HARMONIC PROPERTIES OF THE LOGARITHMIC POTENTIAL AND THE COMPUTABILITY OF ELLIPTIC FEKETE POINTS

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Abstract. We investigate the properties of the function sending each N–tuple of points to minus the logarithm of the product of their mutual distances. We prove that, as a function defined on the product of N spheres, this function is subharmonic and indeed its (Riemannian) Laplacian is constant. We also prove a mean value equality and an upper bound on the derivative of the function. We use these results to get sharp upper bounds for the precision needed to describe an approximation to elliptic Fekete points (in the sense demanded by Smale's 7th problem). We also conclude that Smale's 7th problem has solutions given by rational spherical points of bounded (small) bit length, proving that there exists an exponential running time algorithm which solves it on the Turing machine model.

1. Introduction and main results

Let $\mathbb{S}$ be the Riemann sphere in $\mathbb{R}^3$, that is the sphere of radius $1/2$ centered at $(0,0,1/2)^T$,

$$\mathbb{S} = \{(a,b,c)^T \in \mathbb{R}^3 : a^2+b^2+(c-1/2)^2 = 1/4\}.$$

Equivalently, $\mathbb{S}$ is the set of points $(a,b,c)^T \in \mathbb{R}^3$ such that

$$a^2+b^2+c^2 = c. \tag{1.1}$$

Let

$$\Sigma = \{(x_1,\ldots,x_N) \in \mathbb{S}^N : x_i = x_j \text{ for some } i \neq j\} \subseteq \mathbb{S}^N,$$

and consider the function

$$\begin{array}{cccc} \mathcal{E}: & \mathbb{S}^N \setminus \Sigma & \to & \mathbb{R} \\ & X=(x_1,\ldots,x_N) & \mapsto & -\sum_{i<j} \log \|x_i - x_j\| \quad = -\log \prod_{i<j} \|x_i-x_j\|. \end{array}$$

Let

$$m_N = \min_{X \in \mathbb{S}^N \setminus \Sigma} \mathcal{E}(X)$$

be the global minimum value of $\mathcal{E}$. A N–tuple satisfying $\mathcal{E}(X) = m_N$ is called a set of elliptic Fekete points. In the list of Smale's problems for the XXI Century [20], problem number 7 reads

Date: January 12, 2012.

2010 *Mathematics Subject Classification.* Primary:52A40, 31A05, 68Q25. Secondary: 11B05, 65Y04.

Key words and phrases. Elliptic Fekete points, subharmonic function, logarithmic energy, Smale's 7th problem, harmonic manifold, simply exponential algorithm.

Problem 1.1. *Can one find $x_1, \dots, x_N \in \mathbb{S}$ such that* [1]

$$\mathcal{E}(x_1, \dots, x_N) - m_N \le c \log N, \tag{1.2}$$

c a universal constant?

More specifically, Smale demands a polynomial running time algorithm which, on input N, outputs a N–tuple $(x_1, \dots, x_N)$ satisfying (1.2).

The problem of finding or describing the properties of elliptic Fekete points is indeed a classical problem and there is a wide range of articles dealing with different of its aspects. See for example [21, 19, 4, 16, 17, 24, 15, 9, 23, 1]. Yet, there is not much progress in the understanding of Problem 1.1. The value of m_N is only known to satisfy some limiting inequalities (see [16]) which are accurate to order $O(N \log(N))$ but not even to $O(N)$, let alone the great precision $O(\log(N))$ demanded by Smale.

There have been many numerical attempts to solve Problem 1.1, see for example [16, 17, 2, 3]. Experiments seem to show that the number of local minima of $\mathcal{E}$ in $\mathbb{S}^N$ grows at least exponentially with N (cf. [17, Sec. 4]), which makes it a difficult task to numerically determine a global minimum.

Regarding the existence of a polynomial time machine, the author of these pages is unaware of any reference proving even that Problem 1.1 can be solved by some algorithm (not considering the running time) in any of the two most standard models of computation: the classical Turing model and the real number machine model BSS (see [7, 6] for a description of the BSS model and a comparison with the classical model.)

In this paper we will prove that Problem 1.1 is actually solvable in simple exponential time in both models of computation. This fact will be deduced from the study of the harmonic properties of $\mathcal{E}$. We now summarize our main results, organizing them in several sections.

1.1. Harmonic properties of $\mathcal{E}$.

We equip $\mathbb{S}$ with the Riemannian structure inherited from $\mathbb{R}^3$. Thus, $\mathbb{S}$ is in particular a metric space and the Riemannian distance $d_R(p, q) \in [0, \pi/2]$ for $p, q \in \mathbb{S}$ is the length of the shortest portion of the great circle through p and q in $\mathbb{S}$.

The (Riemannian) Laplacian of $\mathcal{E}$ at $X \in \mathbb{S}^N \setminus \Sigma$ is the trace of the Riemannian (covariant) Hessian, see Section A.1 for details, that is

$$\Delta\mathcal{E}(X) = \mathit{trace}(\mathrm{Hess}(\mathcal{E}))(X) = \sum_{i=1}^{2N} \mathrm{Hess}(\mathcal{E})(X)(v^{(i)}, v^{(i)}), \qquad X \in \mathbb{S}^N \setminus \Sigma,$$

where $\{v^{(1)}, \dots, v^{(2N)}\}$ is any orthonormal basis of $T_X\mathbb{S}^N$ (by bilinearity of $\mathrm{Hess}(\mathcal{E})$, $\Delta\mathcal{E}$ does not depend on the choice of the basis.) Our first result will be proved in Section 2:

Theorem 1.2. *The Laplacian $\Delta\mathcal{E}$ of $\mathcal{E}$ is constant and equal to $2N(N-1)$. In particular, $\mathcal{E}$ is a subharmonic function.*

In Appendix A.2 we recall some well–known results from Riemannian Harmonic Analysis that apply to subharmonic functions and hence to $\mathcal{E}$. For example, the

[1] Smale refers to the unit sphere in $\mathbb{R}^3$, but the two problems are equivalent by sending points $(a, b, c) \in \mathbb{S}$ to $(2a, 2b, 2c - 1)$.

classical strong maximum principle (Theorem A.1) implies the following corollary, which I have not found in the existing literature:

Corollary 1.3. *The function $\mathcal{E}$ has no local maxima.*

Our second result is a mean value equality for $\mathcal{E}$. Opposite to the classical case where averages are computed in (Riemannian) balls, we found it more useful to describe this equality in products of spherical caps. More exactly, for $X = (x_1, \dots, x_N) \in \mathbb{S}^N$ and $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N) \in [0, \pi/2)^N$, let

$$B_\infty(X, \vec{\varepsilon}) = \{(y_1, \dots, y_N) \in \mathbb{S}^N : d_R(x_i, y_i) < \varepsilon_i, 1 \leq i \leq N\} \subseteq \mathbb{S}^N.$$

Thus, $B(X, \vec{\varepsilon})$ is the product of the Riemannian balls in $\mathbb{S}$ centered at x_i with radius ε_i. We use the convention that if $\varepsilon_i = 0$ for some i then the inequality $d_R(x_i, y_i) < \varepsilon_i$ in the formula above must be understood as $x_i = y_i$. Our second result (see Section 3.2 for a proof) is the following:

Theorem 1.4 (**Mean value theorem for B_∞**). *Let $X \in \mathbb{S}^N \setminus \Sigma$ and $\vec{\varepsilon} \in [0, \pi/2)^N$ such that $B_\infty(X, \vec{\varepsilon}) \subseteq \mathbb{S}^N \setminus \Sigma$. Let $⨍_{B_\infty(X,\vec{\varepsilon})} \mathcal{E}(Y)\, dY$ be the average of the values of $\mathcal{E}$ in $B_\infty(X, \vec{\varepsilon})$. Then,*

$$⨍_{B_\infty(X,\vec{\varepsilon})} \mathcal{E}(Y)\, dY = \mathcal{E}(X) + C_N(\vec{\varepsilon}),$$

where

$$C_N(\vec{\varepsilon}) = (N-1) \sum_{j=1}^{N} \left(\frac{1}{2} + \frac{\log(\cos \varepsilon_j)}{\tan^2 \varepsilon_j} \right) \in \left[0, \frac{N(N-1)}{2}\right),$$

with the convention

$$\frac{1}{2} + \frac{\log(\cos 0)}{\tan^2 0} = \lim_{\epsilon \mapsto 0} \left(\frac{1}{2} + \frac{\log(\cos \epsilon)}{\tan^2 \epsilon} \right) = 0.$$

Remark 1.5. *Note that for $0 \leq \varepsilon < \pi/2$ we have*

$$\frac{\varepsilon^2}{5} \leq \frac{1}{2} + \frac{\log(\cos \varepsilon)}{\tan^2 \varepsilon} \leq \frac{\varepsilon^2}{4}.$$

Thus, in the conditions of the theorem we have

$$\mathcal{E}(X) + \frac{N-1}{5}(\varepsilon_1^2 + \dots + \varepsilon_N^2) \leq ⨍_{B_\infty(X,\vec{\varepsilon})} \mathcal{E}(Y)\, dY \leq \mathcal{E}(X) + \frac{N-1}{4}(\varepsilon_1^2 + \dots + \varepsilon_N^2).$$

A similar result for usual Riemannian balls in $\mathbb{S}^N$ (i.e. for sets of the form $\{Y \in \mathbb{S}^N : d_R(X, Y) < \varepsilon_0\}$ for some $\varepsilon_0 > 0$) also holds, see Theorem 4.1 below. The value of the constant in that case is more difficult to compute. Note that Corollary 1.3 can also be (trivially) derived from Theorem 1.4.

1.2. **Gradient estimate and admissible error radius.** We will prove the following result in Section 1.6.

Theorem 1.6. *Let $X \in \mathbb{S}^N$ be such that*

$$B_\infty(X, \vec{\varepsilon}) \subseteq \mathbb{S}^N \setminus \Sigma, \text{ where } \vec{\varepsilon} = (\varepsilon, \dots, \varepsilon), \varepsilon = \sqrt{\frac{2(\mathcal{E}(X) - m_N)}{N-1}}. \tag{1.3}$$

Then,

$$\|D\mathcal{E}(X)\| \leq 2\sqrt{2N(N-1)(\mathcal{E}(X) - m_N)}.$$

Note that hypotheses (1.3) in Theorem 1.6 is equivalent to:

$$\min_{i\neq j} d_R(x_i,x_j)\geq 2\varepsilon.$$

The left–hand term in this last formula is the "separation distance" $\mathrm{d_{sep}}(X)$ of the N-tuple X. Bounds are known since [17] for the separation distances of elliptic Fekete points, see also [16, 10, 9]. The sharpest known bound is that of [9] (note that the result of [9] is written for the unit sphere, so we have to divide by 2 for $\mathbb{S}$):

$$d_N\geq \arcsin\frac{1}{\sqrt{N-1}}\geq\frac{1}{\sqrt{N-1}},\text{ where } d_N=\inf_{X:\mathcal{E}(X)=m_N}\mathrm{d_{sep}}(X). \tag{1.4}$$

Theorem 1.6 and (1.4) can be combined to get an estimate on how close one must be to a set of elliptic Fekete points in order to satisfy (1.2). The result we get can be better understood using the following concept.

Definition 1.7. *The* admissible error function $\mathbf{e}:(0,\infty)\to(0,\infty)$ *is the function defined as*

$$\mathbf{e}(t)=\sup\{\varepsilon : Y\in B_\infty(X,\vec{\varepsilon}),\ \textit{implies } \mathcal{E}(Y)\leq m_N+t\},$$

where $X=(x_1,\ldots,x_N)$ *is a set of elliptic Fekete points and* $\vec{\varepsilon}=(\varepsilon,\ldots,\varepsilon)$.

Note that the supremum is indeed a maximum and that, for fixed $t\in(0,\infty)$, $\mathbf{e}(t)$ is the maximum coordinate–wise error (measured in Riemannian distance in $\mathbb{S}$) that one can permit when writing a set of elliptic Fekete points, if an inequality $\mathcal{E}(Y)\leq m_N+t$ is to be guaranteed. The main result of this section is the following estimate, that will be proved in Section 5.2.1.

Theorem 1.8. *Let $N\geq 3$. The admissible error function satisfies*

$$\mathbf{e}(t)\in\left[\sqrt{\frac{t}{2N^2(N-1)}},\sqrt{\frac{2t}{N(N-1)}}\right],\qquad 0\leq t\leq\frac{N^2(N-1)d_N^2}{2(1+2N)^2}. \tag{1.5}$$

For any $t>d_N^2(N-1)/8$ we also have:

$$\mathbf{e}(t)\in\left[\frac{d_N}{2}\left(1-\left(\frac{d_N^2(N-1)}{8t}\right)^{\frac{1}{2N}}\right),\sqrt{\frac{2t}{N(N-1)}}\right]. \tag{1.6}$$

This theorem can be better understood if we specify some value for t. For example, with $t=1/18$ we get

Corollary 1.9. *Let $N\geq 3$. Then,*

$$\mathbf{e}\left(\frac{1}{18}\right)\in\left[\frac{1/6}{N\sqrt{N-1}},\frac{1/3}{\sqrt{N(N-1)}}\right]. \tag{1.7}$$

Proof. From (1.4), we have

$$\frac{N^2(N-1)d_N^2}{2(1+2N)^2}\geq\frac{N^2}{2(1+2N)^2}\underset{N\geq 3}{\geq}\frac{1}{18},$$

and thus (1.5) implies (1.7). □

Remark 1.10. *The meaning of (1.7) is the following: if a N–tuple Y satisfying $\mathcal{E}(Y)\leq m_N+1/18$ is desired, then we can make an error of approximately $1/(6N^{3/2})$ in the description of each coordinate, but if we make an error greater than approximately $1/(3N)$ then we risk that our N–tuple will not satisfy the desired inequality.*

1.3. **Computability and complexity of Smale's 7th problem.** The set of points with rational coordinates is known to be dense in $\mathbb{S}$ (this follows from the fact that the stereographic projection sends rational points in the plane to rational points in the sphere). Moreover, bounds on the size of the integers representing rational points within a certain radius are also known, see [18] and references therein. In Section 6, from [18] and Theorem 1.8 we will prove:

Proposition 1.11. *There is a universal constant $c \geq 0$ ($c = 17$ suffices) with the following property: for every $N \geq 3$ there exists $Z = (z_1, \ldots, z_N) \in \mathbb{S}^N$ such that:*

(1) $\mathcal{E}(Z) \leq m_N + 1/18$.
(2) *For* $1 \leq i \leq N$,

$$z_i = \left(\frac{p_i^{(1)}}{q_i^{(1)}}, \frac{p_i^{(2)}}{q_i^{(2)}}, \frac{p_i^{(3)}}{q_i^{(3)}} \right) \in \mathbb{S} \cap \mathbb{Q}^3, \qquad p_i^{(j)}, q_i^{(j)} \in \mathbb{Z},\ 1 \leq j \leq 3,$$

where

$$0 \leq |p_i^{(j)}| \leq q_i^{(j)} \leq (cN)^6, \qquad 1 \leq i \leq N,\ 1 \leq j \leq 3,$$

It follows that the number of binary digits needed to write down the coordinates of a rational solution to the problem of finding Z such that $\mathcal{E}(Z) \leq m_N + 1/18$ is $O(\log_2 N)$. Simply by testing all possible options we get:

Corollary 1.12. *There exists a Turing machine which, on input $N \in \{3, 4, 5, \ldots\}$, outputs a set of N rational points in $\mathbb{S}$ satisfying (1.2), and such that the running time is simply exponential* [2] *in N.*

1.4. **Acknowledgements.** Thanks to Cecilia Pola for many conversations, and to Jean Pierre Dedieu, Luis Miguel Pardo, Mike Shub and two anonymous referees for comments and suggestions. Thanks also to Joaquim Ortega Cerdà for his comments on harmonic manifolds.

2. Proof of Theorem 1.2

We will use two technical lemmas.

Lemma 2.1. *Let $f : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be twice differentiable at 0 and $f(0) \neq 0$. Then, $\log \|f(t)\|$ is twice differentiable at 0 and*

$$\frac{d^2}{dt^2} \Big|_{t=0} (\log \|f(t)\|) = \frac{\langle f(0), f''(0) \rangle + \|f'(0)\|^2}{\|f(0)\|^2} - 2 \frac{\langle f(0), f'(0) \rangle^2}{\|f(0)\|^4}.$$

Proof. We first note that $\|f(t)\|$ is twice differentiable at 0 and

$$\frac{d}{dt} \Big|_{t=0} \|f(t)\| = \frac{\langle f(0), f'(0) \rangle}{\|f(0)\|},$$

$$\frac{d^2}{dt^2} \Big|_{t=0} \|f(t)\| = \frac{\langle f(0), f''(0) \rangle + \|f'(0)\|^2}{\|f(0)\|} - \frac{\langle f(0), f'(0) \rangle^2}{\|f(0)\|^3}. \tag{2.1}$$

On the other hand, using the chain rule, if $h : (-\varepsilon, \varepsilon) \to (0, \infty)$ is twice differentiable at 0 then so is $\log(h(t))$ and

$$\frac{d^2}{dt^2} \Big|_{t=0} (\log(h(t))) = \frac{d}{dt} \Big|_{t=0} \left(\frac{h'(t)}{h(t)} \right) = \frac{h''(0)h(0) - h'(0)^2}{h(0)^2}. \tag{2.2}$$

[2] More precisely, the running time of our procedure is $\mathit{polynomial}(N) \cdot (20N)^{36N}$.

From (2.1) and (2.2) we get that if $f:(-\varepsilon,\varepsilon)\to\mathbb{R}^3$ such that $f(0)\neq 0$ is twice differentiable at 0, then so is $\log\|f(t)\|$ and

$$\frac{d^2}{dt^2}\mid_{t=0}\log\|f(t)\| =$$

$$\frac{\left(\frac{\langle f(0),f''(0)\rangle+\|f'(0)\|^2}{\|f(0)\|}-\frac{\langle f(0),f'(0)\rangle^2}{\|f(0)\|^3}\right)\|f(0)\|-\left(\frac{\langle f(0),f'(0)\rangle}{\|f(0)\|}\right)^2}{\|f(0)\|^2} =$$

$$\frac{\langle f(0),f''(0)\rangle+\|f'(0)\|^2}{\|f(0)\|^2}-2\frac{\langle f(0),f'(0)\rangle^2}{\|f(0)\|^4},$$

as desired.

□

I want to thank one referee for pointing out that the following lemma is well–known in mathematical physics (see for example [12, Sec. 15.6.1]). A proof of it is anyway included for completeness.

Lemma 2.2. *Let $q\in\mathbb{S}$ be fixed and let*

$$\begin{array}{rccc} F_q: & \mathbb{S}\setminus\{q\} & \to & \mathbb{R} \\ & p & \mapsto & \log\|p-q\|^{-1} \end{array}$$

Then,

$$\Delta F_q(p)=2\ \ \forall\, p\in\mathbb{S}\setminus\{q\}.$$

Proof. Let $p\in\mathbb{S}$ and let $\{v,w\}$ be an orthonormal basis of $T_p\mathbb{S}$. Let $\gamma_{p,v}$ and $\gamma_{p,w}$ be given by (A.1). Thus, they are the geodesics in $\mathbb{S}$ with base point p at $t=0$ and respective tangent vectors v,w at $t=0$. Let $e_3=(0,0,1)^T$. Take

$$f_v(t)=\gamma_{p,v}(t)-q=p\cos(2t)+\frac{v}{2}\sin(2t)+\frac{1}{2}(0,0,1-\cos(2t))^T-q$$

in Lemma 2.1. Note that

$$f_v(0)=p-q,\qquad f_v'(0)=v,\qquad f_v''(0)=2e_3-4p.$$

Thus,

$$\langle f_v(0),f_v''(0)\rangle=\langle p-q,2e_3-4p\rangle,\qquad \|f_v'(0)\|^2=1,$$
$$\langle f_v(0),f_v'(0)\rangle^2=\langle p-q,v\rangle^2,\qquad \|f_v(0)\|^2=\|p-q\|^2.$$

Hence, from Lemma 2.1,

$$\frac{d^2}{dt^2}\mid_{t=0}(\log\|f_v(t)\|)=\frac{\langle p-q,2e_3-4p\rangle+1}{\|p-q\|^2}-2\frac{\langle p-q,v\rangle^2}{\|p-q\|^4},$$

and the same formula holds if we change v for w. Now, because $\{v,w\}$ is an orthogonal basis of $T_p\mathbb{S}=\{u\in\mathbb{R}^3:\langle u,2p-e_3\rangle=0\}$, denoting by Π the orthogonal projection onto $T_p\mathbb{S}$ we have that

$$\langle p-q,v\rangle^2+\langle p-q,w\rangle^2=\|\Pi(p-q)\|^2=\|p-q\|^2-\langle 2p-e_3,p-q\rangle^2.$$

We thus conclude,

$$-\Delta F_q=\frac{d^2}{dt^2}\mid_{t=0}(\log\|f_v(t)\|)+\frac{d^2}{dt^2}\mid_{t=0}(\log\|f_w(t)\|)=$$

$$\frac{2\langle p-q,2e_3-4p\rangle+2}{\|p-q\|^2}-2\frac{\|p-q\|^2-\langle 2p-e_3,p-q\rangle^2}{\|p-q\|^4}=$$

$$2\frac{\langle p-q,2p-e_3\rangle(\langle p-q,2p-e_3\rangle-2\|p-q\|^2)}{\|p-q\|^4}=$$
$$2\frac{\langle p-q,2p-e_3\rangle\langle p-q,2q-e_3\rangle}{\|p-q\|^4}.$$

Now, note that $p=(p_1,p_2,p_3)^T\in\mathbb{S}$ implies $p_1^2+p_2^2+p_3^2=p_3$ and hence
$$\langle p,2p-e_3\rangle=2p_1^2+2p_2^2+2p_3^2-p_3=\|p\|^2.$$
Thus, using (1.1),
$$\langle p-q,2p-e_3\rangle=\|p\|^2-2\langle q,p\rangle+\|q\|^2=\|p-q\|^2.$$
A symmetric argument shows that $\langle p-q,2q-e_3\rangle=-\|p-q\|^2$. We have then proved that
$$-\Delta F_q=2\frac{-\|p-q\|^4}{\|p-q\|^4},$$
and the lemma follows.

□

2.1. **Proof of Theorem 1.2.** The proof now consists simply in summing the result of Lemma 2.2 over pairs. More precisely, let $X=(x_1,\ldots,x_N)\in\mathbb{S}^N\setminus\Sigma$. For $i\in\{1,\ldots,N\}$, let v_i,w_i be an orthonormal basis of $T_{x_i}\mathbb{S}$. Then, the $2N$ vectors
$$\begin{array}{rcl} v^{(1)} &=& (v_1,0,\ldots,0),\\ w^{(1)} &=& (w_1,0,\ldots,0),\\ v^{(2)} &=& (0,v_2,0,\ldots,0),\\ w^{(2)} &=& (0,w_2,0,\ldots,0),\\ \cdots && \end{array}$$
are linearly independent vectors of $T_X\mathbb{S}^N$ and are thus a basis of $T_X\mathbb{S}^N$. Thus,

$$\Delta\mathcal{E}(X)=\sum_{i=1}^N\left(\frac{d^2}{dt^2}\mid_{t=0}\left(\mathcal{E}(\gamma_{X,v^{(i)}}(t))\right)+\frac{d^2}{dt^2}\mid_{t=0}\left(\mathcal{E}(\gamma_{X,w^{(i)}}(t))\right)\right).\tag{2.3}$$

We claim that, for $i\in\{1,\ldots,N\}$ we have
$$\gamma_{X,v^{(i)}}(t)=(x_1,\ldots,x_{i-1},\gamma_{x_i,v_i}(t),x_{i+1},\ldots,x_N),$$
where $\gamma_{x_i,v_i}(t)$ is given by (A.1). Indeed, the curve in the right hand term has same value and tangent vector than $\gamma_{X,v^{(i)}}(t)$ at $t=0$, and is a geodesic because $\mathbb{S}^N$ has the product Riemannian structure. By the uniqueness of geodesics, both expressions are equal. Similarly,
$$\gamma_{X,w^{(i)}}(t)=(x_1,\ldots,x_{i-1},\gamma_{x_i,w_i}(t),x_{i+1},\ldots,x_N).$$
Thus, for $i\in\{1,\ldots,N\}$,
$$\frac{d^2}{dt^2}\mid_{t=0}\left(\mathcal{E}(\gamma_{x,v^{(i)}}(t)\right)=\frac{d^2}{dt^2}\mid_{t=0}\left(\mathcal{E}(x_1,\ldots,\gamma_{x_i,v_i}(t),x_{i+1},\ldots,x_N)\right)=$$
$$-\sum_{j\neq i}\frac{d^2}{dt^2}\mid_{t=0}\log\|\gamma_{x_i,v_i}(t)-x_j\|,$$
and the same formula holds changing $v^{(i)}$ for $w^{(i)}$ and v_i for w_i. Thus,

$$\frac{d^2}{dt^2}\mid_{t=0}\left(\mathcal{E}(\gamma_{X,v^{(i)}}(t)\right)+\frac{d^2}{dt^2}\mid_{t=0}\left(\mathcal{E}(\gamma_{X,w^{(i)}}(t)\right)=\tag{2.4}$$

$$-\sum_{j\neq i}\frac{d^2}{dt^2}\,|_{t=0}\left(\log\|\gamma_{x_i,v_i}(t)-x_j\|+\log\|\gamma_{x_i,w_i}(t)-x_j\|\right)=\sum_{j\neq i}\Delta F_{x_j}$$

in the notation of Lemma 2.2, and this last expression equals $2(N-1)$. From (2.3) and (2.4) we then have

$$\Delta\mathcal{E}(X)=\sum_{i=1}^{N}2(N-1)=2N(N-1),$$

as desired.

3. Coordinate-wise mean value properties of $\mathcal{E}$

In this section we prove the following result, from which Theorem 1.4 will easily follow. For $p\in\mathbb{S}$ and $\varepsilon>0$ let

$$B(p,\varepsilon)=\{q\in\mathbb{S}:d_R(p,q)<\varepsilon\}\qquad\text{and}\qquad S(p,\varepsilon)=\{q\in\mathbb{S}:d_R(p,q)=\varepsilon\}$$

be respectively the Riemannian open ball and sphere in $\mathbb{S}$ of radius ε.

Theorem 3.1 (**Coordinate-wise mean value theorem).** *Let $(x_1,\dots,x_N)\in\mathbb{S}^N\setminus\Sigma$, and let $\varepsilon>0$ be such that*

$$\{x_1\}\times\cdots\times\{x_{k-1}\}\times B(x_k,\varepsilon)\times\{x_{k+1}\}\times\cdots\times\{x_N\}\subseteq\mathbb{S}\setminus\Sigma.$$

Then, for $1\leq k\leq N$,

$$⨍_{x\in B(x_k,\varepsilon)}\mathcal{E}(x_1,\dots,x_{k-1},x,x_{k+1},\dots,x_N)\,dx=\mathcal{E}(x_1,\dots,x_N)+(N-1)\left(\frac{1}{2}+\frac{\log(\cos\varepsilon)}{\tan^2\varepsilon}\right),$$

$$⨍_{x\in S(x_k,\varepsilon)}\mathcal{E}(x_1,\dots,x_{k-1},x,x_{k+1},\dots,x_N)\,dx=\mathcal{E}(x_1,\dots,x_N)-(N-1)\log(\cos\varepsilon).$$

For the proof we will consider the stereographic projection

$$\begin{array}{rccc}\varphi: & \mathbb{R}^2 & \to & \mathbb{S}\setminus\{0\}\\ & (u,v) & \mapsto & \frac{(u,v,1)}{1+u^2+v^2},\end{array}$$

which is a bijection from $\mathbb{R}^2$ to $\mathbb{S}\setminus\{0\}$. The Jacobian of this mapping is $(1+u^2+v^2)^{-2}$.

The following result is proved using Theorem A.2, which is an easy consequence of classical results from harmonic analysis on manifolds. A more elementary proof can also be obtained by computing some integrals.

Proposition 3.2. *Let $p_0,q\in\mathbb{S}$ and let $0<\varepsilon\leq d_R(p_0,q)\leq\pi/2$. Then,*

$$\begin{aligned}Vol(B(p_0,\varepsilon)) &= \pi\sin^2\varepsilon,\\ ⨍_{p\in B(p_0,\varepsilon)}\log\|p-q\|^{-1}\,dp &= \log\|p_0-q\|^{-1}+\frac{1}{2}+\frac{\log(\cos\varepsilon)}{\tan^2\varepsilon},\\ ⨍_{p\in S(p_0,\varepsilon)}\log\|p-q\|^{-1}\,dp &= \log\|p_0-q\|^{-1}-\log(\cos\varepsilon).\end{aligned}$$

Moreover, let $0<d_R(p_0,q)<\varepsilon\leq\pi/2$. Then,

$$⨍_{p\in B(p_0,\varepsilon)}\log\|p-q\|^{-1}\,dp = \frac{1}{2\sin^2\varepsilon}+\frac{\cot^2\varepsilon}{2}\left(\log(1-\|p_0-q\|^2)-1\right)-\log(\sin\varepsilon).$$

Proof. Since $\|p-q\|$ is invariant under rotation of the sphere, we can consider that $p_0 = (0,0,1)$ is the north pole, and $q = (x,0,z)$, $x \geq 0$, is in the xz–plane. The Riemannian distance in the Riemann Sphere is given by

$$d_R(p_0,p) = \frac{1}{2}\arccos\frac{\langle (0,0,1/2), p-(0,0,1/2)\rangle}{\|(0,0,1/2)\|\|p-(0,0,1/2)\|} =$$

$$\frac{1}{2}\arccos(4\langle (0,0,1/2), p-(0,0,1/2)\rangle).$$

If we let $p = (a,b,c)$ we then have

$$d_R(p_0,p) = \frac{1}{2}\arccos(2c-1) \in [0,\pi/2].$$

Thus, $p \in B(p_0,\varepsilon)$ reads

$$\frac{1}{2}\arccos(2c-1) \leq \varepsilon \quad \text{that is} \quad c \geq \frac{\cos(2\varepsilon)+1}{2} = \cos^2(\varepsilon).$$

We conclude that

$$\varphi^{-1}(B(p_0,\varepsilon)) = \left\{(u,v) \in \mathbb{R}^2 : \frac{1}{1+u^2+v^2} \geq \cos^2(\varepsilon)\right\} = \left\{(u,v) \in \mathbb{R}^2 : u^2+v^2 \leq \delta^2\right\},$$

where $\delta = \tan(\varepsilon)$. The change of variables formula then yields

$$Vol(B(p_0,\varepsilon)) = \int_{u^2+v^2\leq\delta^2} \frac{1}{(1+u^2+v^2)^2}\, d(u,v) = 2\pi\int_0^\delta \frac{\rho}{(1+\rho^2)^2}\, d\rho = \frac{\pi\delta^2}{1+\delta^2}.$$

This proves the first claim of the proposition. The third claim follows from Theorem A.2 and Lemma 2.2. Integrating by polar coordinates and using that $S(p_0,t)$ is a circle of radio $\sin(2t)/2$ we then get

$$\int_{p\in B(p_0,\varepsilon)} \log\|p-q\|^{-1}\, dp = \int_0^\varepsilon \int_{p\in S(p_0,t)} \log\|p-q\|^{-1}\, dp\, dt =$$

$$\int_0^\varepsilon \pi\sin(2t) ⨍_{p\in S(p_0,t)} \log\|p-q\|^{-1}\, dp\, dt.$$

From the third claim we have

$$\int_{p\in B(p_0,\varepsilon)} \log\|p-q\|^{-1}\, dp = \pi\int_0^\varepsilon \sin(2t)\left(\log\|p_0-q\|^{-1} - \log(\cos t)\right)\, dt =$$

$$\pi\sin^2\varepsilon\log\|p_0-q\|^{-1} - \pi\int_0^\varepsilon 2\sin t\cos t\log(\cos t)\, dt =$$

$$\pi\sin^2\varepsilon\log\|p_0-q\|^{-1} + \pi\left(\cos^2\varepsilon\log(\cos\varepsilon) + \frac{1}{2}\sin^2\varepsilon\right).$$

From the first claim, we finally get

$$⨍_{p\in B(p_0,\varepsilon)} \log\|p-q\|^{-1}\, dp = \frac{1}{\pi\sin^2\varepsilon}\int_{p\in B(p_0,\varepsilon)} \log\|p-q\|^{-1}\, dp =$$

$$\log\|p_0-q\|^{-1} + \frac{1}{2} + \frac{\log(\cos\varepsilon)}{\tan^2\varepsilon},$$

and the second claim of the proposition is now proved.

For the last claim, let $\hat{p}_0$ be the conjugate of p_0 in $\mathbb{S}$ (that is, $\hat{p}_0$ is the symmetric point of p_0 with respect to the central isometry with center $(0,0,1/2)$) and note that

$$\int_{B(p_0,\varepsilon)} \log\|p-q\|^{-1}\,dp = \int_{\mathbb{S}} \log\|p-q\|^{-1}\,dp - \int_{B(\hat{p}_0,\pi/2-\varepsilon)} \log\|p-q\|^{-1}\,dp =$$

$$\frac{\pi}{2} - \int_{B(\hat{p}_0,\pi/2-\varepsilon)} \log\|p-q\|^{-1}\,dp.$$

Applying the first part of the proposition to this last integral and using

$$\cos(\pi/2-\varepsilon) = \sin\varepsilon, \qquad \sin(\pi/2-\varepsilon) = \cos\varepsilon,$$

we get

$$⨍_{B(p_0,\varepsilon)} \log\|p-q\|^{-1}\,dp = \frac{1}{2\sin^2\varepsilon} - \frac{\cos^2\varepsilon}{\sin^2\varepsilon}\left(\log\|\hat{p}_0-q\|^{-1} + \frac{1}{2} + \tan^2\varepsilon\log(\sin\varepsilon)\right) =$$

$$\frac{1}{2\sin^2\varepsilon} + \frac{\cot^2\varepsilon}{2}\left(2\log\|\hat{p}_0-q\| - 1\right) - \log(\sin\varepsilon).$$

The last formula of the proposition follows from the elementary fact that

$$\|\hat{p}_0-q\|^2 + \|p_0-q\|^2 = 1,$$

for any $p_0, q \in \mathbb{S}$. $\square$

3.1. **Proof of Theorem 3.1.** It suffices to prove the result for $k=1$. Note that

$$⨍_{x\in B(x_1,\varepsilon)} \mathcal{E}(x,x_2,\ldots,x_N)\,dx =$$

$$⨍_{x\in B(x_1,\varepsilon)} \left(\sum_{2\le i<j} \log\|x_i-x_j\|^{-1} + \sum_{1<j} \log\|x-x_j\|^{-1}\right) dx =$$

$$\sum_{2\le i<j} \log\|x_i-x_j\|^{-1} + \sum_{1<j} ⨍_{x\in B(x_1,\varepsilon)} \log\|x-x_j\|^{-1}\,dx.$$

On the other hand,

$$\mathcal{E}(x_1,\ldots,x_N) = \sum_{2\le i<j} \log\|x_i-x_j\|^{-1} + \sum_{1<j} \log\|x_1-x_j\|^{-1}.$$

Thus,

$$⨍_{x\in B(x_1,\varepsilon)} \mathcal{E}(x,x_2,\ldots,x_N)\,dx - \mathcal{E}(x_1,\ldots,x_N) =$$

$$\sum_{1<j}\left(⨍_{x\in B(x_1,\varepsilon)} \log\|x-x_j\|^{-1}\,dx - \log\|x_1-x_j\|^{-1}\right).$$

From the second equality of Proposition 3.2 this last equals

$$(N-1)\left(\frac{1}{2} + \frac{\log(\cos\varepsilon)}{\tan^2\varepsilon}\right),$$

and the first claim of the theorem follows. The second claim is proved the same way, using the third equality of Proposition 3.2 for the last step.

3.2. **Proof of Theorem 1.4.** We will prove that the following equality is valid for $k \in \{1, \ldots, N\}$.

$$\fint_{B(x_1,\varepsilon_1)\times\cdots\times B(x_k,\varepsilon_k)} \mathcal{E}(y_1, \ldots, y_k, x_{k+1}, \ldots, x_N)\, d(y_1, \ldots, y_k) =$$
$$\mathcal{E}(x_1, \ldots, x_N) + (N-1)\sum_{j=1}^{k}\left(\frac{1}{2} + \frac{\log(\cos\varepsilon_j)}{\tan^2\varepsilon_j}\right). \tag{3.1}$$

Note that Theorem 1.4 is the case $k = N$ of this equality; we will prove (3.1) by induction on k. The base case $k = 1$ of our induction is Theorem 3.1. Now, assume (3.1) is true for $k-1$ where $k \in \{2, \ldots, N\}$. Then, from Fubini's Theorem we have

$$\int_{B(x_1,\varepsilon_1)\times\cdots\times B(x_k,\varepsilon_k)} \mathcal{E}(y_1, \ldots, y_k, x_{k+1}, \ldots, x_N)\, d(y_1, \ldots, y_k) =$$

$$\int_{B(x_1,\varepsilon_1)\times\cdots\times B(x_{k-1},\varepsilon_{k-1})} \int_{y_k\in B(x_k,\varepsilon_k)} \mathcal{E}(y_1, \ldots, y_{k-1}, y_k, x_{k+1}, \ldots, x_N)\, dy_k\, d(y_1, \ldots, y_{k-1}).$$

From Theorem 3.1, this last equals

$$Vol(B(x_k,\varepsilon_k)) \int_{B(x_1,\varepsilon_1)\times\cdots\times B(x_{k-1},\varepsilon_{k-1})} \mathcal{E}(y_1, \ldots, y_{k-1}, x_k, \ldots, x_N)\, d(y_1, \ldots, y_{k-1}) +$$

$$Vol(B(x_k,\varepsilon_k))(N-1)\left(\frac{1}{2} + \frac{\log(\cos\varepsilon_k)}{\tan^2\varepsilon_k}\right) Vol(B(x_1,\varepsilon_1))\cdots Vol(B(x_{k-1},\varepsilon_{k-1})).$$

Hence,

$$\fint_{B(x_1,\varepsilon_1)\times\cdots\times B(x_k,\varepsilon_k)} \mathcal{E}(y_1, \ldots, y_k, x_{k+1}, \ldots, x_N)\, d(y_1, \ldots, y_k) =$$

$$(N-1)\left(\frac{1}{2} + \frac{\log(\cos\varepsilon_k)}{\tan^2\varepsilon_k}\right) + \fint_{B(x_1,\varepsilon_1)\times\cdots\times B(x_{k-1},\varepsilon_{k-1})} \mathcal{E}(y_1, \ldots, y_{k-1}, x_k, \ldots, x_N)\, d(y_1, \ldots, y_{k-1}) =$$

$$(N-1)\left(\frac{1}{2} + \frac{\log(\cos\varepsilon_k)}{\tan^2\varepsilon_k}\right) + (N-1)\sum_{j=1}^{k-1}\left(\frac{1}{2} + \frac{\log(\cos\varepsilon_j)}{\tan^2\varepsilon_j}\right) + \mathcal{E}(x_1, \ldots, x_N).$$

The induction step is proved and (3.1) follows. This finishes the proof of our Theorem 1.4.

If instead of the first equality of Theorem 3.1 we use the second one, we get

Corollary 3.3. *Let $X \in \mathbb{S}^N \setminus \Sigma$ and $\vec{\varepsilon} \in [0, \pi/2)^N$ such that $B_\infty(X, \vec{\varepsilon}) \subseteq \mathbb{S}^N \setminus \Sigma$. Then,*

$$\fint_{S(x_1,\varepsilon_1)\times\cdots\times S(x_N,\varepsilon_N)} \mathcal{E}(Y)\, dY = \mathcal{E}(X) - (N-1)\sum_{j=1}^{N}\log(\cos\varepsilon_j).$$

4. Mean value property for Riemannian balls in $\mathbb{S}^N$

Theorem 1.4 can be stated for Riemannian balls in $\mathbb{S}^N$ instead of products of balls in $\mathbb{S}$. More exactly we have the following result.

Theorem 4.1 (**Mean value theorem for B_R**). *There exists a constant $D_N(\varepsilon) > 0$ with the following property. Let $X = (x_1, \ldots, x_N) \in \mathbb{S}^N \setminus \Sigma$ and let $\varepsilon > 0$ be such that $B_R(X, \varepsilon) \subseteq \mathbb{S}^N \setminus \Sigma$ where*

$$B_R(X,\varepsilon) = \{(y_1, \ldots, y_N) \in \mathbb{S}^N : d_R(y_1, x_1)^2 + \cdots + d_R(y_N, x_N)^2 \leq \varepsilon^2\}$$

is the Riemannian ball in $\mathbb{S}^N$ of radius ε centered at X. Then,

$$⨍_{B_R(X,\varepsilon)} \mathcal{E}(Y)\, dY = \mathcal{E}(X) + D_N(\varepsilon).$$

Proof. Theorem 4.1 follows taking $j = N$ in the next proposition. $\square$

Let

$$B^j(X,\varepsilon) = \{(y_1, \ldots, y_j) \in \mathbb{S}^j : d_R(y_1, x_1)^2 + \cdots + d_R(y_j, x_j)^2 \leq \varepsilon^2\}, \quad 1 \leq j \leq N,$$

and

$$\phi^j(X,\varepsilon) = \int_{B^j(X,\varepsilon)} \mathcal{E}(y_1, \ldots, y_j, x_{j+1}, \ldots, x_N)\, d(y_1, \ldots, y_j).$$

Proposition 4.2. *Let $1 \leq j \leq N$. Then, there exists a constant $D_N^j(\varepsilon)$ with the following property. Let $X = (x_1, \ldots, x_N) \in \mathbb{S}^N \setminus \Sigma$ and let $\varepsilon > 0$ be such that $B^j(X,\varepsilon) \subseteq \mathbb{S}^N \setminus \Sigma$. Then,*

$$\phi^j(X,\varepsilon) = D_N^j(\varepsilon) + V^j(\varepsilon)\mathcal{E}(x_1, \ldots, x_N),$$

where

$$V^j(\varepsilon) = Vol(B^j(X,\varepsilon)).$$

Proof. The proof is by induction on j. Note that the case $j = 1$ is Theorem 3.1. For the general case, by Fubini's Theorem we have,

$$\phi^{j+1}(X,\varepsilon) = \int_{\rho = d_R(y_{j+1}, x_{j+1}) \leq \varepsilon} \phi^j((x_1, \ldots, x_j, y_{j+1}, x_{j+2}, \ldots, x_N), \sqrt{\varepsilon^2 - \rho^2})\, dy_{j+1}.$$

By induction hypotheses, this equals

$$\int_{\rho = d_R(y_{j+1}, x_{j+1}) \leq \varepsilon} D_N^j(\sqrt{\varepsilon^2 - \rho^2}) + V^j(\sqrt{\varepsilon^2 - \rho^2})\mathcal{E}(x_1, \ldots, x_j, y_{j+1}, x_{j+2}, \ldots, x_N)\, dy_{j+1} =$$

$$\int_0^\varepsilon \int_{d_R(y_{j+1}, x_{j+1}) = \rho} D_N^j(\sqrt{\varepsilon^2 - \rho^2})\, dy_{j+1}\, d\rho +$$

$$\int_0^\varepsilon V^j(\sqrt{\varepsilon^2 - \rho^2}) \int_{d_R(y_{j+1}, x_{j+1}) = \rho} \mathcal{E}(x_1, \ldots, x_j, y_{j+1}, x_{j+2}, \ldots, x_N)\, dy_{j+1}\, d\rho.$$

From the third claim of Theorem 3.1 this equals

$$\int_0^\varepsilon \int_{d_R(y_{j+1}, x_{j+1}) = \rho} D_N^j(\sqrt{\varepsilon^2 - \rho^2})\, dy_{j+1}\, d\rho +$$

$$\int_0^\varepsilon V^j(\sqrt{\varepsilon^2 - \rho^2})\pi \sin(2\rho) \left(\mathcal{E}(x_1, \ldots, x_N) - (N-1)\log(\cos\rho)\right)\, d\rho =$$

$$C_N^{j+1}(\varepsilon) + \left(\int_0^\varepsilon V^j(\sqrt{\varepsilon^2 - \rho^2})\pi \sin(2\rho)\, d\rho\right) \mathcal{E}(x_1, \ldots, x_N).$$

Similarly, we have

$$Vol(B^{j+1}(X,\varepsilon)) = \int_{\rho=d_R(y_{j+1},x_{j+1})\le\varepsilon} Vol(B^j((x_1,\dots,x_N),\sqrt{\varepsilon^2-\rho^2}))\,dy_{j+1} =$$

$$\int_0^\varepsilon \int_{d_R(y_{j+1},x_{j+1})=\rho} V^j(\sqrt{\varepsilon^2-\rho^2})\,dy_{j+1}\,d\rho = \int_0^\varepsilon \pi\sin(2\rho)V^j(\sqrt{\varepsilon^2-\rho^2})\,d\rho.$$

We conclude that

$$\phi^{j+1}(X,\varepsilon) = D_N^{j+1}(\varepsilon) + V^{j+1}(\varepsilon)\mathcal{E}(x_1,\dots,x_N),$$

for some constant $D_N^{j+1}(\varepsilon)$.

□

5. Consequences of the mean value equality

The results stated in this section follow from Theorem 1.4 and are thus satisfied by *any* function defined in $\mathbb{S}^N\setminus\Sigma$ and satisfying a mean value property like that of Theorem 1.4. We will however state the results just for $\mathcal{E}$. For two vectors

$$\vec{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_N), \qquad \vec{t} = (t_1,\dots,t_N) \in (0,\pi/2)^N,$$

we denote $\vec{t}\prec\vec{\varepsilon}$ if $t_i<\varepsilon_i$ for $1\le i\le N$ (meaning $t_i=\varepsilon_i$ if $\varepsilon_i=0$). We also denote

$$\vec{\varepsilon}-\vec{t} = (\varepsilon_1-t_1,\dots,\varepsilon_N-t_N).$$

5.1. **Norm of the derivative.** We start with the following result, whose proof mimics that of the classical Harnack's inequality (see for example [11, p. 33]).

Corollary 5.1. *Let $X\in\mathbb{S}^N\setminus\Sigma$ and let $\vec{\varepsilon},\vec{t}\in[0,\pi/2)^N$ be such that $B_\infty(X,\vec{\varepsilon})\subseteq\mathbb{S}^N\setminus\Sigma$ and $\vec{t}\prec\vec{\varepsilon}$. Let*

$$\alpha = \inf\left(\mathcal{E}(Z) : Z\in B_\infty(X,\vec{\varepsilon})\right).$$

Then, for every $Z\in B_\infty(X,\vec{t})$ we have:

$$\mathcal{E}(Z) \le \alpha + \frac{Vol(B_\infty(X,\vec{\varepsilon}))}{Vol(B_\infty(X,\vec{\varepsilon}-\vec{t}))}(\mathcal{E}(X)-\alpha+C_N(\vec{\varepsilon})) - C_N(\vec{\varepsilon}-\vec{t}).$$

Proof. Let $G: B_\infty(X,\vec{\varepsilon})\to\mathbb{R}$ be given by $G(Y)=\mathcal{E}(Y)-\alpha$. From Theorem 1.4 we have

$$⨍_{B_\infty(X,\vec{\varepsilon})} G(Y)\,dY = G(X) + C_N(\vec{\varepsilon}).$$

Moreover, by the triangle inequality for d_R, $B_\infty(Z,\vec{\varepsilon}-\vec{t})\subseteq B_\infty(X,\vec{\varepsilon})\subseteq\mathbb{S}^N\setminus\Sigma$ and we thus have

$$⨍_{B_\infty(Z,\vec{\varepsilon}-\vec{t})} G(Y)\,dY = G(Z) + C_N(\vec{\varepsilon}-\vec{t}).$$

We then get

$$G(Z) = \frac{1}{Vol(B_\infty(Z,\vec{\varepsilon}-\vec{t}))}\int_{B_\infty(Z,\vec{\varepsilon}-\vec{t})} G(Y)\,dY - C_N(\vec{\varepsilon}-\vec{t}) \underset{G\ge 0}{\le}$$

$$\frac{1}{Vol(B_\infty(Z,\vec{\varepsilon}-\vec{t}))}\int_{B_\infty(X,\vec{\varepsilon})} G(Y)\,dY - C_N(\vec{\varepsilon}-\vec{t}) =$$

$$\frac{Vol(B_\infty(X,\vec{\varepsilon}))}{Vol(B_\infty(Z,\vec{\varepsilon}-\vec{t}))}⨍_{B_\infty(X,\vec{\varepsilon})} G(Y)\,dY - C_N(\vec{\varepsilon}-\vec{t}) =$$

$$\frac{Vol(B_\infty(X,\vec{\varepsilon}))}{Vol(B_\infty(Z,\vec{\varepsilon}-\vec{t}))}\left(G(X)+C_N(\vec{\varepsilon})\right)-C_N(\vec{\varepsilon}-\vec{t}).$$

The corollary follows from $G(X)=\mathcal{E}(X)-\alpha$. □

Corollary 5.2. *Let $X\in\mathbb{S}^N\setminus\Sigma$ and let $\varepsilon\in(0,\pi/2)$ be such that*

$$B_\infty(X,(\varepsilon,\dots,\varepsilon))\subseteq\mathbb{S}^N\setminus\Sigma.$$

Let

$$\alpha=\inf\left(\mathcal{E}(Z):Z\in B_\infty(X,(\varepsilon,\dots,\varepsilon))\right).$$

Then,

$$\|D\mathcal{E}(X)\|\leq\frac{2\sqrt{N}}{\tan\varepsilon}\left(\mathcal{E}(X)-\alpha-(N-1)\log(\cos\varepsilon)\right).$$

Proof. First, note that

$$B_\infty(X,(\varepsilon,0,\dots,0))\subseteq B_\infty(X,(\varepsilon,\dots,\varepsilon))\subseteq\mathbb{S}^N\setminus\Sigma.$$

For a unit norm vector $v_1\in T_{x_1}\mathbb{S}$, let $v=(v_1,0,\dots,0)$. Then,

$$D\mathcal{E}(X)(v)=\lim_{s\to0^+}\frac{\mathcal{E}(\gamma_{X,v}(s))-\mathcal{E}(X)}{s}.$$

Now, for $0\leq s<\varepsilon$ we have $\gamma_{X,v}(s)\in B_\infty(X,(\varepsilon,0,\dots,0))$ and thus taking $\vec{\varepsilon}=(\varepsilon,0,\dots,0)$ and $\vec{t}=(s,0,\dots,0)$ in Corollary 5.1, we conclude that this last limit is at most

$$\limsup_{s\to0}\frac{\alpha+\frac{Vol(B_\infty(X,(\varepsilon,0,\dots,0))}{Vol(B_\infty(X,(\varepsilon-s,0,\dots,0)))}\left(\mathcal{E}(X)-\alpha+C_N(\vec{\varepsilon})\right)-C_N(\varepsilon-s,0,\dots,0)-\mathcal{E}(X)}{s}=$$

$$\limsup_{s\to0}\frac{\alpha+\frac{\sin^2\varepsilon}{\sin^2(\varepsilon-s)}\left(\mathcal{E}(X)-\alpha+C_N(\vec{\varepsilon})\right)-C_N(\varepsilon-s,0,\dots,0)-\mathcal{E}(X)}{s}=$$

$$\limsup_{s\to0}\left((\mathcal{E}(X)+C_N(\varepsilon)-\alpha)\frac{\left(\frac{\sin^2\varepsilon}{\sin^2(\varepsilon-s)}-1\right)}{s}+\frac{C_N(\vec{\varepsilon})-C_N(\varepsilon-s,0,\dots,0)}{s}\right)=$$

$$(\mathcal{E}(X)+C_N(\varepsilon)-\alpha)\phi_1'(0)-\phi_2'(0),$$

where ϕ_1,ϕ_2 are the real functions defined by

$$\phi_1(s)=\frac{\sin^2\varepsilon}{\sin^2(\varepsilon-s)},\qquad\phi_2(s)=C_N(\varepsilon-s,0,\dots,0)=(N-1)\left(\frac{1}{2}+\frac{\log(\cos(\varepsilon-s))}{\tan^2(\varepsilon-s)}\right).$$

Now, elementary calculus yields

$$\phi_1'(0)=\frac{2}{\tan\varepsilon},\qquad\phi_2'(0)=\frac{2(N-1)}{\tan\varepsilon}\left(\frac{1}{2}+\frac{\log(\cos\varepsilon)}{\sin^2\varepsilon}\right).$$

We thus conclude,

$$D\mathcal{E}(X)(v)\leq\frac{2}{\tan\varepsilon}\left(\mathcal{E}(X)+(N-1)\left(\frac{1}{2}+\frac{\log(\cos\varepsilon)}{\tan^2\varepsilon}\right)-\alpha-(N-1)\left(\frac{1}{2}+\frac{\log(\cos\varepsilon)}{\sin^2\varepsilon}\right)\right)=$$

$$\frac{2}{\tan\varepsilon}\left(\mathcal{E}(X)-\alpha-(N-1)\log(\cos\varepsilon)\right).$$

Because v_1 is arbitrary, we can assert the same claim about $-v_1$ which yields

$$|D\mathcal{E}(X)(v)|\leq\frac{2}{\tan\varepsilon}\left(\mathcal{E}(X)-\alpha-(N-1)\log(\cos\varepsilon)\right).$$

Finally, a symmetric argument proves the same inequality for $v = (0, \dots, 0, v_i, 0, \dots, 0)$ with v_i in the $i - th$ position being a unit norm vector in $T_{x_i}\mathbb{S}$.

Now, let $v = (v_1, \dots, v_N) \in T_X\mathbb{S}^N$. Then, we have

$$|D\mathcal{E}(X)v|^2 \leq N \sum_{i=1}^{N} |D\mathcal{E}(X)(0, \dots, 0, \overset{(i^{th} position)}{v_i}, 0, \dots, 0)|^2 \leq$$

$$N \left(\frac{2}{\tan \varepsilon} \left(\mathcal{E}(X) - \alpha - (N-1) \log(\cos \varepsilon) \right) \right)^2 \sum_{i=1}^{N} \|v_i\|^2 =$$

$$N \left(\frac{2}{\tan \varepsilon} \left(\mathcal{E}(X) - \alpha - (N-1) \log(\cos \varepsilon) \right) \right)^2 \|v\|^2,$$

which implies the claim of the theorem.

□

Corollary 5.3. *Let $X \in \mathbb{S}^N$ and let $\varepsilon > 0$ be such that*

$$B_\infty(X, (\varepsilon, \dots, \varepsilon)) \subseteq \mathbb{S}^N \setminus \Sigma,$$

and let

$$\alpha = \inf \left(\mathcal{E}(Z) : Z \in B_\infty(X, \vec{\varepsilon}) \right).$$

Then,

$$\|D\mathcal{E}(X)\| \leq \frac{2\sqrt{N}}{\varepsilon} \left(\mathcal{E}(X) - \alpha \right) + \sqrt{N}(N-1)\varepsilon.$$

Moreover, if $t \leq \alpha$ and

$$B_\infty(X, \vec{\varepsilon}) \subseteq \mathbb{S}^N \setminus \Sigma \text{ for } \varepsilon = \sqrt{\frac{2(\mathcal{E}(X) - t)}{N-1}},$$

then

$$\|D\mathcal{E}(X)\| \leq 2\sqrt{2(\mathcal{E}(X) - t)N(N-1)}.$$

Proof. The first part of the corollary is direct from Corollary 5.2 using that

$$\tan \varepsilon \geq \varepsilon, \qquad -\frac{2}{\tan \varepsilon} \log(\cos \varepsilon) \leq \varepsilon, \qquad 0 \leq \varepsilon \leq \frac{\pi}{2}.$$

For the second part, note that

$$\|D\mathcal{E}(X)\| \leq \frac{2\sqrt{N}}{\varepsilon} \left(\mathcal{E}(X) - \alpha \right) + \sqrt{N}(N-1)\varepsilon \leq \frac{2\sqrt{N}}{\varepsilon} \left(\mathcal{E}(X) - t \right) + \sqrt{N}(N-1)\varepsilon.$$

and plug in the value $\varepsilon = \sqrt{\frac{2(\mathcal{E}(X)-t)}{N-1}}$ in this last formula. □

5.1.1. *Proof of Theorem 1.6.* Just apply Corollary 5.3 with $t = m_N$.

5.2. **Admissible error for Smale's 7th problem.** Now we prove Theorem 1.8. We need some intermediate results.

Lemma 5.4. *Let $A \geq 0$, $T > 0$ and $\alpha : [0, T) \to \mathbb{R}$ be an absolutely continuous function, such that*

$$\alpha'(t) \leq 2\sqrt{A(\alpha(t) - \alpha(0))}, \qquad a.e.\ t \in (0, T).$$

Then, for $t \in [0, T]$ we have

$$\alpha(t) \leq \alpha(0) + At^2.$$

Proof. Let $\delta > 0$ be an arbitrary number. Then, note that

$$\alpha'(t) \leq 2\sqrt{A(\alpha(t) - \alpha(0) + \delta)}, \qquad a.e.\ t \in (0,T),$$

which we may rewrite as

$$\frac{d}{dt}\sqrt{\alpha(t) - \alpha(0) + \delta} = \frac{\alpha'(t)}{2\sqrt{\alpha(t) - \alpha(0) + \delta}} \leq \sqrt{A}, \qquad a.e.\ t \in (0,T).$$

The fundamental theorem of Calculus then yields

$$\sqrt{\alpha(t) - \alpha(0) + \delta} \leq \sqrt{\delta} + \sqrt{A}\, t, \qquad t \in [0,T].$$

Thus,

$$\alpha(t) \leq \alpha(0) - \delta + \left(\sqrt{\delta} + \sqrt{A}\, t\right)^2 = \alpha(0) + At^2 + 2\sqrt{\delta}\sqrt{A}t, \qquad t \in [0,T].$$

As δ is arbitrary, we may take the limit as δ goes to 0 and the claim of the lemma follows. □

Corollary 5.5. *Let $U \subseteq \mathbb{S}^N \setminus \Sigma$ be an open set. Let*

$$X = (x_1, \ldots, x_N) \in \operatorname{argmin}_U(\mathcal{E}), \qquad \textit{if it exists.}$$

Let $t_0 > 0$ be such that $B_\infty(X, (\varepsilon, \ldots, \varepsilon)) \subseteq U$ where $\varepsilon = t_0(1 + 2N)$. Then, for every $Z \in B_\infty(X, \vec{t_0})$ we have

$$\mathcal{E}(Z) \leq \mathcal{E}(X) + 2N^2(N-1)t_0^2$$

Proof. We consider the coordinates of X and Z, $X = (x_1, \ldots, x_N)$ and $Z = (z_1, \ldots, z_N)$. Now, let

$$Z(t) = (z_1(t), \ldots, z_N(t)),$$

where $z_i(t)$ is the geodesic in $\mathbb{S}$ such that $z_i(0) = x_i, z_i(d_R(x_i, z_i)) = z_i$. Also, let $z_i(t) = z_i$ for $t \geq d_R(x_i, z_i)$. Thus, we have

$$\left\| \frac{d}{dt} z_i(t) \right\| \leq 1, \qquad \left\| \frac{d}{dt} Z(t) \right\| \leq \sqrt{N},$$

and

$$d_R(z_i(t), x_i) \leq t, \qquad \forall\, t \in [0, \infty),\ i \in \{1, \ldots, N\}.$$

We now let

$$\varphi(s) = \sqrt{\frac{2(\mathcal{E}(Z(s)) - \mathcal{E}(X))}{N-1}}, s \geq 0,$$
$$T = \sup_{[0,\infty)} \left(t : s + \varphi(s) \leq \varepsilon\ \forall s \in [0,t)\right).$$

We first note that $\varphi(0) = 0 < \varepsilon$ and thus $T > 0$. We claim that

$$T \geq \frac{\varepsilon}{1 + 2N} = t_0.$$

Note that for $t \in [0,T)$ we have:

$$B_\infty\left(Z(t), (\varphi(t), \ldots, \varphi(t))\right) \subseteq B_\infty\left(X, (t + \varphi(t), \ldots, t + \varphi(t))\right) \subseteq$$
$$B_\infty\left(X, (\varepsilon, \ldots, \varepsilon)\right) \subseteq U.$$

We are thus under the hypothesis of the second part of Corollary 5.3 (with $t = \mathcal{E}(X)$ in the notations of that corollary) and we have:

$$\frac{d}{dt}(\mathcal{E}(Z(t))) \leq \|D\mathcal{E}(Z(t))\| \left\| \frac{d}{dt} Z(t) \right\| \leq 2\sqrt{2N^2(\mathcal{E}(Z(t)) - \mathcal{E}(X))(N-1)}.$$

Taking $\alpha(t) = \mathcal{E}(Z(t))$ in Lemma 5.4 we conclude that

$$\mathcal{E}(Z(t)) \leq \mathcal{E}(X) + 2N^2(N-1)t^2, \qquad \forall\, t \in [0, T), \tag{5.1}$$

and by continuity of these mappings we get

$$\mathcal{E}(Z(T)) \leq \mathcal{E}(X) + 2N^2(N-1)T^2.$$

This implies that

$$T + \varphi(T) \leq T + \sqrt{\frac{2(2N^2(N-1))T^2}{N-1}} = (1+2N)T.$$

Now, if $T < \varepsilon/(1+2N)$ then

$$T + \varphi(T) \leq (1+2N)T < \varepsilon,$$

and by continuity we have that

$$t + \varphi(t) < \varepsilon \text{ for } t \text{ in some interval } [T, T+\delta),$$

which contradicts the definition of T as a supremum. We thus conclude that

$$T \geq \frac{\varepsilon}{1+2N} = t_0,$$

and (5.1) then finishes the proof. □

The following result is a version of the well–known Gronwall's Inequality. See [25, Th. 4.1] for a general version.

Lemma 5.6. *Let $\alpha : [a,b] \to \mathbb{R}$ be an absolutely continuous function and assume that*

$$\alpha'(t) \leq f_1(t)\alpha(t) + f_2(t) \qquad \text{a.e. } t \in (a,b)$$

for some continuous functions $f_1, f_2 : [a,b] \to \mathbb{R}$. Then,

$$\alpha(t) \leq \frac{1}{\rho(t)} \left(\alpha(a) + \int_a^t \rho(s) f_2(s)\, ds \right) \qquad \forall\, t \in [a,b],$$

where

$$\rho(t) = e^{-\int_a^t f_1(s)\, ds}, \qquad t \in [a,b].$$

Corollary 5.7. *Let $U \subseteq \mathbb{S}^N \setminus \Sigma$ be an open set. Let*

$$X = (x_1, \ldots, x_N) \in \operatorname{argmin}_U(\mathcal{E}), \qquad \text{if it exists.}$$

Let $t > 0$ be such that $B_\infty(X, \vec{t}) \subseteq U$. Then, for every $Z \in B_\infty(X, \vec{t})$ we have

$$\mathcal{E}(Z) \leq \mathcal{E}(X) + \left(\frac{d}{d-2t} \right)^{2N} \frac{N-1}{8} d^2,$$

where d is any positive number such that $2t < d \leq \mathrm{d_{sep}}(X)$.

Proof. As in the proof of Corollary 5.5 we let

$$Z(t) = (z_1(t), \ldots, z_N(t)),$$

where $z_i(t)$ is the geodesic in $\mathbb{S}$ such that $z_i(0) = x_i$, $z_i(d_R(x_i, z_i)) = z_i$, and $z_i(t) = z_i$ for $t \geq d_R(x_i, z_i)$. Thus, we have

$$\left\| \frac{d}{dt} z_i(t) \right\| \leq 1,$$

and

$$d_R(z_i(t), x_i) \leq t, \qquad \forall\, t \in [0, \infty),\ i \in \{1, \ldots, N\}.$$

Because $0 \le t < d/2 \le \mathrm{d}_{\mathrm{sep}}(X)/2$, we have

$$B_\infty\left(Z(t), \left(\frac{d}{2}-t,\dots,\frac{d}{2}-t\right)\right) \subseteq B\left(X, \left(\frac{\mathrm{d}_{\mathrm{sep}}(X)}{2},\dots,\frac{\mathrm{d}_{\mathrm{sep}}(X)}{2}\right)\right) \subseteq \mathbb{S}^N \setminus \Sigma,$$

which from Corollary 5.3 implies

$$\|D\mathcal{E}(Z)\| \le \frac{2\sqrt{N}}{\frac{d}{2}-t}\left(\mathcal{E}(Z(t)) - \alpha\right) + \sqrt{N}(N-1)\left(\frac{d}{2}-t\right),$$

where

$$\alpha = \inf\left(\mathcal{E}(Y) : Y \in B_\infty\left(Z(t), \left(\frac{d}{2}-t,\dots,\frac{d}{2}-t\right)\right)\right) \ge \mathcal{E}(X)$$

Thus,

$$\frac{d}{dt}\mathcal{E}(Z(t)) \le \|D\mathcal{E}(Z)\| \left\|\frac{d}{dt}Z(t)\right\| \le$$

$$\left(\frac{2}{\frac{d}{2}-t}\left(\mathcal{E}(Z(t)) - \mathcal{E}(X)\right) + (N-1)\left(\frac{d}{2}-t\right)\right) N.$$

Taking $\alpha(t) = \mathcal{E}(Z(t)) - \mathcal{E}(X)$, this last reads

$$\alpha'(t) \le f_1(t)\alpha(t) + f_2(t), \qquad \alpha(0) = 0,$$

where

$$f_1(t) = \frac{2N}{\frac{d}{2}-t}, \qquad f_2(t) = (N-1)\left(\frac{d}{2}-t\right)N.$$

Note that

$$\int_0^t f_1(s)\,ds = 2N \log\frac{d}{d-2t}.$$

Thus,

$$e^{-\int_0^t f_1(s)\,ds} = \left(\frac{d-2t}{d}\right)^{2N}.$$

From Lemma 5.7 we conclude that

$$\alpha(t) \le \left(\frac{d}{d-2t}\right)^{2N}\left(\frac{N(N-1)}{2d^{2N}}\int_0^t (d-2s)^{2N+1}\,ds\right) =$$

$$\left(\frac{d}{d-2t}\right)^{2N}\left(\frac{N(N-1)d^2}{8(N+1)}\left(1 - \left(\frac{d-2t}{d}\right)^{2N+2}\right)\right) \le$$

$$\left(\frac{d}{d-2t}\right)^{2N}\frac{N-1}{8}d^2,$$

and the corollary follows. □

Lemma 5.8. *Let X be a global minimizer of $\mathcal{E}$. Let $a > 0$ and let*

$$t \ge \sqrt{\frac{2a}{N(N-1)}}.$$

Then, there exists $Z \in B_\infty(X, \vec{t})$ such that

$$\mathcal{E}(X) > m_N + a.$$

Proof. If $B_\infty(X,(t,\ldots,t))\cap\Sigma\neq\emptyset$ then the claim is trivial because there exists a sequence X_n contained in $B_\infty(X,(t,\ldots,t))$ such that $\mathcal{E}(X_n)\overset{n}{\to}\infty$. Assume thus that $B_\infty(X,(t,\ldots,t))\subseteq\mathbb{S}^N\setminus\Sigma$. Let

$$S_\infty(X,\vec{t})=S(x_1,t)\times\cdots\times S(x_N,t)\subseteq B_\infty(X,\vec{t}).$$

From Corollary 3.3 we have

$$\fint_{S_\infty(X,\vec{t})}\mathcal{E}(Y)\,dY=m_N-N(N-1)\log(\cos t)>m_N+\frac{N(N-1)t^2}{2},$$

where the last inequality follows from the fact that

$$\frac{d}{dt}\left(-\log(\cos t)-\frac{t^2}{2}\right)=\tan t-t>0,\qquad t>0.$$

From the mean value theorem for integrals, there exists a point $Z\in S_\infty(X,\vec{t})$ such that $\mathcal{E}(Z)>m_N+N(N-1)t^2/2=a$ as demanded. $\square$

5.2.1. *Proof of Theorem 1.8.* We first prove the lower bound in (1.5). Let $U=\mathbb{S}^N\setminus\Sigma$ and X a global minimizer of $\mathcal{E}$ (i.e. $\mathcal{E}(X)=m_N$). Let

$$t\le\frac{d_N}{2(1+2N)},\ \text{which implies } B_\infty(X,(1+2N)\vec{t})\subseteq B_\infty(X,d_N/2)\subseteq\mathbb{S}^N\setminus\Sigma.$$

Then, from Corollary 5.5 we have

$$Z\in B_\infty(X,\vec{t})\implies\mathcal{E}(Z)\le m_N+2N^2(N-1)t^2.$$

Namely,

$$\mathbf{e}(2N^2(N-1)t^2)\ge t,\qquad\text{for } t\le\frac{d_N}{2(1+2N)}.$$

Let $s=2N^2(N-1)t^2$. Then, the last sentence reads

$$\mathbf{e}(s)\ge\sqrt{\frac{s}{2N^2(N-1)}},\qquad\text{for } s\le\frac{N^2(N-1)d_N^2}{2(1+2N)^2},$$

and the lower bound in (1.5) follows. The upper bound follows from Lemma 5.9.

Finally, (1.6) is proved the same way using Corollary 5.8 (with $d=d_N$) instead of Corollary 5.5.

6. Computability and complexity of Smale's 7th problem.

6.1. **Proof of Proposition 1.11.** Let X be a global minimizer of $\mathcal{E}$ and let

$$\varepsilon=\frac{1}{3\cdot1.733\cdot N\sqrt{N-1}}.$$

Let $x_i=(x_i^{(1)},x_i^{(2)},x_i^{(3)})$ where $x_i^{(j)}\in[-1,1]$ for $1\le i\le N$, $1\le j\le3$. Let

$$\tilde{x}_i=2x_i-(0,0,1)\in\{(a,b,c)\in\mathbb{R}^3:a^2+b^2+c^2=1\}.$$

From [18, Th. 2.1], for $1\le i\le N$ there exists a point with rational coordinates

$$\tilde{z}_i=\left(\frac{\tilde{p}_i^{(1)}}{\tilde{q}_i^{(1)}},\frac{\tilde{p}_i^{(2)}}{\tilde{q}_i^{(2)}},\frac{\tilde{p}_i^{(3)}}{\tilde{q}_i^{(3)}}\right)\in\mathbb{Q}^3,\ \|\tilde{z}_i\|=1,$$

such that for $1\le j\le3$ we have

$$\left|\frac{\tilde{p}_i^{(j)}}{\tilde{q}_i^{(j)}}-\tilde{x}_i^{(j)}\right|\le\varepsilon,\qquad0\le|\tilde{p}_i^{(j)}|\le\tilde{q}_i^{(j)}\le\left(\frac{128}{\varepsilon^2}\right)^2.$$

Note that this implies

$$2-2\langle\tilde z_i,\tilde x_i\rangle=\|\tilde z_i-\tilde x_i\|^2\leq 3\varepsilon^2,\qquad\Rightarrow\qquad \langle\tilde z_i,\tilde x_i\rangle\geq 1-\frac{3\varepsilon^2}{2},$$

that is

$$d_R(\tilde z_i,\tilde x_i)=\arccos(\langle\tilde z_i,\tilde x_i\rangle)\leq\arccos\left(1-\frac{3\varepsilon^2}{2}\right)\underset{\varepsilon<.003}{\leq}1.733\varepsilon.$$

Let $z_i=(\tilde z_i+(0,0,1))/2\in\mathbb{S}$. Then,

$$d_R(z_i,x_i)=\frac{1}{2}d_R(\tilde z_i,\tilde x_i)\leq\frac{1.733\varepsilon}{2}=\frac{1}{6N\sqrt{N-1}}.$$

From Corollary 1.9 this implies that

$$\mathcal{E}(Z)\leq m_N+\frac{1}{18}.$$

Now, note that

$$z_i=\left(\frac{p_i^{(1)}}{q_i^{(1)}},\frac{p_i^{(2)}}{q_i^{(2)}},\frac{p_i^{(3)}}{q_i^{(3)}}\right),$$

where

$$\begin{array}{rcll}\frac{p_i^{(j)}}{q_i^{(j)}}&=&\frac{\tilde p_i^{(j)}}{2\tilde q_i^{(j)}}, & 1\leq j\leq 2,\\ \frac{p_i^{(3)}}{q_i^{(3)}}&=&\frac{\tilde p_i^{(3)}+\tilde q_i^{(3)}}{2\tilde q_i^{(3)}}.\end{array}$$

We thus have

$$|p_i^{(j)}|\leq q_i^{(j)}\leq 2\tilde q_i^{(j)}\leq 2\left(\frac{128}{\varepsilon^2}\right)^2\leq (cN)^6,$$

($c=17$ suffices) and the corollary follows.

6.2. **Proof of Corollary 1.12.** Given N, let $Z_1,Z_2,\ldots,Z_l$ be all the possible ways to choose N points in $\mathbb{S}$ with rational coordinates of the form p/q where $0\leq|p|\leq q\leq(17N)^6$. Thus, each Z_i is a N–tuple of vectors in $\mathbb{Q}^3$, and we can bound the number of such N–tuples as follows:

(1) There are at most $2(17N)^6+1$ choices for each of the integer numbers in the numerator of the coordinates.
(2) There are at most $(17N)^6+1$ choices for each of the integer numbers in the denominator of the coordinates.
(3) Thus, there are at most $(20N)^{12}$ (the constant 20 is an overestimate, but we don't search for the best number here) choices for each coordinate in each vector in each Z_i, and hence there are $(20N)^{36}$ choices for such vectors.
(4) Thus, there are at most $(20N)^{36N}$ choices of Z, and hence, we can enumerate them all as $Z_1,\ldots,Z_l$ with $l\leq(20N)^{36N}$.

Consider a Turing machine that on input N, generates $Z_1,\ldots,Z_l$. For each of them, it computes the product of the square of the mutual distances of the points in $\mathbb{S}$ it defines. Finally, the machine outputs the N–tuple Z_k with biggest such value. From Proposition 1.11, we have that $\mathcal{E}(Z_k)\leq m_N+1/18$ and thus satisfies (1.2).

The running time of this machine is *polynomial*$(N)\cdot(20N)^{36N}$, for each loop takes polynomial time: one just has to perform $N(N-1)/2$ multiplications of numbers obtained as the squared norms of vectors with rational coordinates of bit length bounded by $O(\log_2 N)$. This can be done in running time polynomial in N. This finishes the proof.

Appendix A. Topics from Riemannian Geometry and Harmonic Analysis in manifolds

To facilitate the reading of this manuscript, we include two short sections with some results from Riemannian Geometry and Harmonic Analysis that have been used in the paper. The contents of this appendix are mainly taken from [8, 14].

A.1. **Riemannian Geometry.** By a "Riemannian manifold" $\mathcal{M}$ we mean here a smooth (C^∞) differentiable manifold with a smooth Riemannian structure, that is a smooth section $\langle\cdot,\cdot\rangle : T\mathcal{M}\times T\mathcal{M}\to\mathbb{R}$ where $T\mathcal{M}$ is the tangent bundle of $\mathcal{M}$ and for each $p\in\mathcal{M}$ $\langle\cdot,\cdot\rangle_p : T_p\mathcal{M}\times T_p\mathcal{M}\to\mathbb{R}$ is a definite positive, symmetric bilinear map. We denote by n the dimension of $\mathcal{M}$. Associated to $\langle\cdot,\cdot\rangle_p$ we consider the norm $\|v\|_p=\langle v,v\rangle_p^{1/2}$ for $v\in T_p\mathcal{M}$. We denote by $C^2(U)$ the set of C^2 functions defined in some open set $U\subseteq\mathcal{M}$.

Given a collection $\mathcal{M}^{(1)},\ldots,\mathcal{M}^{(r)}$ of Riemannian manifolds we can define in $\mathcal{M}=\mathcal{M}^{(1)}\times\cdots\times\mathcal{M}^{(r)}$ a product Riemannian structure as follows: Let $p=(p^{(1)},\ldots,p^{(r)})\in\mathcal{M}$ and let $v^{(i)},w^{(i)}\in T_{p^{(i)}}\mathcal{M}^{(i)}$, $1\le i\le r$ be a collection of tangent vectors. Then, define

$$\langle (v^{(1)},\ldots,v^{(r)}),(w^{(1)},\ldots,w^{(r)})\rangle_p=\langle v^{(1)},w^{(1)}\rangle_{p^{(1)}}+\cdots+\langle v^{(r)},w^{(r)}\rangle_{p^{(r)}}.$$

The length of a piecewise C^1 curve $\gamma:[a,b]\to\mathcal{M}$ with tangent vector $\dot\gamma$ (i.e. $\dot\gamma(t)\in T_{\gamma(t)}\mathcal{M}, t\in[a,b]$) is defined as

$$L(\gamma)=\int_a^b\|\dot\gamma\|\,dt.$$

The distance between two points $p,q\in\mathcal{M}$ is then defined as the infimum of the lengths of piecewise C^1 curves with extremes p,q. This gives $\mathcal{M}$ a structure of metric space and allows us to define open and closed balls as usual.

Given a coordinate chart $\phi : U\subseteq\mathcal{M}\to V\subseteq\mathbb{R}^n$, the Riemannian product is represented by a positive definite, symmetric matrix

$$G(x)=(g_{ij}(x)),\qquad x\in V,$$

such that given $v,w\in T_p\mathcal{M}$ with $p\in U$, $v=(v_1,\ldots,v_n)$ and $w=(w_1,\ldots,w_n)$ in coordinates, we have

$$\langle v,w\rangle_x=w^TG(x)v,$$

where w^T is the transpose of w. A function $f:U\to\mathbb{R}$ is called (Lebesgue–) measurable or integrable if $f(\phi^{-1}(x))|\det(DG(x))|^{1/2}$ is (Lebesgue–) measurable or integrable as a function of $x\in V$. In that case, the integral of f in U is defined as

$$\int_U f(p)\,dp=\int_V f(\phi^{-1}(x))|\det(DG(x))|^{1/2}\,dx$$

Given $f:\mathcal{M}\to\mathbb{R}$ we define

$$\int_{\mathcal{M}} f(p)\,dp=\sum_\alpha\int_{U_\alpha} f(p)\rho_\alpha(p)\,dp,$$

where $\{\rho_\alpha\}$ is a partition of the unity subordinate to some open cover of $\mathcal{M}$ by coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ (f is called measurable or integrable it is so in each U_α). The volume of some measurable subset $U \subseteq \mathcal{M}$ is

$$\mathrm{Vol}(U) = \int_{\mathcal{M}} \chi_U(p)\, dp, \qquad \chi_U(p) = \begin{cases} 1 & p \in U, \\ 0 & \textit{otherwise.} \end{cases}$$

If $\mathrm{Vol}(U) < \infty$ we define the expected value of $f : U \to \mathbb{R}$ in U as

$$\fint_U f(p)\, dp = \frac{1}{\mathrm{Vol}(U)} \int_U f(p)\, dp.$$

We denote by g^{ij} the components of the inverse matrix $G(x)^{-1}$. The Chirstoffel symbols associated to ϕ are then

$$\Gamma^i_{jk} = \frac{1}{2} \sum_{l=1}^{n} g^{il} \left(\frac{\partial g_{jl}}{\partial x_k} + \frac{\partial g_{kl}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_l} \right).$$

A smooth curve $\gamma : [a, b] \to \mathcal{M}$ is a geodesic if it is a critical point of the energy functional $\int_a^b \|\dot\gamma\|^2\, dt$. In coordinates, denoting $x(t) = (x_1(t), \ldots, x_n(t)) = \phi(\gamma(t))$, for $i \in \{1, \ldots n\}$,

$$\ddot{x}_i(t) = -\dot{x}(t)^T \Gamma^i(x(t)) \dot{x}(t), \qquad \Gamma^i = (\Gamma^i_{jk})_{j,k=1\ldots n}.$$

Given $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$ there exists $\varepsilon > 0$ and a unique geodesic $\gamma : [0, \varepsilon] \to \mathcal{M}$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. From the geodesic equation above we can easily see that if $\mathcal{M}^{(1)} \times \cdots \times \mathcal{M}^{(r)}$ has the product structure, then a curve $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(r)}) : [a, b] \to \mathcal{M}$ is a geodesic in $\mathcal{M}$ if and only if $\gamma^{(i)} : [a, b] \to \mathcal{M}^{(i)}$ is a geodesic in $\mathcal{M}^{(i)}$ for $i \in \{1, \ldots, r\}$. If $\mathcal{M} \subseteq \mathbb{R}^k$ is a smooth submanifold of $\mathbb{R}^k$ with the Riemannian structure inherited from $\mathbb{R}^k$ (that is, $\langle v, w \rangle_p = \langle v, w \rangle$ the usual inner product in $\mathbb{R}^k$), the geodesic equation reads

$$\|\dot\gamma(t)\| = \text{constant}, \qquad \ddot\gamma(t) \perp T_{\gamma(t)}\mathcal{M} \subseteq \mathbb{R}^k, \qquad \text{for all } t.$$

We are most interested in the cases $\mathcal{M} = \mathbb{S}$ with the Riemannian structure inherited from $\mathbb{R}^3$ and $\mathcal{M} = \mathbb{S}^N = \mathbb{S} \times \cdots \times \mathbb{S}$ (N times) with the product Riemannian structure. Of course, this product structure is also the Riemannian structure of $\mathbb{S}^N$ as a submanifold of $\mathbb{R}^{3N}$. Note that $\mathbb{S}$ has dimension 2 and $\mathbb{S}^N$ has dimension $2N$. Geodesics in $\mathbb{S}$ are great circles parametrized with constant speed. More exactly, the geodesic $\gamma(t)$ such that $\gamma(0) = p \in \mathbb{S}$ and $\dot\gamma(0) = v \in T_p\mathbb{S}$ where $\|v\| = 1$ is given by

$$\gamma_{p,v}(t) = p \cos(2t) + \frac{v}{2} \sin(2t) + \frac{1}{2}(0, 0, 1 - \cos(2t))^T. \tag{A.1}$$

Indeed, the reader may check that $\gamma(0) = p, \dot\gamma(0) = v, \dot\gamma(t) \perp T_{\gamma(t)}\mathbb{S}, \|\dot\gamma(t)\| = 1$ for all t.

A.2. **Harmonic Analysis in manifolds.** The Hessian of a function $f \in C^2(U)$ is a bilinear form defined as

$$\mathrm{Hess}(\mathcal{E})(p)(v, w) = X(Y(f)) - (\nabla_X Y)(f), \qquad p \in U, v, w \in T_p\mathcal{M},$$

where ∇ is the Levi–Civita connection and X, Y are vector fields such that $X(p) = v$, $Y(p) = w$. In coordinates,

$$\operatorname{Hess}(f)(p(x))(v,w) = w^t(h_{ij}(x))v, \qquad h_{ij}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^n \frac{\partial f}{\partial x_k}\Gamma_{ij}^k.$$

If $\gamma(t)$ is the unique geodesic such that $\gamma(0) = p$ and $\dot\gamma(0) = v$ then

$$\operatorname{Hess}(f)(p)(v,v) = \frac{d^2}{dt^2}\mid_{t=0}(f(\gamma(t)).$$

The Laplacian[3] $\Delta f = \operatorname{div}(\operatorname{grad}(\mathcal{E}))$ of f is the trace of $\operatorname{Hess}(f)$ as a bilinear operator. That is, the trace of (h_{ij}) or in coordinates,

$$\Delta f = \sum_{i=1}^n h_{ii}(x) = \sum_{i=1}^n\left(\frac{\partial^2 f}{\partial x_i \partial x_i}\right) - \sum_{k=1}^n \frac{\partial f}{\partial x_k}\left(\sum_{i=1}^n \Gamma_{ii}^k\right).$$

A function $f \in C^2(U)$, $U \subseteq \mathcal{M}$ open, is called harmonic (subharmonic) if $\Delta f = 0$ ($\Delta f \geq 0$). Note that Δ is a uniformly elliptic operator and satisfies the hypotheses of [13, Sec. 24]. Thus, the classical strong maximum principle applies. Namely,

Theorem A.1 (Strong maximum principle)**.** *Let $f \in C^2(U)$ be a non–constant subharmonic function. Then for any open set $\Omega \subseteq U$ such that $\overline{\Omega} \subseteq U$ we have*

$$x \in \Omega \Rightarrow f(x) < \sup_{x\in\Omega} f.$$

In particular, the supremum is reached only in the boundary of Ω.

The trace of (h_{ij}) is equal to the trace of $Q^*(h_{ij})Q$ for any orthogonal matrix Q. Hence, we also have

$$\Delta f = \sum_{v^{(1)},\ldots,v^{(2N)} \text{ an orthonormal basis of } T_x\mathbb{S}^N} \operatorname{Hess}(f)(x)(v^{(i)}, v^{(i)})$$

The classical mean value theorem for harmonic functions in $\mathbb{R}^n$ (see for example [11, sec. 2.2]) claims that, if $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is harmonic (U and open set) then $f(x) = ⨍_{B(x,\varepsilon)} f(y)\,dy$ for each ball $B(x,\varepsilon) \subseteq U$. Unfortunately, no similar equality is valid in general for harmonic functions on Riemannian manifolds, but there is a class of manifolds for which it holds. These are called locally harmonic manifolds, and the corresponding mean value equality was first proved by Willmore [22] (see also [5, Ch. 6].) Following the proof of [5, Prop. 6.21], if $\mathcal{M}$ is a locally harmonic manifold and $f : \mathcal{M} \to \mathbb{R}$ is such that $\Delta f = C$ is constant, then for small enough $t > 0$ we have

$$\text{(A.2)} \qquad \frac{d}{dt} ⨍_{S(x,t)} f = C\frac{Vol(B(x,t))}{Vol(S(x,t))}.$$

The sphere $\mathbb{S} \subseteq \mathbb{R}^3$ is locally harmonic, and moreover (A.2) can be stated for every $t > 0$ such that f is defined in $B(x,t)$. Thus, the classical mean value theorem is valid in the case of $\mathbb{S} = \mathcal{M}$:

[3] As a linear operator in $C^2(\mathcal{M})$, Δ is called the Laplace–Beltrami operator. Note that for some authors $\Delta f = -\operatorname{div}(\operatorname{grad}(f))$ and hence has a minus sign, see for example [14, p. 88].

Theorem A.2. *Let $f : U \to \mathbb{R}$ ($U \subseteq \mathbb{S}$ an open set) be such that $\Delta f = C$. Let $p \in U$. Then, for every $\varepsilon > 0$ such that $B_p(\varepsilon) \subseteq U$, we have*

$$\fint_{S(p,\varepsilon)} f = f(p) + C \int_0^{\varepsilon} \frac{\pi \sin^2 t}{\pi \sin(2t)} = f(p) - \frac{C}{2} \log(\cos \varepsilon).$$

References

[1] D. Armentano, C. Beltrán, and M. Shub, *Minimizing the discrete logarithmic energy on the sphere: The role of random polynomials*, Trans. Amer. Math. Soc. **363** (2011), no. 6, 2955–2965.

[2] E. Bendito, A. Carmona, A. M. Encinas, J. M. Gesto, A. Gómez, C. Mouriño, and M. T. Sánchez, *Computational cost of the Fekete problem. I. The forces method on the 2-sphere*, J. Comput. Phys. **228** (2009), no. 9, 3288–3306.

[3] E. Bendito, A. Carmona, A.M. Encinas, and J.M. Gesto, *Computational cost of the Fekete Problem II: on Smale's 7th Problem*, To appear. Available at *http://www-ma3.upc.es/users/bencar/papers.html*

[4] B. Bergersen, D. Boal, and P. Palffy-Muhoray, *Equilibrium configurations of particles on a sphere: the case of logarithmic interactions*, J. Phys. A: Math. Gen. **27** (1994), 2579–2586.

[5] A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 93, Springer-Verlag, Berlin, 1978, With appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J. L. Kazdan.

[6] L. Blum, F. Cucker, M. Shub, and S. Smale, *Complexity and Real Computation*, Springer-Verlag, New York, 1998.

[7] L. Blum, M. Shub, and S. Smale, *On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines*, Bull. Amer. Math. Soc. (N.S.) **21** (1989), no. 1, 1–46.

[8] M.P. do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty.

[9] P. D. Dragnev, *On the separation of logarithmic points on the sphere*, Approximation Theory, X (St. Louis, MO, 2001), Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, 2002, pp. 137–144.

[10] A. Dubickas, *On the maximal product of distances between points on a sphere*, Liet. Mat. Rink. **36** (1996), no. 3, 303–312.

[11] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[12] P. J. Forrester, *Log-gases and random matrices*, Princeton University Press, Princeton, 2010.

[13] J. Jost, *Postmodern analysis*, third ed., Universitext, Springer-Verlag, Berlin, 2005.

[14] ———, *Riemannian geometry and geometric analysis*, fifth ed., Universitext, Springer-Verlag, Berlin, 2008.

[15] A. B. J. Kuijlaars and E. B. Saff, *Asymptotics for minimal discrete energy on the sphere*, Trans. Amer. Math. Soc. **350** (1998), 523–538.

[16] E. A. Rakhmanov, E. B. Saff, and Y. M. Zhou, *Minimal discrete energy on the sphere*, Math. Res. Letters **1** (1994), 647–662.

[17] ———, *Electrons on the sphere*, Computational Methods and Function Theory 1994 (Penang), Ser. Approx. Decompos., vol. 5, World Sci. Publ., River Edge, NJ, 1995, pp. 293–309.

[18] E. Schmutz, *Rational points on the unit sphere*, Cent. Eur. J. Math. **6** (2008), no. 3, 482–487.

[19] M. Shub and S. Smale, *Complexity of Bezout's theorem. III. Condition number and packing*, J. Complexity **9** (1993), no. 1, 4–14, Festschrift for Joseph F. Traub, Part I.

[20] S. Smale, *Mathematical problems for the next century*, Mathematics: Frontiers and Perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 271–294.

[21] L. L. Whyte, *Unique arrangements of points on a sphere*, Amer. Math. Monthly **59** (1952), 606–611.

[22] T. J. Willmore, *Mean value theorems in harmonic Riemannian spaces*, J. London Math. Soc. **25** (1950), 54–57.

[23] Q. Zhong, *Energies of zeros of random sections on Riemann surfaces*, Indiana Univ. Math. J. **57** (2008), no. 4, 1753–1780.
[24] Y. Zhou, *Arrangements of points on the sphere*, Ph. D. Thesis. Math. Department, University of South Florida, 1995.
[25] P. Hartman, *Ordinary differential equations*, John Wiley & Sons Inc., New York,1964.

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