{A}}(f)} \|x\|^r \geq \varepsilon^{-r} \right] \leq \text{Prob}_\Delta \left[ \sum_{x \in V_{\mathbb{A}}(f)} \|x\|^r \geq \varepsilon^{-r} \right].$$

Finally, using Markov's Inequality (19 above) and Lemma 37 (with  $n = m$ ) we have:

$$\text{Prob}_\Delta \left[ \sum_{x \in V_{\mathbb{A}}(f)} \|x\|^r \geq \varepsilon^{-r} \right] \leq \mathcal{D}^{1/2} \varepsilon^r \beta \left( \frac{n+r}{2}, \frac{1-r}{2} \right).$$

Now it is very easy to prove that for every  $n \geq 3$  and for every  $r, 0 < r < 1$ :

$$\beta \left( \frac{n+r}{2}, \frac{1-r}{2} \right) \leq \Gamma \left( \frac{1-r}{2} \right),$$

and this finishes the proof.

The Corollary 2 follows simply using  $r = 1/2$ .

## 5 Estimates on the probability distribution of the Condition Number.

In this Section we prove Theorem 4 and then Corollaries 6 and 5.

### 5.1 Proof of Theorem 4.

Let  $\chi_\varepsilon : V \rightarrow \{0, 1\}$  be the characteristic function of the set:

$$A_\varepsilon = \{(f, \zeta) \in V : \mu_{norm}^m(f, \zeta) \geq \varepsilon^{-1}\}.$$

We will compute upper and lower bounds estimates for the following quantity:

$$I = \int_{\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})} \text{Vol}_\varepsilon^m(f) \vartheta_\Delta = \int_{\mathbb{P}(\mathcal{H}_{(d)}^{\mathbb{R}})} \left( \int_{\pi_1^{-1}(f)} \chi_\varepsilon(f, \zeta) d\pi^{-1}(f) \right) d\nu_\Delta.$$

As the condition number  $\mu_{norm}^m$  is invariant under the action of  $\mathcal{O}_{n+1}(\mathbb{R})$  on  $V$  (cf. Proposition 9), we apply Proposition 30 to conclude:

$$I = 2\vartheta_n \vartheta_{N-(n+1)m-1} J(\varepsilon, n, m, (d)),$$

where

$$\begin{aligned} J(\varepsilon, n, m, (d)) &= \\ &= \int_{B^1(L_{e_0})} \frac{\chi_\varepsilon(f, e_0)}{\|f\|_\Delta} \left| \det \left( (T_{e_0}f)(T_{e_0}f)^t \right) \right|^{1/2} \left( 1 - \|f\|_\Delta^2 \right)^{\frac{N-(n+1)m-2}{2}} dB^1 \end{aligned}$$

and  $B^1(L_{e_0})$  is the unit ball in  $L_{e_0}$  with respect to the norm  $\|\cdot\|_\Delta$ .

Let  $\rho : B^1(L_{e_0}) \longrightarrow B^1(\mathcal{M}_{n \times m}(\mathbb{R}))$  be the isometry given by the following identity:

$$\rho(f) := \Delta((d)^{1/2})T_{e_0}f,$$

We then conclude:

$$\begin{aligned} J(\varepsilon, n, m, (d)) &= \\ &= \mathcal{D}^{1/2} \int_{B^1(\mathcal{M}_{n \times m}(\mathbb{R}))} \frac{\chi'_\varepsilon(M)}{\|M\|_F} \left| \det(MM^t) \right|^{1/2} \left( 1 - \|M\|_F^2 \right)^{\frac{N-(n+1)m-2}{2}} dB^1, \end{aligned}$$

where  $\chi'_\varepsilon : \mathcal{M}_{n \times m}(\mathbb{R}) \longrightarrow \{0, 1\}$  is the characteristic function of the subset:

$$B_\varepsilon = \{M \in \mathcal{M}_{n \times m}(\mathbb{R}) : \mu(M) \geq \varepsilon^{-1}\}$$

Note that  $\chi'_\varepsilon$  is homogeneous of degree zero and hence a projective function. We now integrate by spherical coordinates to conclude:

$$\begin{aligned} J(\varepsilon, n, m, (d)) &= \\ &= \mathcal{D}^{1/2} \int_{\mathbb{S}} \int_0^1 \chi'_\varepsilon(rM) \left| \det \left( (rM)(rM)^t \right) \right|^{1/2} r^{nm-2} \left( 1 - \|rM\|_F^2 \right)^{\frac{N-(n+1)m-2}{2}} dr d\mathbb{S} \end{aligned}$$

Then we have:

$$J(\varepsilon, n, m, (d)) = \mathcal{D}^{1/2} \int_0^1 r^{(n+1)m-2} (1 - r^2)^{\frac{N-(n+1)m-2}{2}} dr H(\varepsilon, n, m),$$

where

$$\begin{aligned} H(\varepsilon, n, m) &= \\ &= \int_{\mathbb{S}} \chi'_\varepsilon(M) \left| \det(MM^t) \right|^{1/2} d\mathbb{S} = 2 \int_{\mathbb{P}(\mathcal{M}_{n \times m}(\mathbb{R}))} \chi'_\varepsilon(M) \left| \det(MM^t) \right|^{1/2} d\nu_{\mathbb{P}} \end{aligned}$$

Now we consider the projective space  $\mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R}))$  and the following incidence variety

$$W = \{(M, x) \in \mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R})) \times \mathbb{P}_n(\mathbb{R}) : Mx = 0\}.$$

Note that  $\mathcal{M}_{m \times (n+1)}(\mathbb{R}) = \mathcal{H}_{(1)}^{\mathbb{R}}$ , where  $(1) = (1, \dots, 1) \in \mathbb{N}^m$ . Then  $W$  is a

Riemannian manifold with the usual projection discussed in Subsection 2.2.

$$\begin{array}{ccc} & W & \\ \pi_1^{(1)} \swarrow & & \searrow \pi_2^{(1)} \\ \mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R})) & & \mathbb{P}_n(\mathbb{R}). \end{array}$$

Note that  $\pi_2^{(1)-1}(e_0)$  can be identified with the projective space  $\mathbb{P}(\mathcal{M}_{m \times n}(\mathbb{R}))$ . Namely,  $\pi_2^{(1)-1}(e_0) = W_{e_0} = \mathbb{P}(\mathcal{M}_{m \times n}(\mathbb{R}))$ . Moreover, for every matrix  $M \in \pi_2^{(1)-1}(e_0)$ ,  $T_{e_0}M = M$ . Then, from Proposition 26 the following equality also hold:

$$H(\varepsilon, n, m) = \frac{2}{\vartheta_n} \int_{x \in \mathbb{P}_n(\mathbb{R})} \left( \int_{M \in \pi_2^{(1)-1}(x)} \chi_\varepsilon''(M, x) \frac{\text{NJ}_{(M,x)} \pi_1^{(1)}}{\text{NJ}_{(M,x)} \pi_2^{(1)}} d\pi_2^{(1)-1}(x) \right) d\nu_{\mathbb{P}},$$

where  $\chi_\varepsilon'' : W \longrightarrow \{0, 1\}$  is the characteristic function of the set

$$D_\varepsilon = \{(M, x) \in W : \mu_{norm}^m(M, x) > \varepsilon^{-1}\}$$

Thus, we conclude

$$H(\varepsilon, n, m) = \frac{2}{\vartheta_n} \int_W \chi_\varepsilon''(M, x) \text{NJ}_{(M,x)} \pi_1^{(1)} dW,$$

and

$$H(\varepsilon, n, m) = \frac{2}{\vartheta_n} \int_{\mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R}))} \chi_\varepsilon'''(M) d\nu_{\mathbb{P}},$$

where  $\chi_\varepsilon''' : \mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R})) \longrightarrow \{0, 1\}$  is the characteristic function of  $\pi_1^{(1)}(D_\varepsilon)$ .

Now observe that there is a Zariski closed proper subset  $S \subseteq \mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R}))$  such that for all  $M \in \mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R})) \setminus S$  the fiber  $\pi_1^{(1)-1}(M)$  is a single projective point. Namely, for all  $M \in \mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R})) \setminus S$ , the kernel of  $M$  is a vector subspace of  $\mathbb{R}^{n+1}$  of dimension 1 and  $M$  defines an onto linear mapping. In particular, for all  $M \in \mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R})) \setminus S$ ,  $\mu_{norm}^m(M, x) = \mu(M)$  and we conclude that

$$H(\varepsilon, n, m) = \frac{2}{\vartheta_n} \int_{\mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R}))} \chi_\varepsilon^{iv}(M) d\nu_{\mathbb{P}},$$

where  $\chi_\varepsilon^{iv} : \mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R})) \longrightarrow \{0, 1\}$  is the characteristic function of the following set

$$F_\varepsilon = \{M \in \mathbb{P}(\mathcal{M}_{m \times (n+1)}(\mathbb{R})) : \mu(M) \geq \varepsilon^{-1}\}.$$

In particular, from the results of Chen-Dongarra (cf. Corollary 22 above) we

conclude

$$\begin{aligned} \frac{2}{\vartheta_n} \frac{\vartheta_{m(n+1)-1}}{\sqrt{2\pi}} \left( \frac{(n+1)c\varepsilon}{n-m+2} \right)^{n-m+2} &\leq \\ &\leq H(\varepsilon, n, m) \leq \\ &\leq \frac{2}{\vartheta_n} \frac{\vartheta_{m(n+1)-1}}{\sqrt{2\pi}} \left( \frac{(n+1)^{3/2}C\varepsilon}{n-m+2} \right)^{n-m+2} \end{aligned}$$

where  $0.245 \leq c \leq 2$  and  $5.013 \leq C \leq 6.414$ .

On the other hand, as

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2 \int_0^1 t^{2x-1}(1-t^2)^{y-1} dt,$$

we have that

$$\int_0^1 r^{(n+1)m-2}(1-r^2)^{\frac{N-(n+1)m-2}{2}} dr = \beta \left( \frac{(n+1)m-1}{2}, \frac{N-(n+1)m+1}{2} \right),$$

Using the Gautchi's Inequality (see [23] for example) is very simple to obtain that

$$\begin{aligned} \left( \frac{2}{(n+1)m-1} \right)^{1/2} \beta \left( \frac{(n+1)m}{2}, \frac{N-(n+1)m+1}{2} \right) &\leq \\ &\leq \beta \left( \frac{(n+1)m-1}{2}, \frac{N-(n+1)m+1}{2} \right) \leq \\ &\leq \left( \frac{N}{2} \right)^{1/2} \beta \left( \frac{(n+1)m}{2}, \frac{N-(n+1)m+1}{2} \right), \end{aligned}$$

and

$$\begin{aligned} \left( \frac{(n+1)m-1}{2} \right)^{-1/2} \vartheta_N &\leq \\ &\leq \vartheta_{N-(n+1)m} \vartheta_{m(n+1)-1} \beta \left( \frac{(n+1)m-1}{2}, \frac{N-(n+1)m+1}{2} \right) \leq \\ &\leq \left( \frac{N}{2} \right)^{1/2} \vartheta_N. \quad (4) \end{aligned}$$

Thus combining all these equalities and inequalities we conclude the following estimates.

$$I = 2\vartheta_n \vartheta_{N-(n+1)m} J(\varepsilon, n, m, (d)) = 2\vartheta_n \vartheta_{N-(n+1)m} \mathcal{D}^{1/2} \beta H(\varepsilon, n, m),$$

where

$$\beta = \beta \left( \frac{(n+1)m-1}{2}, \frac{N-(n+1)m+1}{2} \right).$$

Now we use lower and upper bounds of  $H(\varepsilon, n, m)$

$$\begin{aligned} 2\vartheta_n \vartheta_{N-(n+1)m} \mathcal{D}^{1/2} \beta \frac{2}{\vartheta_n} \frac{\vartheta_{m(n+1)-1}}{\sqrt{2\pi}} \left( \frac{(n+1)c\varepsilon}{n-m+2} \right)^{n-m+2} &\leq \\ &\leq I \leq \\ &\leq 2\vartheta_n \vartheta_{N-(n+1)m} \mathcal{D}^{1/2} \beta \frac{2}{\vartheta_n} \frac{\vartheta_{m(n+1)-1}}{\sqrt{2\pi}} \left( \frac{(n+1)^{3/2} C \varepsilon}{n-m+2} \right)^{n-m+2}. \end{aligned}$$

Using Inequalities (4),

$$\begin{aligned} \frac{4\mathcal{D}^{1/2} \vartheta_N}{\sqrt{2\pi}} \left( \frac{2}{(n+1)m-1} \right)^{1/2} \left( \frac{(n+1)c\varepsilon}{n-m+2} \right)^{n-m+2} &\leq \\ &\leq I \leq \\ &\leq \frac{4\mathcal{D}^{1/2} \vartheta_N}{\sqrt{2\pi}} \left( \frac{N}{2} \right)^{1/2} \left( \frac{(n+1)^{3/2} C \varepsilon}{n-m+2} \right)^{n-m+2}, \end{aligned}$$

where  $0.245 \leq c \leq 2$  and  $5.013 \leq C \leq 6.414$ .

Then recall that

$$P = \mathbb{E}_\Delta [\text{Vol}_\varepsilon^m(f)] = \frac{I}{\vartheta_N}.$$

Thus we conclude

$$\begin{aligned} \frac{4}{\sqrt{\pi}} \left( \frac{\mathcal{D}}{(n+1)m-1} \right)^{1/2} \left( \frac{(n+1)c\varepsilon}{n-m+2} \right)^{n-m+2} &\leq \\ &\leq P \leq \\ &\leq \frac{2}{\sqrt{\pi}} (N\mathcal{D})^{1/2} \left( \frac{(n+1)^{3/2} C \varepsilon}{n-m+2} \right)^{n-m+2}. \end{aligned}$$

## 5.2 Proof of Corollary 6.

Corollary 6 follows immediately. Simply use the bound of  $\nu_{n-m} [V_{\mathbb{P}}(f)]$  provided by Corollary 18 in Theorem 4.

**Remark 39** *Note that the lower bound is also valid in the case where the degree list  $(d) = (d_1, \dots, d_n)$  has an even degree.*

### 5.3 Proof of Corollary 5.

Let  $n = m$ . For the first part, Theorem 4 now state

$$\begin{aligned} \mathbb{E}_\Delta [Vol_\varepsilon^n(f)] &= \frac{1}{\vartheta_N} \int_{\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})} \sum_{\zeta \in V_{\mathbb{P}}(f)} \chi_\varepsilon(f, \zeta) dV_{\mathbb{P}}(f) d\nu_\Delta \leq \\ &\leq \left( \frac{N\mathcal{D}}{\pi} \right)^{1/2} \frac{(n+1)^3 (C\varepsilon)^2}{2}, \end{aligned}$$

where  $\chi_\varepsilon : V \longrightarrow \{0, 1\}$  is the characteristic function of the set

$$A_\varepsilon = \{f \in \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R}) : \mu_{norm}(f, \zeta) \geq \varepsilon^{-1}\}.$$

On the other hand

$$\text{Prob}_\Delta [\mu_{worst}(f) \geq \varepsilon^{-1}] = \frac{1}{\vartheta_N} \int_{\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})} \chi'_\varepsilon d\nu_\Delta,$$

where  $\chi'_\varepsilon : \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R}) \longrightarrow \{0, 1\}$  is the characteristic function of the set

$$B_\varepsilon = \{f \in \mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R}) : \mu_{worst}(f) \geq \varepsilon^{-1}\}.$$

The Corollary simply follows noting that

$$\frac{1}{\vartheta_N} \int_{\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})} \chi'_\varepsilon d\nu_\Delta \leq \frac{1}{\vartheta_N} \int_{\mathbb{P}(\mathcal{H}_{(d)}^\mathbb{R})} \sum_{\zeta \in V_{\mathbb{P}}(f)} \chi_\varepsilon(f, \zeta) dV_{\mathbb{P}}(f) d\nu_\Delta.$$

The second part simply follow applying Theorem 21.

### 5.4 Proof of Corollary 7

Let  $\varepsilon < \min\{1/3, \frac{1}{n-m+2}\}$  then from Theorem 25 we have that

$$\min \left\{ \frac{2u_0}{d^{3/2} \mu_{norm}^m(f, \zeta)}, \frac{1}{3} \right\} \leq R(f, \zeta).$$

Now as

$$\nu_{n-m} [\zeta \in V_{\mathbb{P}}(f) : R(f, \zeta) < \varepsilon] \leq \nu_{n-m} \left[ \zeta \in V_{\mathbb{P}}(f) : \frac{2u_0}{d^{3/2} \varepsilon} \leq \mu_{norm}^m(f, \zeta) \right].$$

Hence, using the bounds in Corollary 4

$$\mathbb{E}_\Delta [\nu_{n-m} [\zeta \in V_{\mathbb{P}}(f) : R(f, \zeta) < \varepsilon]] \leq 2 \left( \frac{N\mathcal{D}}{\pi} \right)^{1/2} \left( \frac{(n+1)^{3/2}C}{n-m+2} \right)^{n-m+2} \left( \frac{2u_0}{d^{3/2}} \varepsilon \right)^{n-m+2},$$

### 5.5 Proof of Corollary 8

From Theorem 25 we can get the following inequality

$$\min \left\{ \frac{2u_0}{d^{3/2} \mu_{\text{worst}}^m(f)}, \frac{1}{3} \right\} \leq R_{\text{worst}}(f).$$

On the other hand, we have that:

$$\min\{a, b\} \geq \frac{ab}{a+b}.$$

So we conclude

$$R_{\text{worst}}(f) \geq \frac{2u_0}{6u_0 + d^{3/2} \mu_{\text{worst}}(f)}.$$

Then, using Corollary 5 and 20 we conclude:

$$\mathbb{E}_\Delta [R_{\text{worst}}] \geq \frac{2u_0 \pi^{1/4}}{6u_0 \pi^{1/4} + 2\sqrt{2} d^{3/2} (N\mathcal{D})^{1/4} (n+1)^{3/2} C}.$$

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