The Moment of Truth, in Automatic Theorem Proving in Elementary Geometry

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(Extended Abstract)

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1 Truly true

While still being an active field of research, automatic theorem proving in elementary geometry has reached a certain status of maturity. Software and hardware requirements to successfully (and quickly) perform hundreds of interesting examples are now widespread available (such as GEX, Geother, GDI, etc.)

Yet, in recent years, a few papers ([BDR],[BFS], [CT1], [CT2]) have repeatedly expressed the need for reviewing some foundational aspects of the algebraic geometry via to automatic geometry theorem proving. Among these papers we will like to highlight the one by P. Conti and C. Traverso ([CT2]), devoted to unveil truth's inherent manifold structure in this context.

Actually, unraveling the notion of truth has always been a recurrent task in the realm of automatic proving of geometry theorems by algebraic geometry means. Practically all major contributors to this topic have felt obliged to devote quite a few lines to discuss on what (and, then, why) should be a correct definition for a "true theorem", we refer to the classic articles and books by D. Kapur ([Ka86], [Ka88]), by W.-T. Wu ([Wu78], [Wu82], [Wu84]), by D. Wang ([Wa98]) and by S.-C Chou ([Ch88]). In fact, what is at stake is not a mere scholastic digression, but a crucial issue: after all, automatic proving is expected to deal with proving *true* statements (and disproving false ones).

Roughly speaking, the algebraic geometry method towards theorem proving, proceeds translating thesis and hypothesis about geometric entities into systems of equations, say $\boldsymbol{H} = \{h_1 = 0, \ldots, h_r = 0\}$ and $\boldsymbol{T} = \{t_1 = 0, \ldots, t_s = 0\}$. Solutions (in a suitable field: there will be different interpretations for different choices of this field) for $\boldsymbol{H}$ (respectively, for $\boldsymbol{T}$), can be interpreted as geometric

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instances verifying the hypothesis (respectively, the thesis). In this scheme, it seems that a theorem will have to be declared true if and only if the algebraic set defined by $\boldsymbol{H}$ is included in the solution set for $\boldsymbol{T}$, since this inclusion expresses that all instances verifying the hypothesis satisfy, as well, the thesis. This is clear... but only apparently!

The problem originates within the process of translating geometric facts into algebraic formulae, and conversely. To mention a simple instance, when we visualize the situation behind a common expression such as "Given any triangle...", that appears in so many theorems, we probably mean much more than a collection of three arbitrary points and the segments joining them (which, in algebraic terms could be translated as a mere list of independent variables, two for each point). Often, we also want to predicate, implicitly, that none of the three vertices should coincide (which, in algebraic terms, requires imposing some inequations involving the indeterminates). And the problem is that, in many other, less simple, instances, it is not trivial at all to exhaustively list all the degenerate cases we are supposed to exclude from our formulation. Thus, if we adopt a straightforward definition of truth, such as having all solutions for $\boldsymbol{H}$ included in the solution for $\boldsymbol{T}$, briefly, $\boldsymbol{H} \subset \boldsymbol{T}$, certain (in fact, most) statements that "should" have got to be true (in geometric terms), will happen to be false when expressed in algebraic terms, because the inclusion will not hold over some degenerate cases that we might have forgotten to eliminate in the algebraic translation of the statement.

On the other hand, aiming towards a more realistic theory, we could accept considering as "true" those statements that are verified in some, but not all, instances; we might allow some exceptions to occur on a (well controlled) degenerate set of cases. For example, we could label as true those thesis that hold over "almost all" instances verifying our hypothesis; where the "almost" should be measured according to some precise, computational, criterion, such as

– considering all irreducible components, or just irreducible components of some special kind (so called non-degenerate components), of the algebraic set of solutions for the hypothesis equations.
– accepting as true those thesis that include some irreducible component, or some irreducible component of the special kind, or all irreducible components of the special kind (according to different proposals)

Let us add that, roughly speaking, the distinguished feature of the special components is related to the choice of a set of independent variables for the hypothesis variety, ie. a set of variables that freely parametrize the collection of all (or most of) given instances.

This approach, thought useful, does not provide a satisfactory solution to the truth definition issue. Because, the core of the problem is to be automatically confident about the correspondence of each of the named cases with our intuition. For instance, as in [BFS] the authors themselves point out, "generic truth reflects the idea that the conclusion holding on the special irreducible components of the hypotheses is what is "really intended" by the author of a theorem". It is evident that the meaning of the expression "really intended" is rather vague. That is why

Conti-Traverso have expressed the need to revisit and to make explicit not only the truth test, but also some other, perhaps less automatizable, steps in the proving method: that is, the complete *protocol* (as they named it) that starts with an informally stated theorem, produces a formal statement, decides its algebraic translation, tests (according to some chosen criterion) the truth or falsity of the algebraic version, produces side conditions that were overlooked in the formal statement and proposes a corrected version of the theorem as well as a formal proof for it. They have also made an important observation: ie. that it could be useful to consider different approaches to the concept of truth for different types of geometric theorems (of constructible type, of declarative type, and so on). Yet, the whole proposal is in its initial stages...

Our contribution, far from providing a complete answer to all involved issues, brings yet a different point of view into this problem, that of automatic discovery, that could yield light on the automatic proving case as well. Next section introduces a concrete procedure in this context and shows, through a couple of examples, the relevance (for discovery) of the idea of "interesting" variables. Last section attempts proposing through general goals (namely, by considering we will like to have a method that will do this and that...) a protocol for discovery. This leads to the concept of "full set of discovering interesting conditions" (FSDIC), that is discussed in that section, showing how it supports, under certain conditions, the procedure outlined in section 2. Finally, some limitations and counterexamples are also mentioned.

2 Automatic discovery and deduction

Automatic discovery addresses ([Ch84], [Ch90][RV96], [Wa98]) two different situations. First, the case in which one has a collection of hypotheses and wants to deduce some interesting properties whatsoever, that could be correctly derived from it. Second, when a collection of hypotheses and theses is given, and the statement might be wrong, but one wants to learn how to modify it –if needed– in order to have a true (and interesting) result.

Notice that the word "interesting" is, by no means, crucial in this approach. For one can always deduce a true consequence of any given set of hypothesis: namely, considering any of the given hypotheses as a valid consequence (since $\boldsymbol{H} \Rightarrow \boldsymbol{H}$ is always true). And, if we are given some hypotheses and an arbitrary thesis, adding the thesis itself to the collection of hypotheses turns the given statement into one trivially true (since $\{\boldsymbol{H} \wedge \boldsymbol{T}\} \Rightarrow \boldsymbol{T}$). But this is surely not the kind of answer we are expecting to receive from automatic discovery.

So we must tackle with the elusive concept of "interesting". Searching for a true result when the given statement is wrong, or deducing, without a given thesis, whatever correct consequences from the hypotheses, it seems that the user of an automatic proving program should point out, as part of user's preferences, some hints about his/her areas of interest.

Example 1. For instance, if we take as input any given triangle, with vertices $(a, b), (c, d), (e, f)$ and we consider the its sides lengths l, m, n and its area s, we might ask for some formula that relates both magnitudes (lengths and area). In this case, the hypotheses equations form the ideal

$$H := Ideal(l^2 - (a-c)^2 - (b-d)^2, m^2 - (c-e)^2 - (d-f)^2,$$
$$n^2 - (e-a)^2 - (f-b)^2, 2s - (cf + eb + ad - cb - de - af));$$

and it is clear we want to privilege the variables l, m, n, s, since we are searching for a relation among them.

In fact, eliminating all variables except the "interesting" ones, namely l, m, n, s, we get using CoCoA [CNR99] the following:

```
Use RR ::=Q[abcdeflmns];
H:=Ideal(l^2-(a-c)^2-(b-d)^2,m^2-(c-e)^2-(d-f)^2,
        n^2-(e-a)^2-(f-b)^2,2s -(cf+eb+ad-cb-de-af))
Elim(a..f, H);
Ideal(1/2l^4 - l^2m^2 + 1/2m^4 - l^2n^2 - m^2n^2 + 1/2n^4 +
8s^2)
```

i.e. $16s^2 = 2l^2n^2 + 2m^2n^2 + 2l^2m^2 - l^4 - m^4 - n^4$, which is an interesting result, since it is a rearrangement of Heron's well known formula ([Ch90], [NR], [P]).

One important remark deriving from the consideration of this example is that, in the context of automatic deduction and discovery, the variables of interest are not, in general, independent with respect to the data variety (ie. the set of constraints that are given in order to deduce or to discover some property). In fact, we may want to find some relation holding among the "apparently" unrelated variables. This opposes to the standard approach for automatic proving, where the search for non-degeneracy conditions is restricted to free variables with respect to the hypothesis variety.

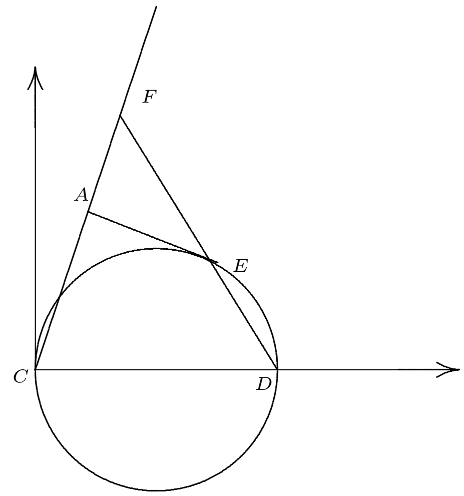
In ([RV96]) a method for automatic discovery[4] is devised, summarily described as follows:

– Given some thesis and hypotheses, check, first, whether the theorem is true (according to some criterion)
– If it is true, exit. Else, add the thesis to the hypothesis set and eliminate, in the ideal of thesis+hypotheses, all variables except those that are independent (or specially relevant, according to the different criteria) for the hypothesis ideal
– Finally, consider the generators of the elimination ideal as supplementary hypotheses and start over again (but this time the collection of relevant variables might be a subset of the previously considered ones, since we might have added new constraints to the original hypothesis set).

Clearly, the hypothesis and the thesis ideals are, respectively, the ideals generated by the set of the hypothesis equations and by set of the thesis equations. Let us show how this procedure works, through the following example.

[4] A suitable modification works, as well, for the case of automatic deduction of formulas, as in the above example.

Example 2. (From Example 91 in [Ch88]). Let us consider as given data a circle and two diametral opposed points on it (say, take a circle centered at $(1,0)$ with radius 1, and let $C = (0,0), D = (2,0)$ the two ends of a diameter), plus an arbitrary point $A = (u_1, u_2)$. Then trace a tangent from A to the circle and let $E = (x_1, x_2)$ be the tangency point. Let $F = (x_3, x_4)$ be the intersection of DE and CA. Then we claim that $AE = AF$. Moreover, in order to be able to define the lines DE, CA, we require that $D \neq E$ (ie. $u_1 \neq 2$) and that $C \neq A$ (ie. $u_1 \neq 0$ or $u_2 \neq 0$).



Now, using CoCoA and its package TP (for Theorem Proving), we translate the given situation as follows

```
Alias  TP   := $contrib/thmproving;

Use R::=Q[x[1..4],u[1..2]];

A:=[u[1],u[2]];
E:=[x[1],x[2]];
D:=[2,0];
F:=[x[3],x[4]];
C:=[0,0];

Ip2:=TP.Perpendicular([E,A],[E,[1,0]]);
Ip3:=TP.LenSquare([E,[1,0]])-1;
Ip4:=TP.Collinear([0,0],A,F);
Ip5:=TP.Collinear(D,E,F);
```

```
H:=Saturation(Ideal(Ip2,Ip3,Ip4,Ip5),Ideal(u[1]-2)*
                                 Ideal(u[1], u[2]));

T:=Ideal(TP.LenSquare([A,E])-TP.LenSquare([A,F]));
```

where T is the thesis and H describes the hypothesis ideal. Notice that $Ip2$ expresses that the segments $[E, A], [E,(1,0)]$ are perpendicular; $Ip3$ states that the square of the length of $[E,(1,0)]$ is 1 (so $Ip2, Ip3$ imply E is the tangency point from A); and the next two hypotheses express that the corresponding three points are collinear. The hypothesis ideal H is here constructed by using the *Saturation* command (see [KR00])[5], since it is a compact form of stating that the hypothesis variety is the closure of the set defined by all the conditions $Ip[i] = 0, i = 2 \ldots 5$ minus the union $\{u[1] = 2\} \cup \{u[1] = 0, u[2] = 0\}$, as declared in the formulation of this example. Finally, the thesis expresses that the two segments $[AE], [AF]$ have equal non oriented length.

Now we check that the statement $\boldsymbol{H} \Rightarrow \boldsymbol{T}$ is not algebraically true in any conceivable way. For instance, applying the protocol in [CT2], it turns that

```
Saturation(H, Saturation(H,T));
 >Ideal(1)
-------------------------------
```

and this computation shows that all possible non-degeneracy conditions (those polynomials $p(\mathbf{u}, \mathbf{x})$ that could be added to the hypotheses as conditions of the kind $p(\mathbf{u}, \mathbf{x}) \neq \mathbf{0}$) lie in the hypothesis ideal, yielding, therefore to an empty set of conditions of the kind $p \neq 0 \wedge p = 0$. This implies, in particular, that the same negative result would be obtained if we restrict the computations to some subset of variables, since the thesis does not vanish on any irreducible component of the hypothesis variety.

Thus we must switch on to the discovery protocol, checking before hand that $u[1], u[2]$ actually is a (maximal) set of independent variables for our construction:

```
Dim(R/H);
 >2
-------------------------------

Elim([x[1],x[2],x[3],x[4]],H);
 >Ideal(0)
-------------------------------
```

Then we add the thesis to the hypothesis ideal and we eliminate all variables except $u[1], u[2]$[6]:

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5 Given two ideals I, J, its Saturation(I, J) is defined as Saturation$(I, J) = I : J^{\infty}$, which gives, under certain conditions, the Zariski closure of the algebraic set $V(I) \backslash V(J)$.

6 If $U' \subseteq U \subseteq \{x_1, \ldots, x_n = X\}$ and $A \subseteq \mathbb{Q}[U']$, $B \subseteq \mathbb{Q}[U]$, we denote by $A^{e'} = A\mathbb{Q}[U]$, by $A^e = A\mathbb{Q}[X]$, and by $B^e = B\mathbb{Q}[X]$. Clearly $(A^{e'})^e = A^e$.

```
H':=Elim([x[1],x[2],x[3],x[4]],H+T);
H';
 >Ideal(-1/2u[1]^5 - 1/2u[1]^3u[2]^2 + u[1]^4)
-------------------------------
Factor(-1/2u[1]^5 - 1/2u[1]^3u[2]^2 + u[1]^4);
 >[[u[1]^2 + u[2]^2 - 2u[1], 1], [u[1], 3], [-1/2, 1]]
-------------------------------
```

yielding as complementary hypotheses the conditions $u[1]^2 + u[2]^2 - 2u[1] = 0 \vee u[1] = 0$ that can be interpreted by saying that either point A lies on the given circle or (when $u[1] = 0$) triangle $\Delta(A, C, D)$ is rectangle at C. In the next step of the discovery procedure we consider as new hypothesis ideal the set $H + H'^e$, which is of dimension 1 and where both $u[2]$ or $u[1]$ can be taken as independent variables ruling the new construction.

```
Dim(R/(H+H'^e));
 >1
-------------------------------
Elim([x[1],x[2],x[3],x[4],u[1]],H+H'^e);
 >Ideal(0)
-------------------------------
Elim([x[1],x[2],x[3],x[4],u[2]],H+H'^e);
 >Ideal(0)
```

Choosing, for example, $u[2]$ as relevant variable, we check that the new statement $\boldsymbol{H} \cap V(H'^e) \Rightarrow \boldsymbol{T}$ is correct under the non-degeneracy condition $u[2] \neq 0$:

```
H'':=Elim([x[1],x[2],x[3],x[4],u[1]], Saturation(H+H'^e,T));
H'';
 >Ideal(u[2]^3)
--------------------------------
```

Thus we have arrived to the following statement: Given a circle of radius 1 and centered at $(1, 0)$, and a point A not in the X-axis and not in the line $X = 2$, the segments AE, AF (where E is the tangency point from A to the circle and F is the intersection of the lines passing by $(2, 0), E$ and $A, (0, 0)$) are of equal length if A is on the Y-axis or in the circle. The latter case is quite trivial, since it means that $A = E = F$.

Remark that if we choose $u[1]$ as the privileged variable, what we get is

```
H'':=Elim([x[1],x[2],x[3],x[4],u[2]], Saturation(H+H'^e,T));
H'';
 >Ideal(2u[1]^4)
-------------------------------
```

a more trivial statement, since here A is subject to the conditions of being "not in the Y-axis" and not in the line $X = 2$ and A is not the origin, plus A on the Y-axis or in the circle; which can be summarized as A is in the circle and A is neither the origin nor the point $(0, 2)$ (for the equality of lengths of the segments AE, AF).

3 Full Set of Discovering Interesting Conditions

In the previous section we have recalled and exemplified an algebraic method for discovering new results starting from a wrong statement (for instance, when a thesis that does not hold over some irreducible component, maybe of a distinguished kind, of the hypothesis variety). It is also known that such method is not complete (see [RV96], Example 4), namely, there are certain cases where $H'' = 0$ (with the notation as in the precedent example) and we arrive to a dead end (since applying the outlined method we will conclude with a set of new hypotheses including $0 \neq 0$. Is there a different method that will succeed?

Speaking in all generality, let us assume that we have already a procedure that deals with checking statements that hold over all irreducible components (maybe of special kind) of a hypothesis variety. In the remaining cases what we would like to present is a protocol that finds

- a set of complementary hypotheses expressed in terms of the declared meaningful variables for the data, that are necessary for the thesis to hold,
- and such that, adding them to the given set of hypotheses, they become sufficient for the thesis, under certain non-degeneracy conditions.

Thus we are lead to the following definition for theorems stated by means of polynomials in $K[x_1, \ldots, x_n]$, where K is the field of rational numbers or a finite extension of this field.

Definition 1. *Let H, $T \subseteq K[x_1, \ldots, x_n]$ the hypothesis ideal and thesis ideal, and $U' \subseteq U \subseteq \{x_1, \ldots, x_n\}$. Then a couple (R', R'') of ideals respectively in $K[U]$ and $K[U']$ is a* **Full Set of Discovering Interesting Conditions (FSDIC)** *with respect to U and U' if the following items hold:*

a) $R' \subseteq K[U]$ and $R'' \subseteq K[U']$;
b) $V(H + R'^e) \backslash V(R''^e) \subseteq V(T)$;
c) $V(H + T) \subseteq V(R'^e)$;
d) if $f \in K[U']$ is such that $V(H + R'^e) \backslash V((f)^e) \subseteq V(T)$, then $f \in \sqrt{R''}$;
e) $V(H + R'^e) \backslash V(R''^e) \neq \emptyset$.

Notice that the condition d) is equivalent to the following condition:

d') if $R''' \subseteq K[U']$ is an ideal such that $V(H + (R')^e) \backslash V((R''')^e) \subseteq V(T)$, then $\mathbb{C}^n \setminus V((R''')^e) \subseteq \mathbb{C}^n \setminus V((R'')^e)$.

Let us go back to the example in the previous section. It suggests us that a good candidate to be a **FSDIC** is the couple (H', H'') where $H' = (H+T) \cap K[U]$ and $H'' = ((H + H'^e) : (T)^\infty) \cap K[U']$. In fact, we have proved the following theorems.

Theorem 1. *Let $H' = (H + T) \cap K[U]$ and $H'' = ((H + H'^e) : (T)^\infty) \cap K[U']$. Then there exist two ideals R', R'' such that (R', R'') is a* **FSDIC** *with respect to U and U' if and only if (H', H'') is a* **FSDIC** *for T with respect to U and U'.*

Theorem 2. *(H', H'') is a* **FSDIC** *with respect to U and U' if and only if $1 \notin (H' \cap K[U']) : H''^{\infty}$. Moreover, if U' is a set of algebraically independent variables for H' then (H', H'') is a* **FSDIC** *for T with respect to U and U' if and only if $H'' \neq (0)$.*

Remark that in the definition of a **FSDIC** we do not assume the independence of the variables U and U'. In particular if $U = U' = X$ then $H' = H + T$ and $H'' = (1)$, and clearly this case is not interesting. The choice of taking U and U' as sets of independent variables is exclusively related to the elusive concept of "interesting". In our full paper we give a geometric interpretation about the **FSDIC**, in order to understand the idea of truth we are speaking of.

In this direction, here we merely state a collection of properties that hold in the case U is a set of algebraically independent variables for H and U' is a set of algebraically independent variables for $H + H'^e$.

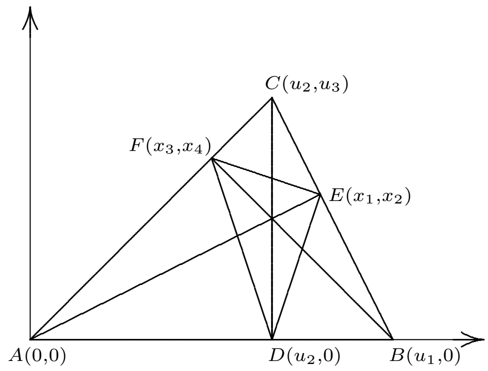
- $H'' \neq (0)$ iff T is contained in all the minimal primes of $H + H'^e$ where U' are independent.
- if T is contained in some but not all the minimal primes of $H + H'^e$ where U' are independent then $H'' = (0)$.
- Suppose U' a maximal set of algebraically independent variables for $H + H'^e$; if $H'' = (0)$ then T is contained in some but not all the minimal primes of $H + H'^e$ where U' are independent variables.

Things are quite subtle. Notice that such U' may not exist: consider $H = (a+1) \cap (b+1)$ in $K[a, b, c]$, $T = (a + b + 1, c)$, and $U = \{b, c\}$. Then $H' = (H + T) \cap K[b, c] = (c, b^2 + b)$ and there is not $U' \subset U$ a set of algebraically independent variables for $H + H'^e$.

Therefore we believe we have got a good translation of the idea of truth in the case we choose U' as a of algebraically independent variables for $H + H'^e$, and a complete description if the set of variables U' is maximal.

On the other hand, what does it happen if U' are dependent on $H + H'^e$, in particular if $U' = U$? Clearly $H' \neq (0)$, since we suppose that U are independent on H. If $1 \notin (H' \cap K[U']) : H''^{\infty}$ then T is contained in some minimal prime of $H + H'^e$ but we do not have more informations about these components. We can produce examples where there does not exist $V \subset U$ a set of algebraically independent variables for $H + H'^e$, examples where $V \subset U$ are independent (but not a maximal set!!), and T is contained just in the components where all the variables V are dependent. Only in the case $V \subset U$ is a maximal set of algebraically independent variables for $H + H'^e$, then T results contained in some minimal prime of $H + H'^e$ where V are independent. But also in this case we cannot conclude (as shown by different examples) that T is contained in all the minimal primes of $H + H'^e$ where V are independent, even if $1 \notin H' : H''^{\infty}$.

Example 3. Given an arbitrary triangle, we construct the associated orthic triangle, that is, the triangle wihose vertices are the endpoints of the altitudes of the given triangle. Then we conjecture the orthic triangle is equilateral.



As in the previous example, we translate the given situation as follows:

```
Alias  TP :=  $contrib/thmproving;

Use R::=Q[x[1..4],u[1..3],z];

A:=[0,0];
B:=[u[1],0];
C:=[u[2],u[3]];
D:=[u[2],0];
E:=[x[1],x[2]];
F:=[x[3],x[4]];



Ip1:=TP.Perpendicular([E,A],[B,C]);
Ip2:=TP.Perpendicular([F,B],[A,C]);
Ip3:=TP.Collinear(A,C,F);
Ip4:=TP.Collinear(B,C,E);


H:=Ideal(Ip1,Ip2,Ip3,Ip4,u[1]u[3]z-1);


T:=Ideal(TP.LenSquare([D,E])-TP.LenSquare([E,F]),
        TP.LenSquare([D,E])-TP.LenSquare([D,F]));
```

Then we check that $u[1], u[2], u[3]$ actually is a (maximal) set of independent variables for our construction:

```
Elim([x[1],x[2],x[3],x[4],z],H);
```

```
Ideal(0)
-------------------------------
Dim(R/H);
3
-------------------------------
```

As the theorem is obviously false, we turn over to discovery conditions, adding the thesis to the hypotheses ideal.

```
H':=Elim([x[1],x[2],x[3],x[4],z],H+T);

H';
Ideal(1/4u[1]^2u[2] - 3/4u[1]u[2]^2 + 1/2u[2]^3 + 1/4u[1]u[3]^2 -
1/2u[2]u[3]^2,
3/4u[1]u[2]^3 - 3/4u[2]^4 - 3/4u[1]^2u[3]^2 - 1/4u[1]u[2]u[3]^2 +
u[2]^2u[3]^2 + 3/4u[3]^4,
3/16u[2]^4 - 5/8u[2]^2u[3]^2 + 3/16u[3]^4,
-3/16u[1]^3 + 9/8u[1]u[2]^2 - 3/4u[2]^3 - 1/8u[1]u[3]^2 +
1/4u[2]u[3]^2)
```

Then we take $U' = u[1]$ and we check that U' is a maximal set of independent variables for $H + H'$.

```
Elim([x[1],x[2],x[3],x[4],z,u[2],u[3]],H+H');
Ideal(0)
-------------------------------
Dim(R/(H+H'));
1
-------------------------------
```

Following our protocol, we compute next the ideal of conditions H''

```
H'':=Elim([x[1],x[2],x[3],x[4],z,u[2],u[3]],Saturation(H+H',T));
H'';
Ideal(1)
-------------------------------
```

Therefore H', H'' is a **FSDIC** for our problem. But what do we have discovered? In order to look for an answer, we find a decomposition of the new hypotheses ideal

```
NewH:=H+H';
```

as follows:

```
H1;
Ideal(
64x[4]^4z^2-27,
-8x[4]^3z+3x[3],
192x[3]^2x[4]^3z^2-512x[4]^5z^2-48x[3]x[4]^2z+6x[2]+21x[4],
-256x[4]^7z^3+108x[3]^2x[4]z-96x[4]^3z+27x[1]-54x[3],
288x[3]^2x[4]^3z^2-544x[4]^5z^2-144x[3]x[4]^2z+21x[4]+9u[3],
1024x[4]^7z^3-108x[3]^2x[4]z-156x[4]^3z+27u[2],
-1024x[4]^7z^3+216x[3]^2x[4]z-24x[4]^3z-108x[3]+27u[1])
-------------------------------
H2;
Ideal(
64x[4]^4z^2-3,
-8x[4]^3z+x[3],
192x[3]^2x[4]^3z^2-512x[4]^5z^2-48x[3]x[4]^2z+6x[2]+21x[4],
-256x[4]^7z^3+108x[3]^2x[4]z-96x[4]^3z+27x[1]-54x[3],
288x[3]^2x[4]^3z^2-544x[4]^5z^2-144x[3]x[4]^2z+21x[4]+9u[3],
1024x[4]^7z^3-108x[3]^2x[4]z-156x[4]^3z+27u[2],
-1024x[4]^7z^3+216x[3]^2x[4]z-24x[4]^3z-108x[3]+27u[1])
-------------------------------
H3;
Ideal(
64x[4]^4z^2-3,
8x[4]^3z+3x[3],
192x[3]^2x[4]^3z^2-512x[4]^5z^2-48x[3]x[4]^2z+6x[2]+21x[4],
-256x[4]^7z^3+108x[3]^2x[4]z-96x[4]^3z+27x[1]-54x[3],
288x[3]^2x[4]^3z^2-544x[4]^5z^2-144x[3]x[4]^2z+21x[4]+9u[3],
1024x[4]^7z^3-108x[3]^2x[4]z-156x[4]^3z+27u[2],
-1024x[4]^7z^3+216x[3]^2x[4]z-24x[4]^3z-108x[3]+27u[1])
-------------------------------
H4;
Ideal(
64x[4]^4z^2-3,
-8x[4]^3z+3x[3],
192x[3]^2x[4]^3z^2-512x[4]^5z^2-48x[3]x[4]^2z+6x[2]+21x[4],
-256x[4]^7z^3+108x[3]^2x[4]z-96x[4]^3z+27x[1]-54x[3],
288x[3]^2x[4]^3z^2-544x[4]^5z^2-144x[3]x[4]^2z+21x[4]+9u[3],
1024x[4]^7z^3-108x[3]^2x[4]z-156x[4]^3z+27u[2],
-1024x[4]^7z^3+216x[3]^2x[4]z-24x[4]^3z-108x[3]+27u[1])
-------------------------------
NewH=Intersection(H1,H2,H3,H4);
TRUE
-------------------------------
```

Therefore the hypotheses for the orthic triangle to be equilateral are the disjunction of the equations given by the generators of these ideals. But is is clear that such hypotheses should be stated in terms of the given data (ie. the variables $u[1], u[2], u[3]$). Thus we proceed eliminating the rest of the variables in this collection of ideals:

```
Elim([z,x[1],x[2],x[3],x[4]],H1);
Ideal(
  27/2048u[1] - 27/1024u[2],
  4u[2]^2 - 12u[3]^2)
-------------------------------
Elim([z,x[1],x[2],x[3],x[4]],H2);
Ideal(
  81/2048u[1] - 27/1024u[2],
  4/9u[2]^2 - 4/3u[3]^2)
-------------------------------
Elim([z,x[1],x[2],x[3],x[4]],H3);
Ideal(
  -27/2048u[1] - 27/1024u[2],
  4u[2]^2 - 4/3u[3]^2)
-------------------------------
Elim([z,x[1],x[2],x[3],x[4]],H4);
Ideal(
  27/2048u[1] - 27/1024u[2],
  4u[2]^2 - 4/3u[3]^2)
-------------------------------
```

which means that the angles of the given triangle should be of one the following set of degrees:
$H1$: $A = 30$, $B = 30$, $C = 120$
$H2$: $A = 30$, $B = 120$, $C = 30$
$H3$: $A = 120$, $B = 30$, $C = 30$
$H4$: $A = 60$, $B = 60$, $C = 60$

Therefore we have discovered that the orthic triangle would be equilateral if the given triangle is itself equilateral or isosceles (and then of the particular type with angles equal to 120, 30 and 30 degrees, respectively). We have not been able to find a reference in the literature to this, somehow surprising, result.

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