Estimates on the Distribution of the Condition Number of Singular Matrices

C. Beltrán [1,2,3], L.M. Pardo [1,2]

January 31, 2006

Abstract

We exhibit some new techniques to study volumes of tubes about algebraic varieties in complex projective spaces. We prove the existence of relations between volumes and Intersection Theory in the presence of singularities. In particular, we can exhibit an average Bézout Equality for equi–dimensional varieties. We also state an upper bound for the volume of a tube about a projective variety. As a main outcome, we prove an upper bound estimate for the volume of the intersection of a tube with an equi–dimensional projective algebraic variety. We apply these techniques to exhibit upper bounds for the probability distribution of the generalized condition number of singular complex matrices.

Keywords: Condition number, complex matrices, volume, tubes, algebraic projective varieties. **AMS Classification:** Primary 15A12, 14N05. Secondary: 53C65.

1 Introduction.

In these pages we exhibit some upper bound estimates of the probability distribution of the condition number of singular complex matrices. These estimates are immediate consequences of some more general techniques dealing with volumes of tubes about projective algebraic varieties. This Introduction is devoted to state the main outcomes and the motivations of this study.

Condition numbers in Linear Algebra were introduced by A. Turing in [44]. They were also studied by J. von Neumann and collaborators (cf. [32]) and by J.H. Wilkinson (cf. also [48]). Variations of these condition numbers may be found in the literature of Numerical Linear Algebra (cf. [7], [17], [25], [43] and references therein).

A relevant breakthrough was the study of the probability distribution of these condition numbers. The works by Steve Smale (cf. [38]), J. Renegar (cf. [33]), J. Demmel (cf. [6], [7]) and mainly the works by A. Edelman (cf. [9], [10]) showed the exact values of the probability distribution of the condition number of dense complex matrices.

From a computational point of view, these statements can be translated in the following terms. Let $\mathcal{P}$ be a numerical analysis procedure whose space of input data is the space of

arbitrary square complex matrices $\mathcal{M}_n(\mathbb{C})$. Then, Edelman's statements mean that *the probability that a randomly chosen dense matrix in $\mathcal{M}_n(\mathbb{C})$ is a well–conditioned input for $\mathcal{P}$ is high* (cf. also [3]).

Sometimes however we deal with procedures $\mathcal{P}$ whose input space is a proper subset $\mathcal{C} \subseteq \mathcal{M}_n(\mathbb{C})$. Additionally such procedures with particular data lead to particular condition numbers $\kappa_{\mathcal{C}}$ adapted both for the procedure $\mathcal{P}$ and the input space $\mathcal{C}$. Renegar's, Demmel's, Edelman's and Smale's results do not apply to these new conditions. In these pages we introduce a new technique to study the probability distribution of condition numbers $\kappa_{\mathcal{C}}$. Namely, we introduce a technique to exhibit upper bound estimates of the quantity

$$\frac{vol[\{A \in \mathcal{C} : \kappa_{\mathcal{C}}(A) > \varepsilon^{-1}\}]}{vol[\mathcal{C}]}, \tag{1}$$

where $\varepsilon > 0$ is a positive real number, and $vol[\cdot]$ is some suitable measure on the space $\mathcal{C}$ of acceptable inputs of $\mathcal{P}$.

As an example of how these questions arise, let $\mathcal{C} := \Sigma^{n-1} \subseteq \mathcal{M}_n(\mathbb{C})$ be the class of all singular complex matrices. From [27] and [40], a condition number for singular matrices $A \in \mathcal{C}$ is introduced. This condition number measures the precision required to perform kernel computations (cf. Section 4 for precise details). For every singular matrix $A \in \Sigma^{n-1}$ of corank 1, the condition number $\kappa_D^{n-1}(A) \in \mathbb{R}$ is defined by the following identity

$$\kappa_D^{n-1}(A) := \|A\|_F \|A^\dagger\|_2,$$

where $\|\cdot\|_F$ is the Frobenius norm of a matrix A, $A^\dagger$ is the Moore–Penrose pseudo–inverse of A and $\|A^\dagger\|_2$ is the norm of $A^\dagger$ as a linear operator.

As Σ^{n-1} is a complex homogeneous hypersurface in $\mathcal{M}_n(\mathbb{C})$ (i.e. a cone of complex codimension 1), it is endowed with a natural volume vol induced by the $2(n^2-1)$−dimensional Hausdorff measure of its intersection with the unit disk (cf. Section 2 for details). We then wish to have upper bound estimates for the following quantity:

$$\frac{vol[A \in \Sigma^{n-1} : \kappa_D^{n-1}(A) > \varepsilon^{-1}]}{vol[\Sigma^{n-1}]} \tag{2}$$

In Section 4 other proper subclasses of $\mathcal{M}_n(\mathbb{C})$ are also discussed. Upper bound estimates for the quantity in (2) belong to a wider class of results we state in Theorem 2 below.

First of all, most condition numbers are by nature projective functions. For instance, the classical condition number κ of Numerical Linear Algebra is naturally defined as a function on the complex projective space $\mathbb{P}(\mathcal{M}_n(\mathbb{C}))$ defined by the complex vector space $\mathcal{M}_n(\mathbb{C})$. Namely, we may see κ as a function

$$\kappa : \mathbb{P}(\mathcal{M}_n(\mathbb{C})) \longrightarrow \mathbb{R}_+ \cup \infty.$$

Secondly, statements like the Schmidt–Mirsky–Eckart–Young Theorem (cf. [8],[35], [29]) imply that Smale's, Demmel's and Edelman's estimates are, in fact, estimates of the volume of a tube about a concrete projective algebraic variety in $\mathbb{P}(\mathcal{M}_n(\mathbb{C}))$ (cf. also Section 4).

We prove a general upper bound for the volume of a tube about any (possibly singular) complex projective algebraic variety (see Theorem 1 below), that slightly improves the constants obtained by Renegar (cf. [33]) and Demmel (cf. [7]) for the same problem.

2

Estimates on volumes of tubes is a classic topic that began with Weyl's Tube Formula for tubes in the affine space (cf. [47]). Formulae for the volumes of some tubes about analytic submanifolds of complex projective spaces are due to A. Gray (cf. [18], [19] and references therein). However, Gray's results do not apply even to Smale's and Edelman's case. They also do not apply to particular classes $\mathcal{C}$ as above. First of all, Gray's statements are only valid for smooth submanifolds and not for singular varieties (as, for instance, Σ^{n-1}). Secondly, Gray's theorems are only valid for tubes of small enough radius (depending on intrinsic features of the manifold under consideration) which may become dramatically small in the case of existence of singularities. These two drawbacks pushed J. Renegar and J. Demmel to look for a general statement concerning upper bound estimates for the volumes of tubes about equidimensional complex projective varieties that may contain some singularities (cf. [33] for the hypersurface case, [6] or [7] for the general case). Here we obtain a slight improvement of Demmel's Theorem 4.2 in [7], that may be summarized as follows.

Let $d\nu_N$ be the volume form associated to the complex Riemannian structure of $\mathbb{P}_N(\mathbb{C})$. Let $V \subseteq \mathbb{P}_N(\mathbb{C})$ be any subset of the complex projective space and let $\varepsilon > 0$ be a positive real number. We define *the tube of radius ε about V in $\mathbb{P}_N(\mathbb{C})$* as the subset $V_\varepsilon \subseteq \mathbb{P}_N(\mathbb{C})$ given by the following identity.

$$V_\varepsilon := \{x \in \mathbb{P}_N(\mathbb{C}) : d_{\mathbf{P}}(x, V) < \varepsilon\},$$

where $d_{\mathbf{P}}(x, y) := \sin d_R(x, y)$ and $d_R : \mathbb{P}_N(\mathbb{C})^2 \longrightarrow \mathbb{R}$ is the Fubini–Study distance.

Theorem 1 *Let $V \subseteq \mathbb{P}_N(\mathbb{C})$ be a (possibly singular) equi–dimensional complex algebraic variety of (complex) codimension r in $\mathbb{P}_N(\mathbb{C})$. Let $0 < \varepsilon \leq 1$ be a positive real number. Then, the following inequality holds*

$$\frac{\nu_N[V_\varepsilon]}{\nu_N[\mathbb{P}_N(\mathbb{C})]} \leq 2 \deg(V) \left(\frac{e \, N \, \varepsilon}{r} \right)^{2r}, \tag{3}$$

where e stands for the basis of the natural logarithms, and $\deg(V)$ is the degree of V (in the sense of [24]).

The proof of this theorem is a by–product of the techniques we introduce to deal with the upper bound estimates of the quantity described in inequality (2). This theorem can be applied to Edelman's conditions to conclude the following estimate:

$$\frac{vol[\{A \in \mathcal{M}_n(\mathbb{C}) : \kappa_D(A) > \varepsilon^{-1}\}]}{vol[\mathcal{M}_n(\mathbb{C})]} \leq 2e^2 n^5 \varepsilon^2,$$

where $\kappa_D(A) := \|A\|_F \|A^{-1}\|_2$, and *vol* is the standard Gaussian measure in $\mathbb{C}^{n^2}$. The reader will observe that this kind of upper bounds is less sharp than Edelman's or Smale's bounds, but they are a particular instance of a more general statement.

Next, observe that neither Renegar's Demmel's, Smale's, Edelman's results nor Theorem 1 above apply to exhibit upper bounds of the quantity described in equation (2) above. Neither does Gray's theorem apply to such kinds of questions. The reason is the following one. The probability space of input data is the projective algebraic variety Σ^{n-1}. As we

said above, this variety is neither smooth nor a complex projective space (i.e. it is not "linear", even at a local level).

In order to deal with this kind of estimates, we need to introduce a brand new technique that combines Intersection Theory and Integral Geometry. Again, the Schmidt–Mirsky–Eckart–Young Theorem implies

$$\kappa_D^{n-1}(A) = \frac{1}{d_{\mathbf{P}}\left(A, \Sigma^{n-2}\right)},$$

where Σ^{n-2} is the projective variety of matrices of rank at most $n-2$ and $d_{\mathbf{P}}$ is the projective distance. Hence, in order to bound the quantity in equation (2), we need to prove some kind of *upper bound for the volume of the intersection of an extrinsic tube about a (possibly singular) projective algebraic subvariety with a proper (possibly singular) projective algebraic variety.*

Hence, the main outcome in this paper is the following theorem.

Theorem 2 *Let $V, V' \subseteq \mathbb{P}_N(\mathbb{C})$ be two (possibly singular) projective equi–dimensional algebraic varieties of respective dimensions $m > m' \geq 1$. Let $0 < \varepsilon \leq 1$ be a positive real number. With the same notations as in Theorem 1 above, the following inequality holds:*

$$\frac{\nu_m[V'_\varepsilon \cap V]}{\nu_m[V]} \leq c \deg(V') N \binom{N}{m'}^2 \left[e \frac{N-m'}{m-m'} \, \varepsilon \right]^{2(m-m')},$$

where $c \leq 4e^{1/3}\pi$, ν_m is the $2m-$dimensional natural measure in the algebraic variety V, and $\deg(V')$ is the degree of V' in the sense of $[24]$.

The occurrence of $deg(V')$ on the right–hand side of the inequality seems to be unavoidable because of Bézout's Theorem, whereas the constants depending on N, m, m' are essentially the square of the multinomial coefficient:

$$\frac{N!}{(m')!(N-m)!(m-m')!}.$$

This statement can finally be applied to show upper bound estimates for the quantity described in equation (2). Noting that the complex projective dimensions of Σ^{n-1} and Σ^{n-2} satisfy $dim(\Sigma^{n-1}) = n^2 - 2$ and $dim(\Sigma^{n-2}) = n^2 - 5$, we immediately conclude (cf. also Corollary 29).

Corollary 3 *With the same notations and assumptions as above, the following inequality holds:*

$$\frac{vol[A \in \Sigma^{n-1} : \kappa_D^{n-1}(A) > \varepsilon^{-1}]}{vol[\Sigma^{n-1}]} \leq 9 \, deg(\Sigma^{n-2}) \left[n^{8/3}\varepsilon \right]^6.$$

Moreover, noting that

$$deg(\Sigma^{n-2}) = \frac{n^2(n^2-1)}{12},$$

we can estimate the upper bound in this last corollary by:

Corollary 4 *With the same notations as in Corollary 3 above, the following inequalities also hold:*

$$\frac{vol[A \in \Sigma^{n-1} : \kappa_D^{n-1}(A) > \varepsilon^{-1}]}{vol[\Sigma^{n-1}]} \leq \frac{9n^4}{12} \left[n^{8/3}\varepsilon\right]^6 \leq \left[n^{10/3}\varepsilon\right]^6.$$

Let the reader observe that the exponent 6 is unavoidable since it is two times the complex codimension of Σ^{n-2} in Σ^{n-1}. In Section 4 other proper subclasses of $\mathcal{M}_n(\mathbb{C})$ are also discussed.

As we have said, the condition number κ_D^{n-1} can be defined as the inverse of the projective distance to the algebraic variety Σ^{n-2} of matrices of rank at most $n-2$. This allows us to consider κ_D^{n-1} defined in the whole space of matrices $\mathcal{M}_n(\mathbb{C})$. We may use Theorem 1 and Corollary 4 to obtain upper bounds for the expected value of κ_D^{n-1} in the respective probability spaces Σ^{n-1} and $\mathcal{M}_n(\mathbb{C})$ (with the Gaussian distribution), and thus compare the different behavior of κ_D^{n-1} when considering as inputs randomly chosen singular matrices or randomly chosen dense matrices. Namely, we have the following result (cf. Corollary 44 for a more technical version).

Corollary 5 *The expected value of κ_D^{n-1} in the space Σ^{n-1} satisfies:*

$$E_{\Sigma^{n-1}}[\kappa_D^{n-1}] \leq 2n^{10/3}.$$

Moreover, the expected value of κ_D^{n-1} in the whole space $\mathcal{M}_n(\mathbb{C})$ satisfies

$$E_{\mathcal{M}_n(\mathbb{C})}[\kappa_D^{n-1}] \leq n^{5/2}.$$

The paper is structured as follows. Section 2 is devoted to stating most of the notations and some basic lemmata to be used in the sequel. Section 3 is devoted to proving Theorem 2. A proof of Theorem 1 is also included in Subsection 3.2. Finally, in Section 4 we prove Corollaries 4, 5 and other applications to other particular classes of complex matrices.

1.1 Appendix to the Introduction

Although Theorem 1 is not the main outcome of these pages, a relevant question about this theorem concerns the optimality of the constants occurring on the right–hand side of equation (3). However, it seems to be a hard result to prove this optimality. For instance, in Proposition 27 of Section 3.2 we prove that the constants are essentially optimal in the case V is a linear subvariety of a complex projective space.

A second approach to understand the optimality of the constants occurring in the upper bound estimate of Theorem 1 will be to compare it with Gray's main theorem in [18] (cf. also [19]). Gray's main theorem can be stated as follows. Assume that the projective algebraic variety V satisfies the following hypothesis:

- The variety V is smooth (i.e., it contains no singularity) and it is a complex submanifold of $\mathbb{P}_N(\mathbb{C})$.

- The variety V is a complete intersection. Namely, there are homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{C}[X_0, \ldots, X_N]$ of respective degrees $deg(f_i) = d_i$, $1 \leq i \leq r$ such that V is the set of common projective zeros of $f_1, \ldots, f_r$ and such that the codimension of V is r (i.e., the number of equations equals the codimension).

Additionally, let us assume that $\varepsilon > 0$ is a positive real number smaller than the minimum of the convergence radius of the Taylor expansion of the normal exponential map of V at any point of V. Under all these conditions, A. Gray proves the following equality (cf. [18]):

$$\frac{\nu_N[V_\varepsilon]}{\nu_N[\mathbb{P}_N(\mathbb{C})]} = \sum_{c=0}^{N-r} \binom{N}{c} \varepsilon^{2(N-c)}(1-\varepsilon^2)^c \prod_{i=1}^{r}\left(1-(1-d_i)^{N-r-c+1}\right).$$

The dominant term in Gray's equality corresponds to the minimum exponent of ε. Then, there is a constant $\rho > 0$ such that the following inequality holds:

$$\frac{\nu_N[V_\varepsilon]}{\nu_N[\mathbb{P}_N(\mathbb{C})]} \geq \rho \binom{N}{N-r} \varepsilon^{2r}(1-\varepsilon^2)^{N-r} \prod_{i=1}^{r} deg(f_i). \qquad (4)$$

Noting that $deg(V) \leq \prod_{i=1}^{r} deg(f_i)$ (Bézout Inequality) and that this inequality is generically an equality, the reader may easily compare the lower bound in equation (4) with the upper bound of Theorem 1. Namely, under the very restrictive conditions of Gray's theorem, the constants in Theorem 1 are given by

$$\left(\frac{eN}{r}\right)^{2r},$$

whereas the "constants" in Gray's lower bound are

$$\binom{N}{N-r}(1-\varepsilon^2)^{N-r}.$$

Constants occurring in inequality (3) are not so far from constants occurring in Gray's lower bound. It does not prove that the bound of Theorem 1 is optimal but it is not so far from being optimal at least in some restrictive cases.

2   Some Intersection Theory in complex projective space.

By $(W, < \cdot, \cdot >_W)$ we denote an hermitian space where W is a complex vector space and $< \cdot, \cdot >_W : W \times W \longrightarrow \mathbb{C}$ is the hermitian product. The norm in $(W, < \cdot, \cdot >_W)$ will be denoted by $\| \cdot \|_W$. In the case $W = \mathbb{C}^{N+1}$, we denote by $< \cdot, \cdot >_2$ the usual hermitian product, and by $\| \cdot \|_2$ the usual norm. We say that a finite set of vectors $S = \{v_1, \ldots, v_s\} \in W$ are mutually orthogonal if $< v_i, v_j >_W = 0$, $i \neq j$. We say that S is an orthonormal frame if its elements are mutually orthogonal and $\|v_i\|_W = 1$ for $1 \leq i \leq s$.

As usual, the terms orthogonal and orthonormal will be used in the case of real inner product spaces.

Let $\mathcal{U}_{N+1}$ be the group of unitary matrices of size $N+1$. Recall that the hermitian product in $\mathbb{C}^{N+1}$ is unitarily invariant. That is, for every $\underline{x}, \underline{y} \in \mathbb{C}^{N+1}$ and every $U \in \mathcal{U}_{N+1}$, the following holds:

$$< \underline{x}, \underline{y} >_2 = < U\underline{x}, U\underline{y} >_2 .$$

We denote by $B_{\mathbb{C}^{N+1}}(\underline{x}, \varepsilon)$ the open ball of radius ε centered at $\underline{x}$. Namely,

$$B_{\mathbb{C}^{N+1}}(\underline{x}, \varepsilon) := \{\underline{y} \in \mathbb{C}^{N+1} : \|\underline{x} - \underline{y}\|_2 < \varepsilon\}.$$

Let $S^{2N+1}(\varepsilon) = \partial B_{\mathbb{C}^{N+1}}(0, \varepsilon)$ be the sphere of radius ε in $\mathbb{C}^{N+1}$. Namely,

$$S^{2N+1}(\varepsilon) := \{\underline{x} \in \mathbb{C}^{N+1} : \|\underline{x}\|_2 = \varepsilon\}.$$

As usual, we denote by $S^{2N+1} := S^{2N+1}(1)$ the sphere of radius 1 centered at 0. Recall that S^{2N+1} is a real differentiable submanifold of $\mathbb{C}^{N+1} \equiv \mathbb{R}^{2N+2}$ of real dimension $2N+1$. We consider S^{2N+1} equipped with the Riemannian structure inherited from that of $\mathbb{C}^{N+1}$. Let $\mathbb{P}_N(\mathbb{C}) := \mathbb{P}(\mathbb{C}^{N+1})$ be the complex projective space of dimension N. We also consider the canonical projection

$$
\begin{array}{cccc}
\pi : & \mathbb{C}^{N+1} \setminus \{0\} & \longrightarrow & \mathbb{P}_N(\mathbb{C}) \\
& \underline{x} & \mapsto & \{\underline{y} : \underline{y} = \lambda \underline{x},\ \lambda \in \mathbb{C}\}.
\end{array}
$$

Let $p := \pi \mid_{S^{2N+1}} \colon S^{2N+1} \longrightarrow \mathbb{P}_N(\mathbb{C})$ be the Hopf Fibration. Then, there exists a unique Riemannian structure in $\mathbb{P}_N(\mathbb{C})$ such that p is a Riemannian submersion, i.e., p is a smooth submersion and for every $\underline{x} \in S^{2N+1}$, $d_{\underline{x}}p$ is an isometry between the orthogonal complement of $(d_{\underline{x}}p)^{-1}(0)$ and $T_x\mathbb{P}_N(\mathbb{C})$ (cf. for example [15, Prop. 2.28]). This defines a Riemannian structure in $\mathbb{P}_N(\mathbb{C})$ (cf. example [15, ex. 2.29] for details). Points in the complex projective space $\mathbb{P}_N(\mathbb{C})$ are usually represented by their *homogeneous coordinates*, which are defined the following way: If $x \in \mathbb{P}_N(\mathbb{C})$ is the class of the point $\underline{x} = (x_0, \ldots, x_N)$, the homogeneous coordinates of x are $(x_0 : \cdots : x_N)$. The Riemannian distance (or Fubini–Study distance) between any two points in the complex projective space is given by the formula:

$$d_R(x, y) := \arccos \frac{|<\underline{x}, \underline{y}>_2|}{\|\underline{x}\|_2 \|\underline{y}\|_2},$$

where $\underline{x}, \underline{y}$ are respective affine representants of x and y. We denote by $d_{\mathbf{P}}$ the projective distance, which is defined to be the sinus of the Riemannian distance. Namely,

$$d_{\mathbf{P}}(x, y) = \sin d_R(x, y).$$

Let $B_{\mathbf{P}}(x, \varepsilon) \subseteq \mathbb{P}_N(\mathbb{C})$ be the open ball of radius ε centered at x with respect to $d_{\mathbf{P}}$. Namely,

$$B_{\mathbf{P}}(x, \varepsilon) := \{y \in \mathbb{P}_N(\mathbb{C}) : d_{\mathbf{P}}(x, y) < \varepsilon\}.$$

For every complex submanifold $M \subset \mathbb{P}_N(\mathbb{C})$ of complex dimension m, we denote by $d\nu_m$ the volume element induced by its Riemannian structure inherited from that of $\mathbb{P}_N(\mathbb{C})$. The following formula is well–known.

$$\nu_N[\mathbb{P}_N(\mathbb{C})] = \frac{1}{2\pi} \mathscr{H}^{2N+1}[S^{2N+1}] = \frac{\pi^N}{N!},$$

where $\mathscr{H}^{2N+1}$ is the $(2N+1)-$dimensional Hausdorff measure. If we consider $\mathbb{P}_m(\mathbb{C})$, $m < N$, as a submanifold of $\mathbb{P}_N(\mathbb{C})$ (i.e. as a linear subvariety of dimension m of $\mathbb{P}_N(\mathbb{C})$), then its volume as submanifold agrees with its volume as a projective space itself.

Since [42] we have a explicit formula for the volume of $B_{\mathbf{P}}(x, \varepsilon)$ (see [4] for a modern reference). Namely,

$$\nu_N[B_{\mathbf{P}}(x, \varepsilon)] = \nu_N[\mathbb{P}_N(\mathbb{C})]\varepsilon^{2N}.$$

The Riemannian structure we have defined in $\mathbb{P}_N(\mathbb{C})$ is unitarily invariant. That is, for any unitary matrix $U \in \mathcal{U}_{N+1}$, the following map is an isometry.

$$U: \quad \mathbb{P}_N(\mathbb{C}) \quad \longrightarrow \quad \mathbb{P}_N(\mathbb{C})$$
$$x \quad \mapsto \quad Ux := \pi(U(\pi^{-1}(x))).$$

Also, the tangent map at $0 \in \mathbb{C}^N$ of the following affine chart is an isometry:

$$\varphi_0: \quad \mathbb{C}^N \quad \longrightarrow \quad \mathbb{A}_0^N \subseteq \mathbb{P}_N(\mathbb{C})$$
$$(z_1, \ldots, z_N) \quad \mapsto \quad (1 : z_1 : \cdots : z_N),$$

where $\mathbb{A}_0^N \subseteq \mathbb{P}_N(\mathbb{C}) := \mathbb{P}_N(\mathbb{C}) \setminus \{x \in \mathbb{P}_N(\mathbb{C}) : x_0 = 0\}$ is the projective space without the hyperplane of infinity, and $\mathbb{C}^N$ is seen as the affine space with the natural Riemannian structure.

As in the Introduction, $\mathcal{M}_{N+1}(\mathbb{C})$ denotes the complex vector space of all $(N+1) \times (N+1)$ complex matrices. It is well–known that $\mathcal{U}_{N+1}$ is a real submanifold of $\mathcal{M}_{N+1}(\mathbb{C})$ of real dimension $(N + 1)^2$. The Riemannian structure of $\mathcal{U}_{N+1}$ is the inherited from that of $\mathcal{M}_{N+1}(\mathbb{C})$, normalized such a way that the volume of $\mathcal{U}_{N+1}$ is equal to 1. The volume element for this Riemannian structure will be simply denoted by $d\mathcal{U}_{N+1}$ and the volume of a measurable subset $T \subseteq \mathcal{U}_{N+1}$ will be denoted by $\nu_{\mathcal{U}_{N+1}}[T]$. We say that some property is satisfied for almost all $U \in \mathcal{U}_{N+1}$ if it is satisfied up to a zero–measure subset of $\mathcal{U}_{N+1}$. The two following mappings are isometries for any $U \in \mathcal{U}_{N+1}$:

$$U_L: \quad \mathcal{U}_{N+1} \quad \longrightarrow \quad \mathcal{U}_{N+1}, \qquad U_R: \quad \mathcal{U}_{N+1} \quad \longrightarrow \quad \mathcal{U}_{N+1}$$
$$U' \quad \mapsto \quad UU' \qquad\qquad\qquad U' \quad \mapsto \quad U'U.$$

We usually refer to the left mapping U_L and we simply denote by $U = U_L : \mathcal{U}_{N+1} \longrightarrow \mathcal{U}_{N+1}$ this left mapping. For every unitary matrix $U \in \mathcal{U}_{N+1}$ and any set $A \subset \mathbb{P}_N(\mathbb{C})$, we denote by $UA \subset \mathbb{P}_N(C)$ the image of A by U in $\mathbb{P}_N(\mathbb{C})$. Namely,

$$UA := \{y \in \mathbb{P}_N(\mathbb{C}) : \exists x \in A : Ux = y\}.$$

A projective algebraic variety (or, simply, a projective variety) is a subset of the complex projective space $\mathbb{P}_N(\mathbb{C})$ given as the set of projective zeros of a collection of homogeneous polynomials. We refer to the reader to [36], [37], [31] for general background on projective varieties.

A *quasi–projective* complex variety is a Zariski open subset of a projective variety (cf. [36] for additional terminology).

Let $V \subseteq \mathbb{P}_N(\mathbb{C})$ be a quasi–projective variety. A simple point in $a \in V$ is a point such that the germ V_a of V at a is a complex submanifold of $\mathbb{P}_N(\mathbb{C})$ of complex dimension equal to $\dim(V)$. We denote by $Reg(V)$ the set of all simple points in V. The Zariski closure of $Reg(V)$ (i.e. the smallest projective variety containing $Reg(V)$) equals to the union of all irreducible components of the Zariski closure of V of dimension equal to $\dim(V)$. In other terms, there is a projective variety $V_1 \subseteq \mathbb{P}_N(\mathbb{C})$ such that $\dim(V_1) < \dim(V)$ and the following equality holds:
$$Reg(V) \setminus V_1 = V \setminus V_1.$$

We shall say that two subsets $A, B \subseteq V$ are generically equal in V if there is $V_1 \subseteq \mathbb{P}_N(\mathbb{C})$ a projective variety satisfying $\dim(V_1) < \dim(V)$ and $A \setminus V_1 = B \setminus V_1$. In other words, V

and $Reg(V)$ are generically equal. If V were equi–dimensional, then $Reg(V)$ is dense (in the standard topology induced by that of $\mathbb{P}_N(\mathbb{C})$ in V).

Let $V \subseteq \mathbb{P}_N(\mathbb{C})$ be a quasi–projective variety of dimension m. Then, $Reg(V) \subseteq \mathbb{P}_N(\mathbb{C})$ is a complex submanifold of complex dimension m, endowed with a volume form $d\nu_m$. We define a measure on V in the following terms:

$$\nu_m[A] := \nu_m[A \cap Reg(V)],$$

for every subset $A \subseteq V$ such that $A \cap Reg(V)$ is measurable for $d\nu_m$. Accordingly, $\int_A f \ d\nu_m$ is the integral of a function $f : A \longrightarrow \mathbb{R}$ (when it can be defined with respect to this measure). Note that given $A, B \subseteq V$ generically equal in V, then $\nu_m[A] = \nu_m[B]$, and $\int_A f \ d\nu_m = \int_B f \ d\nu_m$.

The notion of geometric degree (or, simply, degree) of a projective variety $V \subseteq \mathbb{P}_N(\mathbb{C})$ is a classical notion that comes from the origins of Elimination Theory in the XIX century. The main property satisfied by any accurate notion of degree is a Bézout Inequality. The reader may follow several proofs of Bézout's Inequalities in [24],[45],[14]. Let $W \subseteq \mathbb{P}_N(\mathbb{C})$ be a Zariski open subset in an irreducible projective variety $V \subseteq \mathbb{P}_N(\mathbb{C})$ of Krull dimension m. The geometric degree of W is defined as the following quantity

$$\deg(W) := \max\{\sharp(L \cap W) : L \subseteq \mathbb{P}_N(\mathbb{C}) \text{ linear, } \dim(L) = N - m, \sharp(L \cap W) < +\infty\}.$$

One immediately observes that $\deg(W) = \deg(V)$ for any Zariski open subset W of the irreducible projective variety V. If $V \subseteq \mathbb{P}_N(\mathbb{C})$ is any projective variety, $\deg(V)$ is defined to be the sum of the degrees of its irreducible components. Similarly, for every constructible subset $C \subset \mathbb{P}_N(\mathbb{C})$ we may define $\deg(C)$ as the sum of the degrees of its locally closed irreducible components (cf. [24] for some ideas in this sense). This notion of geometric degree satisfies a Bézout Inequality for locally closed subsets of $\mathbb{P}_N(\mathbb{C})$ (cf. [24]), namely:

$$\deg(W_1 \cap W_2) \leq \deg(W_1) \deg(W_2),$$

for W_1 and W_2 locally closed sets. The following equality immediately follows from the notion of degree.

Proposition 6 *Let $V \subseteq \mathbb{P}_N(\mathbb{C})$ be an equi–dimensional projective subvariety of dimension m. Let $L \subseteq \mathbb{P}_N(\mathbb{C})$ be a fixed projective linear subspace of dimension $N - m$. Then, the following equality holds:*

$$\deg(V) = \max\{\sharp(UL \cap V) : U \in \mathcal{U}_{N+1}, \sharp(UL \cap V) < +\infty\}.$$

The following quantitative estimate is a consequence of Bertini's theorems as used in [28], [16] or [21].

Lemma 7 *Let $V \subseteq \mathbb{P}_N(\mathbb{C})$ be an equi–dimensional projective variety of dimension m. Assume there is a finite subset of homogeneous polynomials $\{f_1, \ldots, f_s\} \subseteq \mathbb{C}[X_0, \ldots, X_N]$ of degree at most d such that*

$$V = V(f_1, \ldots, f_s) = \{x \in \mathbb{P}_N(\mathbb{C}) : f_i(x) = 0, 1 \leq i \leq s\}.$$

Then, the following inequality holds:

$$\deg(V) \leq d^s.$$

The following lemma is probably a well–known fact in Lie Group Theory. We include its proof here for lack of an appropriate reference.

Lemma 8 *Let $x \in \mathbb{C}^{N+1}\backslash\{0\}$ be a non–zero point. The following mapping is a submersion (i.e. its set of critical values is empty):*

$$\psi: \quad \mathcal{U}_{N+1} \quad \longrightarrow \quad S^{2N+1}(\|x\|_2)$$
$$U \quad \mapsto \quad Ux.$$

Proof.– Since ψ is surjective, from Sard's Lemma, we conclude that the set of regular values of ψ is a non–empty dense residual subset of S^{2N+1}. Moreover, given $z, z' \in S^{2N+1}(\|x\|_2)$, let $U_1, U_2 \in \mathcal{U}_{N+1}$ be such that $\psi(U_1) := U_1 x = z$ and $\psi(U_2) = U_2 x = z'$. Let $U' := U_2 U_1^{-1}$ be the unitary matrix such that $U'U_1 = U_2$. Then, $U'z = z'$ and the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{U}_{N+1} & \xrightarrow{\psi} & S^{2N+1}(\|x\|_2) \\
U' \downarrow & & \downarrow Iso_{U'} \\
\mathcal{U}_{N+1} & \xrightarrow{\psi} & S^{2N+1}(\|x\|_2)
\end{array}
$$

where $U'(U) = U'_L(U) = U'U$ is the left translation defined by U' and $Iso_{U'}$ is the isometry defined by U' ($Iso_{U'}(v) = U'v \ \forall v \in S^{2N+1}(\|x\|_2)$). As the differential mappings $d_{U_1}U'_L$ and $d_z Iso_{U'}$ are linear isomorphisms, we also conclude that $d_{U_1}\psi$ is surjective if and only if $d_{U_2}\psi$ is surjective. That is, z is a regular value of ψ if and only if z' is a regular value of ψ. Thus, we conclude that the set of critical values of ψ is empty and the lemma follows.

∎

Lemma 9 *Let M be a complex submanifold of $\mathbb{P}_N(\mathbb{C})$, of complex dimension m. Let M' be a complex submanifold of $\mathbb{P}_N(\mathbb{C})$, of complex dimension p. Then, there is a dense residual subset $W \subset \mathcal{U}_{N+1}$ (depending only on M and M') such that the following properties hold:*

   i) If $m + p < N$, for all $U \in W$, $M \cap UM' = \emptyset$.

   ii) If $m + p \geq N$, for all $U \in W$, $M \cap UM'$ is the empty set or a complex submanifold of $\mathbb{P}_N(\mathbb{C})$ of complex dimension $m + p - N$.

Proof.– Let $\widetilde{M}, \widetilde{M}' \subset \mathbb{C}^{N+1}$ respectively be the cones over M and M'. Namely,

$$\widetilde{M} := \pi^{-1}(M), \qquad \widetilde{M}' := \pi^{-1}(M').$$

Note that $\widetilde{M}$ and $\widetilde{M}'$ are complex submanifolds of $\mathbb{C}^{N+1}$ and their complex dimensions satisfy:

$$\dim(\widetilde{M}) = \dim(M) + 1,$$
$$\dim(\widetilde{M}') = \dim(M') + 1.$$

Let us define the following mapping between (real) submanifolds of $\mathbb{R}^{2(N+1)^2} \times \mathbb{R}^{2(N+1)} \times \mathbb{R}^{2(N+1)}$:

$$\varphi: \quad \mathcal{U}_{N+1} \times \widetilde{M}' \times \widetilde{M} \quad \longrightarrow \quad \mathbb{C}^{N+1}$$
$$(U, \underline{y}, \underline{x}) \quad \mapsto \quad U\underline{y} - \underline{x}.$$

10

We claim that φ is transversal to the submanifold $\{0\}$ of $\mathbb{C}^{N+1}$. Equivalently, we prove that $0 \in \mathbb{C}^{N+1}$ is not a critical value of φ. Let $F := \varphi^{-1}(\{0\})$ be the fiber over $\{0\}$. We then prove that every point $P := (U, \underline{y}, \underline{x}) \in F$ is a regular point of φ. In other words, we just need to prove that the tangent mapping $d_P\varphi$ is surjective, where

$$d_P\varphi : T_U\mathcal{U}_{N+1} \times T_{\underline{y}}\widetilde{M}' \times T_{\underline{x}}\widetilde{M} \longrightarrow T_0\mathbb{C}^{N+1}.$$

Observe that $U\underline{y} = \underline{x}$ implies $\|\underline{y}\|_2 = \|\underline{x}\|_2$. As $\widetilde{M}$ and $\widetilde{M}'$ are cones, identifying $T_{\underline{x}}\widetilde{M}, T_{\underline{y}}\widetilde{M}'$ with subspaces of $\mathbb{C}^{N+1}$ we immediately conclude that $\underline{x} \in T_{\underline{x}}\widetilde{M}$ and $\underline{y} \in T_{\underline{y}}\widetilde{M}'$. Hence, we also have $(0, \underline{y}, 0) \in T_U\mathcal{U}_{N+1} \times T_{\underline{y}}\widetilde{M}' \times T_{\underline{x}}\widetilde{M}$ and

$$d_P\varphi(0, \underline{y}, 0) = U\underline{y} = \underline{x} \in T_0\mathbb{C}^{N+1}.$$

On the other hand, let $\varphi_{\underline{y},\underline{x}}$ be the restriction of φ to $\mathcal{U}_{N+1} \times \{\underline{y}\} \times \{\underline{x}\}$, and let us define the mapping

$$\psi_{\underline{y},\underline{x}} : \begin{array}{ccc} \mathcal{U}_{N+1} & \longrightarrow & S^{2N+1}(\|\underline{y}\|_2) \\ U & \mapsto & U\underline{y}. \end{array}$$

Note that $\psi_{\underline{y},\underline{x}} = t_{\underline{x}} \circ \varphi_{\underline{y},\underline{x}}$, where

$$t_{\underline{x}} : \begin{array}{ccc} \partial B(-\underline{x}, \|\underline{y}\|_2) & \longrightarrow & S^{2N+1}(\|\underline{y}\|_2) \\ v & \mapsto & v + \underline{x} \end{array}$$

is a simple translation, where $\partial B(-\underline{x}, \|\underline{y}\|_2) = \{\underline{z} \in \mathbb{C}^{N+1} : \|\underline{z} + \underline{x}\|_2 = \|\underline{y}\|_2\}$.
From Lemma 8 we know that $\psi_{\underline{y},\underline{x}}$ has no critical values and, hence, $\varphi_{\underline{y},\underline{x}}$ has no critical values. In particular, we have that

$$T_0\partial B(-\underline{x}, \|\underline{y}\|_2) \subseteq Im(d_U\varphi_{\underline{y},\underline{x}}) \subseteq Im(d_P\varphi).$$

Finally, as $\underline{x} + T_0\partial B(-\underline{x}, \|\underline{y}\|_2) = \mathbb{C}^{N+1}$ we conclude that $d_P\varphi$ is a surjective mapping and P is a regular point of φ. Now, we apply the Weak Transversality Theorem (cf. [5]) to conclude that there is a residual subset W of $\mathcal{U}_{N+1}$ such that for every $U \in W$, the mapping

$$\varphi_U : \begin{array}{ccc} \widetilde{M}' \times \widetilde{M} & \longrightarrow & \mathbb{C}^{N+1} \\ (\underline{y}, \underline{x}) & \mapsto & U\underline{y} - \underline{x} \end{array}$$

is transversal to the submanifold $\{0\}$ of $\mathbb{C}^{N+1}$. In particular, the fiber $\varphi_U^{-1}(\{0\})$ is a (possibly empty) complex submanifold of (complex) dimension satisfying the following equality:

$$\dim(\varphi_U^{-1}(\{0\})) = \dim(\widetilde{M}') + \dim(\widetilde{M}) - \text{codim}_{\mathbb{C}^{N+1}}(\{0\}) = m + p - N + 1, \quad (5)$$

for every $U \in W$.
On the other hand, let $U \in W$ be a unitary matrix. Let $M \cap UM' \subset \mathbb{P}_N(\mathbb{C})$ be the projective subset defined by the intersection of M and UM' and let $\widetilde{M \cap UM'}$ be the cone over $M \cap UM'$. Namely

$$\widetilde{M \cap UM'} = \pi^{-1}(M \cap UM').$$

Note that the following is a diffeomorphism between $\varphi_U^{-1}(\{0\})$ and $\widetilde{M \cap UM'}$:

$$\pi_2: \quad \begin{array}{ccc} \varphi_U^{-1}(\{0\}) & \longrightarrow & \widetilde{M \cap UM'} \\ (\underline{y}, \underline{x}) & \mapsto & \underline{x}. \end{array}$$

The inverse of π_2 is obviously given by the following identity

$$\pi_2^{-1}(\underline{x}) = (U^{-1}\underline{x}, \underline{x}).$$

Thus, $\widetilde{M \cap UM'}$ is a complex submanifold of $\mathbb{C}^{N+1}$ of complex dimension $m + p - N + 1$ for every $U \in W$. As $\widetilde{M \cap UM'}$ is the cone over $M \cap UM'$, we also conclude that for every $U \in W$, $M \cap UM'$ is empty or a complex submanifold of $\mathbb{P}_N(\mathbb{C})$ of complex dimension $m + p - N$. Noting that $M \cap UM' = \emptyset$ if and only if $\dim(M \cap UM') = m + p - N < 0$, we have achieved the proof of the lemma. ∎

The following statement is a consequence of the application of the general Poincare's Formula to the complex projective space. It can be read with detail in the paper by Ralph Howard [26, pp. 13-18].

Theorem 10 *Let M, M' be two complex submanifolds of $\mathbb{P}_N(\mathbb{C})$, of respective complex dimensions $m, p \in \mathbb{N}$. Let $f : M \longrightarrow \mathbb{R}$ be a measurable function, such that f is integrable or f is non–negative. Assume that $m + p \geq N$. Then, the following equality holds:*

$$\nu_m[M'] \int_M f \, d\nu_m = \frac{\nu_p[\mathbb{P}_p(\mathbb{C})]\nu_m[\mathbb{P}_m(\mathbb{C})]}{\nu_{p+m-N}[\mathbb{P}_{p+m-N}(\mathbb{C})]} \int_{U \in \mathcal{U}_{N+1}} \int_{x \in UM' \cap M} f(x) \, d\nu_{UM' \cap M} \, d\mathcal{U}_{N+1}.$$

In order to prove this statement we just need to apply Lebesgue's Monotone Convergence Theorem to obtain this result from the very similar one found in [26, pp. 13-18]. A direct proof of this result can also be obtained from Federer's Coarea Formula (cf. [12, Th. 3.2.22]).

Remark 11 *Let the reader observe that the integration on $\mathcal{U}_{N+1}$ in the formula above is done on the residual dense subset W which exists from Lemma 9. Namely,*

$$\int_{U \in \mathcal{U}_{N+1}} \int_{x \in UM' \cap M} f(x) \, d\nu_{UM' \cap M} \, d\mathcal{U}_{N+1} = \int_{U \in W} \int_{x \in UM' \cap M} f(x) \, d\nu_{UM' \cap M} \, d\mathcal{U}_{N+1},$$

where $W \subseteq \mathcal{U}_{N+1}$ is the residual dense subset of these unitary matrices $U \in \mathcal{U}_{N+1}$ such that $M \cap UM'$ is a (possibly empty) complex submanifold of complex dimension $m + p - N$.

Corollary 12 *Let $f : \mathbb{P}_N(\mathbb{C}) \longrightarrow \mathbb{R}$ be an integrable function or a non–negative function. Let $z \in \mathbb{P}_N(\mathbb{C})$ be any point. Then, the following equality holds:*

$$\int_{x \in \mathbb{P}_N(\mathbb{C})} f(x) \, d\mathbb{P}_N(\mathbb{C}) = \nu_N[\mathbb{P}_N(\mathbb{C})] \int_{U \in \mathcal{U}_{N+1}} f(Uz) \, d\mathcal{U}_{N+1}.$$

Proof.– Apply Theorem 10 to $M = \mathbb{P}_N(\mathbb{C})$ and $M' = \{z\}$. ∎

Corollary 13 *Let V, V' be two equi–dimensional complex quasi–projective varieties of respective dimensions m and p. Assume that $m + p - N \geq 0$. Let $A \subset V$, $A' \subset V'$ be two open (for the topology induced by $\mathbb{P}_N(\mathbb{C})$) subsets of V and V'. Then, for almost all $U \in \mathcal{U}_{N+1}$, $V \cap UV'$ is an equi–dimensional quasi–projective variety of dimension $m + p - N$. Moreover, the following equality holds:*

$$\nu_m[A] \; \nu_p[A'] = \frac{\nu_p[\mathbb{P}_p(\mathbb{C})]\nu_m[\mathbb{P}_m(\mathbb{C})]}{\nu_{p+m-N}[\mathbb{P}_{p+m-N}(\mathbb{C})]} \int_{U \in \mathcal{U}_{N+1}} \nu_{m+p-N}[A \cap U A'] \; d\mathcal{U}_{N+1}.$$

Proof.– Let $W_1 \subseteq \mathcal{U}_{N+1}$ be the residual dense subset of Lemma 9. Namely, for all $U \in W_1$, $Reg(V) \cap U Reg(V')$ is a (possibly empty) complex submanifold of complex dimension $m + p - N$. On the other hand, $V \setminus Reg(V)$ can be described as a disjoint union of complex submanifolds of complex dimensions at most $m - 1$. Similarly, $V' \setminus Reg(V')$ can also be described as a disjoint union of complex submanifolds of complex dimension at most $p - 1$. Hence, there is a residual dense subset W_2 of $\mathcal{U}_{N+1}$ such that:

$$U(V' \setminus Reg(V')) \cap Reg(V), \; U Reg(V') \cap (V \setminus Reg(V)) \text{ and}$$

$$U(V' \setminus Reg(V')) \cap (V \setminus Reg(V))$$

are disjoint unions of complex submanifolds of dimension at most $m + p - N - 1$. Then, for every $U \in W = W_1 \cap W_2$ the following properties hold:

- $V \cap UV'$ is a quasi–projective complex variety.

- $V \cap UV'$ is given as a disjoint union of complex submanifolds of dimension at most $m + p - N$.

- $Reg(V) \cap U Reg(V')$ is a complex submanifold of complex dimension $m + p - N$.

- $(V \cap UV') \setminus (Reg(V) \cap U Reg(V'))$ is a constructible subset of dimension at most $m + p - N - 1$.

Hence, $V \cap UV'$ is a quasi–projective variety of dimension $m + p - N$. Now, there exist open subsets $T, T' \subseteq \mathbb{P}_N(\mathbb{C})$ such that $A = V \cap T$, $A' = V' \cap T'$ and so we have:

$$A \cap U A' = (T \cap UT') \cap (V \cap UV').$$

So, for every $U \in W$, $A \cap U A'$ is an open subset of $V \cap UV'$. So, for $U \in W$ we have:

$$\nu_{m+p-N}[A \cap U A'] = \nu_{m+p-N}[(A \cap U A') \cap Reg(V \cap UV')] =$$

$$= \nu_{m+p-N}[(A \cap Reg(V)) \cap (U A' \cap Reg(U A'))].$$

Additionally, we have:

$$\nu_m[A] = \nu_m[A \cap Reg(V)] \qquad \nu_m[A'] = \nu_m[A' \cap Reg(V')].$$

The statement of the corollary follows immediately from Theorem 10 above, applied to the complex manifolds $A \cap Reg(V)$ and $A' \cap Reg(V')$. ∎

The following identity relates the geometric degree of an equi–dimensional quasi–projective variety and its volume. A different proof of this identity for the case that the variety is algebraic and smooth, may be found in [31, Th. 5.22].

Corollary 14 *Let $V \subseteq \mathbb{P}_N(\mathbb{C})$ be an equi–dimensional quasi–projective variety of dimension m. Then, the following equality holds:*

$$\nu_m[V] = \nu_m[\mathbb{P}_m(\mathbb{C})] \deg(V).$$

Proof.– Let $M := Reg(V)$ be the submanifold of all the simple points of V. Let $L^{N-m} \subseteq \mathbb{P}_N(\mathbb{C})$ be a linear subspace of dimension $N - m$. From the proof of Lemma 9 above, there is a dense residual subset W of $\mathcal{U}_{N+1}$ such that for every $U \in W$, UL^{N-m} and M are transversal at any common zero. Namely, for every $U \in W$, $UL^{N-m} \cap M$ is a zero–dimensional complex submanifold and for every $x \in UL^{N-m} \cap M$, the tangent spaces $T_x UL^{N-m}$ and $T_x M$ are transversal. From [31, Th. 5.16], we conclude that for all $U \in W$, $\sharp(UL^{N-m} \cap M) = \sharp(UL^{N-m} \cap V) = \deg(V)$.
From Corollary 13 above we conclude:

$$\nu_m[V] \, \nu_{N-m}[L^{N-m}] = \nu_{N-m}[\mathbb{P}_{N-m}(\mathbb{C})]\nu_m[\mathbb{P}_m(\mathbb{C})] \int_{U \in \mathcal{U}_{N+1}} \sharp(V \cap UL^{N-m}) \, d\mathcal{U}_{N+1}.$$

Thus we conclude

$$\nu_m[V] \, \nu_{N-m}[L^{N-m}] = \nu_{N-m}[\mathbb{P}_{N-m}(\mathbb{C})]\nu_m[\mathbb{P}_m(\mathbb{C})] \deg(V),$$

and hence the equality above. ∎

Corollary 15 *Let V be an equi–dimensional quasi–projective subvariety of $\mathbb{P}_N(\mathbb{C})$, of complex dimension m. Let $A \subseteq V$ be an open subset of V and $0 \leq \varepsilon \leq 1$ be a positive number. The following equality holds:*

$$\nu_m[A]\nu_N[\mathbb{P}_N(\mathbb{C})]\varepsilon^{2N} = \int_{x \in \mathbb{P}_N(\mathbb{C})} \nu_m[B_\mathbb{P}(x, \varepsilon) \cap A] \, d\mathbb{P}_N(\mathbb{C}).$$

Proof.– Apply Corollary 13 to A and $B_\mathbb{P}(e_0, \varepsilon)$, obtaining:

$$\nu_m[A]\nu_N[B_\mathbb{P}(e_0, \varepsilon)] = \frac{\nu_N[\mathbb{P}_N(\mathbb{C})]\nu_m[\mathbb{P}_m(\mathbb{C})]}{\nu_m[\mathbb{P}_m(\mathbb{C})]} \int_{U \in \mathcal{U}_{N+1}} \nu_m[UB_\mathbb{P}(e_0, \varepsilon) \cap A] \, d\mathcal{U}_{N+1}.$$

Now, use Corollary 12 to see that:

$$\int_{U \in \mathcal{U}_{N+1}} \nu_m[UB_\mathbb{P}(e_0, \varepsilon) \cap A] \, d\mathcal{U}_{N+1} = \frac{1}{\nu_N[\mathbb{P}_N(\mathbb{C})]} \int_{x \in \mathbb{P}_N(\mathbb{C})} \nu_m[B_\mathbb{P}(x, \varepsilon) \cap A] \, d\mathbb{P}_N(\mathbb{C}).$$

So, we have obtained:

$$\nu_m[A]\nu_N[B_\mathbb{P}(e_0, \varepsilon)] = \int_{x \in \mathbb{P}_N(\mathbb{C})} \nu_m[B_\mathbb{P}(x, \varepsilon) \cap A] \, d\mathbb{P}_N(\mathbb{C}).$$

Now, the following equality holds:

$$\nu_N[B_\mathbb{P}(e_0, \varepsilon)] = \nu_N[\mathbb{P}_N(\mathbb{C})]\varepsilon^{2N},$$

and we conclude the result. ∎

The following corollary may be understood as a Bézout Theorem on the average.

Corollary 16 *Let V and V' be equi–dimensional quasi–projective subvarieties of $\mathbb{P}_N(\mathbb{C})$ of respective complex dimensions m and p. Suppose that $m + p \geq N$. Then, for almost all $U \in \mathcal{U}_{N+1}$, $V \cap UV'$ is an equi–dimensional quasi–projective variety of dimension $m+p-N$ and the following equality holds:*

$$\deg(V)\deg(V') = \int_{\mathcal{U}_{N+1}} \deg(V \cap UV') \, d\mathcal{U}_{N+1}.$$

Proof.– Apply Corollary 13 to V and V', then use Corollary 14 to replace $\nu_m[V]$ by $\nu_m[\mathbb{P}_m(\mathbb{C})]\deg(V)$, and the same for V' and $UV' \cap V$. ∎

Remark 17 *In [34], a result similar to Corollary 16 is announced without a proof. Combination of the classical Bézout inequality with Corollary 16 yields the following equality:*

$$\nu_{\mathcal{U}_{N+1}}[U \in \mathcal{U}_{N+1} : \deg(V \cap UV') \neq \deg(V)\deg(V')] = 0,$$

for V and V' equi–dimensional varieties of respective dimensions m, p with $m + p \geq N$. A similar result to that of Corollary 13 can be stated for the case that the ambient space is either the real projective space $\mathbb{P}_N(\mathbb{R})$ or the $N-$dimensional sphere $S^N \subset \mathbb{R}^{N+1}$. In these cases, the unitary group turns to be the orthogonal group of matrices, also normalized with total volume 1.
In [26], more consequences of Poincare's Formula in the real case are exhibited.

3   Extrinsic tubes.

In this Section we prove Theorem 2. Namely, we state upper and lower bounds for the volume of the intersection of a projective variety with a tube about another projective variety.

For every two positive integer numbers $1 \leq m < N$, let $C(N, m) \in \mathbb{Q}$ be the number given by

$$C(N, m) := 2\frac{N^{2N}}{m^{2m}(N-m)^{2(N-m)}} \leq 2\left(\frac{e\,N}{N-m}\right)^{2(N-m)},$$

where e stands for the basis of the natural logarithms. Then, for every three positive integer numbers $1 \leq m' < m < N$, let $C(N, m, m') \in \mathbb{Q}$ be the number given by

$$C(N, m, m') := \frac{1}{2}C(N, m')C(N - m', N - m).$$

For every subset $A \subset \mathbb{P}_N(\mathbb{C})$, $N > 1$ and for every positive real number $0 < \varepsilon$, let the *tube of radius ε about A* be the subset $A_\varepsilon \subseteq \mathbb{P}_N(\mathbb{C})$ defined by the following identity:

$$A_\varepsilon := \{z \in \mathbb{P}_N(\mathbb{C}) : d_{\mathbb{P}}(z, A) < \varepsilon\}.$$

That is, A_ε is the set of projective points $z \in \mathbb{P}_N(\mathbb{C})$ such that the projective distance to some point in A is smaller than ε.

The following statement is a more technical and precise version of Theorem 2. Note that the lower bound is a partial answer to the question in the last paragraph of [20, p. 178].

Theorem 18 *Let V, V' be two proper equi–dimensional projective varieties of $\mathbb{P}_N(\mathbb{C})$, of respective dimensions $m > m' \geq 1$. Let $0 < \varepsilon \leq 1$ be a positive real number. Suppose that $m < N$. Then, the following inequality holds:*

$$\frac{\nu_m[V'_\varepsilon \cap V]}{\nu_m[V]} \leq C(N, m, m')\deg(V')\varepsilon^{2(m-m')}. \tag{6}$$

Moreover, if $V' \subseteq V$ and $0 \leq \varepsilon \leq \frac{\sqrt{2}}{2}$, the following also holds:

$$\frac{\nu_m[V'_\varepsilon \cap V]}{\nu_m[\mathbb{P}_m(\mathbb{C})]} \geq \frac{1}{2}\varepsilon^{2(m-m')}. \tag{7}$$

The constant $C(N, m)$ also satisfies the following inequality:

$$C(N, m) = C(N, N - m) \leq 2\left(\frac{e\,N}{m}\right)^{2m}.$$

Moreover, the following estimate is consequence of [39]:

$$2\sqrt{\pi}\frac{\sqrt{m}\sqrt{N-m}}{\sqrt{N}}\binom{N}{m} < C(N,m)^{1/2} < 2e^{1/6}\sqrt{\pi}\frac{\sqrt{m}\sqrt{N-m}}{\sqrt{N}}\binom{N}{m}.$$

Hence, the constant $C(N, m, m')$ is essentially equal to the square of the multinomial coefficient

$$\frac{N!}{(m')!(N - m)!(m - m')!}.$$

We start by some technical results that we will use to prove Theorem 18.

3.1 Some Technical Lemmata.

The first technical result is due to H. Federer [11, Th. 4.2]. A more readable version can be found in [41]. In what follows, $\mathscr{H}^m$ denotes the usual Hausdorff m–dimensional measure (cf. [12, p. 171], for instance). Recall that for every complex equi–dimensional affine algebraic variety $\overline{V} \subseteq \mathbb{C}^N$ of dimension m and for every open subset $A \subseteq \mathbb{C}^N$, $\mathscr{H}^{2m}[\overline{V} \cap A]$ equals the $2m$–volume of the regular part of $\overline{V} \cap A$, considered as a submanifold of $\mathbb{C}^N$.

Lemma 19 (cf. [11],[41]) *Let $\varepsilon > 0$ be a positive real number, and let $\overline{V}$ be an equi–dimensional algebraic subvariety of $\mathbb{C}^N$, of dimension m. Suppose that $0 \in \overline{V}$. Then, the following formula holds:*

$$\mathscr{H}^{2m}[\overline{V} \cap B_{\mathbb{C}^N}(0, \varepsilon)] \geq \mathscr{H}^{2m}[B_{\mathbb{C}^m}(0, 1)]\varepsilon^{2m}.$$

Next statement is a classical formula discovered by Federer that can be found many places in the literature. Some classic references are [12], [30], [34]. Our formulation bellow has been taken from [1, p. 241].

Theorem 20 (Coarea Formula) *Consider a differentiable map $F : \mathsf{M} \longrightarrow \mathsf{N}$, where M, N are Riemannian manifolds of real dimensions $n_1 \geq n_2$. Consider a measurable*

function $f : \mathsf{M} \longrightarrow \mathbb{R}$, such that f is integrable. Then, for every $y \in \mathsf{N}$ except a zero–measure set, $F^{-1}(y)$ is empty or a real submanifold of M of real dimension $n_1 - n_2$. Moreover, the following equality holds (and the integrals appearing on it are well defined):

$$\int_{\mathsf{M}} f N J_x F \, d\mathsf{M} = \int_{y \in \mathsf{N}} \int_{x \in F^{-1}(y)} f(x) \, dF^{-1}(y) d\mathsf{N},$$

where $N J_x F$ is the normal jacobian of F in x, defined as the volume in $T_{F(x)}\mathsf{N}$ of the image by $d_x F$ of an unit cube in $(T_x \mathsf{M}) \cap Ker(d_x F)^{\perp}$ (see [1] for details).

Lemma 21 *Let $\{(\mathbb{A}_i^N, \varphi_i) : 0 \leq i \leq N\}$ be the atlas of $\mathbb{P}_N(\mathbb{C})$ given by the affine charts. Namely,*

$$\varphi_i : \quad \begin{matrix} \mathbb{C}^N \\ (z_1, \ldots, z_N) \end{matrix} \quad \begin{matrix} \longrightarrow \\ \mapsto \end{matrix} \quad \begin{matrix} \mathbb{A}_i^N := \{x \in \mathbb{P}_N(\mathbb{C}) : x_i \neq 0\} \subseteq \mathbb{P}_N(\mathbb{C}) \\ (z_1 : \cdots : z_i : 1 : z_{i+1} : \cdots : z_N). \end{matrix}$$

Then, for every $\underline{z} \in \mathbb{C}^N$ the following properties hold:

i) For every tangent vector $v \in T_{\underline{z}}\mathbb{C}^N$, $\|v\|_{T_{\underline{z}}\mathbb{C}^N} = 1$, we have

$$\frac{1}{1 + \|z\|_{\mathbb{C}^N}^2} \leq \|d_{\underline{z}}\varphi_i(v)\|_{T_{\varphi_i(\underline{z})}\mathbb{P}_N(\mathbb{C})} \leq \frac{1}{(1 + \|z\|_{\mathbb{C}^N}^2)^{1/2}}.$$

ii) The normal jacobian of φ_i satisfies

$$N J_{\underline{z}} \varphi_i = \frac{1}{(1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^{N+1}}.$$

iii) For every complex submanifold $M \subseteq \mathbb{C}^N$ of complex dimension $m \geq 1$, and for every $\underline{z} \in M$, the normal jacobian of $\varphi_i \mid_M : M \longrightarrow \varphi_i(M)$ satisfies

$$\frac{1}{(1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^{m+1}} \leq N J_{\underline{z}}(\varphi_i \mid_M) \leq \frac{1}{(1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^m}.$$

Proof.– First of all, it is enough to prove the claim for $i = 0$. Denote $\varphi := \varphi_0$. Namely,

$$\varphi := \varphi_0 : \quad \begin{matrix} \mathbb{C}^N \\ (z_1, \ldots, z_N) \end{matrix} \quad \begin{matrix} \longrightarrow \\ \mapsto \end{matrix} \quad \begin{matrix} \mathbb{A}_0^N := \{x \in \mathbb{P}_N(\mathbb{C}) : x_0 \neq 0\} \subseteq \mathbb{P}_N(\mathbb{C}) \\ (1 : z_1 : \cdots : z_N). \end{matrix}$$

Let $0 \in \mathbb{C}^N$ be the origin and $e_0 = \varphi(0) = (1 : 0 : \cdots : 0)$ its image. Observe that the tangent mapping

$$d_0\varphi : T_0\mathbb{C}^N \longrightarrow T_{e_0}\mathbb{P}_N(\mathbb{C})$$

is an isometry and, hence, $N J_0\varphi = 1$. Let $\underline{z} \in \mathbb{C}^N$ be any point, $\underline{z} = (z_1, \ldots, z_N)$. Let $U \in \mathcal{U}_{N+1}$, $U = (u_{ij})_{i,j=0\ldots N}$ be an unitary matrix such that $U\varphi(\underline{z}) = e_0$. Namely,

$$U \begin{pmatrix} 1 \\ \underline{z}^t \end{pmatrix} = \begin{pmatrix} (1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^{1/2} \\ 0 \end{pmatrix}. \tag{8}$$

Let $U_0, \ldots, U_N$ be the rows of U. Note that U_0 can be chosen to be the complex vector

$$U_0 = \frac{1}{(1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^{1/2}} \ (1, \overline{z_1}, \ldots, \overline{z_N}),$$

where $\overline{z_i}$ holds for the complex conjugate of z_i. Additionally, $U_1, \ldots, U_N$ are orthogonal to U_0. On the other hand, $U : \mathbb{P}_N(\mathbb{C}) \longrightarrow \mathbb{P}_N(\mathbb{C})$ is also an isometry at any projective point and, hence, $NJ_{\varphi(\underline{z})}U = 1$. Finally, let $\phi : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ be the mapping given by

$$\phi := \varphi^{-1} \circ U \circ \varphi.$$

Observe that $\phi(z) = 0$ and $\varphi \circ \phi = U \circ \varphi$. This yields the following equality between normal jacobians:

$$NJ_0\varphi \ NJ_{\underline{z}}\phi = NJ_{\varphi(\underline{z})}U \ NJ_{\underline{z}}\varphi.$$

Hence, we conclude that $NJ_{\underline{z}}\phi = NJ_{\underline{z}}\varphi$.
Additionally, for every tangent vector $v \in T_{\underline{z}}\mathbb{C}^N$, we have

$$d_{\underline{z}}\phi(v) = \frac{1}{(1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^{1/2}} \left(U_1 \begin{pmatrix} 0 \\ v^t \end{pmatrix}, \ldots, U_N \begin{pmatrix} 0 \\ v^t \end{pmatrix} \right),$$

where v^t is the transpose of the vector v. Let $v, w \in T_{\underline{z}}\mathbb{C}^N$ be two tangent vectors. Then, we have

$$< d_{\underline{z}}\phi(v), d_{\underline{z}}\phi(w) >_{T_0\mathbb{C}^N} = \frac{1}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2} \sum_{i=1}^N U_i \begin{pmatrix} 0 \\ v^t \end{pmatrix} \overline{U_i \begin{pmatrix} 0 \\ w^t \end{pmatrix}},$$

where $\overline{\cdot}$ stands for complex conjugation. Hence,

$$< d_{\underline{z}}\phi(v), d_{\underline{z}}\phi(w) >_{T_0\mathbb{C}^N} = \frac{1}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2} \left[< v, w >_{\mathbb{C}^N} - U_0 \begin{pmatrix} 0 \\ v^t \end{pmatrix} \overline{U_0 \begin{pmatrix} 0 \\ w^t \end{pmatrix}} \right].$$

Assume now that $< v, \underline{z} >_{\mathbb{C}^N} = 0$. Then, we have

$$U_0 \begin{pmatrix} 0 \\ v^t \end{pmatrix} = \frac{1}{(1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^{1/2}} < v, \underline{z} >_{\mathbb{C}^N} = 0.$$

Hence, for every $v \in T_{\underline{z}}\mathbb{C}^N$ such that $< v, \underline{z} >_{\mathbb{C}^N} = 0$, and for every $w \in T_{\underline{z}}\mathbb{C}^N$, the following equality holds:

$$< d_{\underline{z}}\phi(v), d_{\underline{z}}\phi(w) >_{T_0\mathbb{C}^N} = \frac{1}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2} < v, w >_{\mathbb{C}^N}.$$

Now, let $\{b_1, \ldots, b_N\}$ be an orthonormal frame of $T_{\underline{z}}\mathbb{C}^N$ such that $b_N = \frac{1}{\|\underline{z}\|_{\mathbb{C}^N}}\underline{z}$. This implies that $< b_i, \underline{z} >_{T_{\underline{z}}\mathbb{C}^N} = 0$ for $i = 1 \ldots N - 1$. Then, we have

$$< d_{\underline{z}}\phi(b_i), d_{\underline{z}}\phi(b_j) >_{T_0\mathbb{C}^N} = \frac{1}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2} < b_i, b_j >_{\mathbb{C}^N} = 0 \qquad i \neq j.$$

Additionally, for every i, $1 \leq i \leq N - 1$,

$$< d_{\underline{z}}\phi(b_i), d_{\underline{z}}\phi(b_i) >_{T_0\mathbb{C}^N} = \frac{1}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2}.$$

For $i = N$, we have

$$< d_{\underline{z}}\phi(b_N), d_{\underline{z}}\phi(b_N) >_{T_0\mathbb{C}^N} = \frac{1}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2}\left[1 - \frac{1}{\|\underline{z}\|_{\mathbb{C}^N}^2}U_0\begin{pmatrix}0\\\underline{z}^t\end{pmatrix}\overline{U_0\begin{pmatrix}0\\\underline{z}^t\end{pmatrix}}\right].$$

Now, observe that:

$$U_0\begin{pmatrix}0\\\underline{z}^t\end{pmatrix}\overline{U_0\begin{pmatrix}0\\\underline{z}^t\end{pmatrix}} = \left[U_0\begin{pmatrix}1\\\underline{z}^t\end{pmatrix} - u_{00}\right]\overline{\left[U_0\begin{pmatrix}1\\\underline{z}^t\end{pmatrix} - u_{00}\right]} = \frac{\|\underline{z}\|_{\mathbb{C}^N}^4}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2}.$$

Thus, we conclude

$$\|d_{\underline{z}}\phi(b_N)\|_{T_0\mathbb{C}^N}^2 = \frac{1}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2}\left[1 - \frac{\|\underline{z}\|_{\mathbb{C}^N}^2}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2}\right] = \frac{1}{(1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^2}.$$

We immediately obtain claim $ii)$, since

$$NJ_{\underline{z}}\varphi = NJ_{\underline{z}}\phi = \prod_{i=1}^N \|d_{\underline{z}}\phi(b_i)\|_{T_0\mathbb{C}^N}^2 = \frac{1}{(1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^{N+1}}.$$

Now, let $v = T_{\underline{z}}\mathbb{C}^N$, $v = \sum_{i=1}^N \lambda_i v_i$, $\sum_{i=1}^N |\lambda_i|^2 = 1$. Then,

$$\|d_{\underline{z}}\phi(v)\|_{T_0\mathbb{C}^N}^2 = \sum_{i=1}^N |\lambda_i|^2\|d_{\underline{z}}\phi(b_i)\|_{T_0\mathbb{C}^N}^2 = \frac{1}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2}\left(\sum_{i=1}^{N-1}|\lambda_i|^2 + |\lambda_N|^2\frac{1}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2}\right),$$

which implies

$$\frac{1}{1 + \|\underline{z}\|_{\mathbb{C}^N}^2} \leq \|d_{\underline{z}}\phi(v)\|_{T_0\mathbb{C}^N} \leq \frac{1}{(1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^{1/2}}. \qquad (9)$$

Now, since $\phi = \varphi^{-1} \circ U \circ \varphi$, we have

$$d_0\varphi\ d_{\underline{z}}\phi(v) = d_{\varphi(\underline{z})}U\ d_{\underline{z}}\varphi(v),$$

where $d_0\varphi$ and $d_{\varphi(\underline{z})}U$ are linear isometries. Thus, we conclude

$$\|d_{\underline{z}}\phi(v)\|_{T_0\mathbb{C}^N} = \|d_{\underline{z}}\varphi(v)\|_{T_{\varphi(\underline{z})}\mathbf{P}_N(\mathbb{C})},$$

and claim $i)$ follows from inequalities (9) above.

Let us denote by $\{b_1', \ldots, b_N'\}$ the image under $d_{\underline{z}}\varphi$ of the basis $\{b_1, \ldots, b_N\}$. Namely,

$$b_i' = d_{\underline{z}}\varphi(b_i), \qquad i = 1 \ldots N.$$

Then, we have proved that $\{b_1', \ldots, b_N'\}$ is orthogonal. In fact,

$$< b_j', b_i' >_{T_{\varphi(\underline{z})}\mathbf{P}_N(\mathbb{C})} = < d_{\underline{z}}\phi(b_i), d_{\underline{z}}\phi(b_j) >_{T_0\mathbb{C}^N} = 0, \qquad i \neq j.$$

Moreover,

$$\|b_i'\|_{T_{\varphi(\underline{z})}\mathbf{P}_N(\mathbb{C})} = \frac{1}{\sqrt{1 + \|\underline{z}\|_{\mathbb{C}^N}^2}}, \qquad i = 1 \ldots N - 1,$$

and
$$\|b'_N\|_{T_{\varphi(\underline{z})}\mathbf{P}_N(\mathbb{C})} = \frac{1}{1 + \|\underline{z}\|^2_{\mathbb{C}^N}}.$$

Let $M \subseteq \mathbb{C}^N$ be a complex submanifold of complex dimension m, and let $\underline{z} \in M$ be a point. Recall that $T_{\underline{z}}M$ is a $m-$dimensional complex subspace of $T_{\underline{z}}\mathbb{C}^N$, endowed with the Hermitian product inherited from that of $T_{\underline{z}}\mathbb{C}^N$. Then, the following expression defines a linear subspace of $T_{\underline{z}}\mathbb{C}^N$ of complex dimension at least $m-1$:

$$W := T_{\underline{z}}M \cap <\{b_1,\ldots,b_{N-1}\}>,$$

where $<\{b_1,\ldots,b_{N-1}\}>$ is the complex subspace of $\mathbb{C}^N$ generated by these vectors. Then, we can find an orthonormal frame $\{c_1,\ldots,c_m\}$ of $T_{\underline{z}}M$ such that $c_1,\ldots,c_{m-1} \in W$. Hence, for every $i = 1\ldots m-1$ we have

$$\|d_{\underline{z}}(\varphi \mid_M)(c_i)\|_{T_{\varphi(\underline{z})}\mathbf{P}_N(\mathbb{C})} = \frac{1}{(1 + \|\underline{z}\|^2_{\mathbb{C}^N})^{1/2}},$$

and the real number $\|d_{\underline{z}}(\varphi \mid_M)(c_m)\|$ is bounded from equation (9). Without loss of generality we may assume $c_i = b_i$, for $1 \le i \le m-1$. Then, $d_{\underline{z}}(\varphi \mid_M (c_m))$ belongs to the complex subspace $<b'_m,\ldots,b'_N>$ and it is orthogonal to the complex subspace generated by $\{d_{\underline{z}}(\varphi \mid_M)(c_i) : 1 \le i \le m-1\}$. In particular, we have seen that the family of vectors $\{d_{\underline{z}}(\varphi \mid_M)(c_1),\ldots,d_{\underline{z}}(\varphi \mid_M)(c_m)\}$ is orthogonal. Thus, the normal jacobian satisfies the following equality:

$$NJ_{\underline{z}}(\varphi \mid_M) = \prod_{i=1}^{m} \|d_{\underline{z}}(\varphi \mid_M)(c_i)\|^2_{T_{\varphi(\underline{z})}\mathbf{P}_N(\mathbb{C})},$$

and claim iii) follows. $\blacksquare$

Lemma 22 *Let V be an irreducible projective variety in $\mathbf{P}_N(\mathbb{C})$ of dimension $m \ge 1$. Let $x \in V$ be a point in V and $0 < \varepsilon \le 1$ a positive real number. Then, the following inequality holds:*
$$\nu_m[V \cap B_{\mathbf{P}}(x,\varepsilon)] \ge \nu_m[\mathbf{P}_m(\mathbb{C})]\varepsilon^{2m}(1 - \varepsilon^2).$$
In particular, for every $\varepsilon > 0$ such that $\varepsilon \le \frac{\sqrt{2}}{2}$, we have

$$\nu_m[V \cap B_{\mathbf{P}}(e_0,\varepsilon)] \ge \frac{1}{2}\nu_m[\mathbf{P}_m(\mathbb{C})]\varepsilon^{2m}.$$

Proof.– Let $\mathbb{A}_0^N$ and φ_0 be as in the former lemma. Without loss of generality we may assume that
$$x = e_0 = (1:0:\cdots:0) \in V \cap \mathbb{A}_0^N \neq \emptyset.$$

In particular, the variety $V \cap \mathbb{A}_0^N$ is dense in V both for the Zariski and the usual topology. Note that $d_0\varphi_0 : T_0\mathbb{C}^N \longrightarrow T_{e_0}\mathbf{P}_N(\mathbb{C})$ is a linear isometry. Additionally, observe that the following equality holds for every ε, $0 < \varepsilon < 1$:

$$\varphi_0^{-1}(B_{\mathbf{P}}(e_0,\varepsilon)) = B_{\mathbb{C}^N}(0, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}). \tag{10}$$

This equality follows from the following chain of identities:

$$d_{\mathbf{P}}(e_0, \varphi_0(\underline{z})) = \sin \arccos \frac{|<e_0, (1, \underline{z})>_{\mathbb{C}^{N+1}}|}{\|(1, \underline{z})\|_{\mathbb{C}^{N+1}}} = \frac{\|\underline{z}\|_{\mathbb{C}^N}}{\sqrt{1 + \|\underline{z}\|_{\mathbb{C}^N}^2}} = \frac{d_{\mathbb{C}^N}(0, \underline{z})}{\sqrt{1 + d_{\mathbb{C}^N}(0, \underline{z})^2}},$$

and, hence,

$$d_{\mathbb{C}^N}(0, \underline{z}) = \frac{d_{\mathbf{P}}(e_0, \varphi_0(\underline{z}))}{\sqrt{1 - d_{\mathbf{P}}(e_0, \varphi_0(\underline{z}))^2}},$$

which leads to equality (10) above.

Let $W = Reg(V)$ be the complex submanifold of $\mathbb{P}_N(\mathbb{C})$ of complex dimension m consisting of the regular points of V. Let $\overline{W} = \varphi_0^{-1}(W)$ be the inverse image of W by φ_0. $\overline{W}$ is the complex submanifold of $\mathbb{C}^N$ of complex dimension m formed by the regular points of the algebraic variety $\overline{V} = \varphi_0^{-1}(V)$. From Lemma 21 the following inequality holds:

$$NJ_{\underline{z}}(\varphi_0 \mid_{\overline{W}}) \geq \frac{1}{(1 + \|\underline{z}\|_{\mathbb{C}^N}^2)^{m+1}}.$$

Now, Theorem 20 yields the following chain of equalities and inequalities:

$$\nu_m[W \cap B_{\mathbf{P}}(e_0, \varepsilon)] = \int_{\underline{z} \in \overline{W} \cap B_{\mathbb{C}^N}(0, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}})} NJ_{\underline{z}}(\varphi_0 \mid_{\overline{W}}) \, d\overline{W} \geq$$

$$\geq \int_{\underline{z} \in \overline{W} \cap B_{\mathbb{C}^N}(0, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}})} \frac{1}{\left(1 + \|\underline{z}\|_{\mathbb{C}^N}^2\right)^{m+1}} \, d\overline{W} \geq \frac{1}{\left(1 + \frac{\varepsilon^2}{1-\varepsilon^2}\right)^{m+1}} \mathscr{H}^{2m} \left[\overline{W} \cap B_{\mathbb{C}^N}(0, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}})\right] =$$

$$= \mathscr{H}^{2m} \left[\overline{W} \cap B_{\mathbb{C}^N}(0, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}})\right] (1 - \varepsilon^2)^{m+1},$$

where $\mathscr{H}^{2m}$ holds for the usual Hausdorff $2m-$dimensional measure. As $\overline{V} \setminus \overline{W} = \varphi_0^{-1}(V \setminus W)$ is contained in an affine algebraic subvariety of complex dimension at most $m - 1$, we have:

$$\mathscr{H}^{2m} \left[\overline{W} \cap B_{\mathbb{C}^N}(0, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}})\right] = \mathscr{H}^{2m} \left[\overline{V} \cap B_{\mathbb{C}^N}(0, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}})\right].$$

Next, Lemma 19 implies:

$$\mathscr{H}^{2m} \left[\overline{V} \cap B_{\mathbb{C}^N}(0, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}})\right] \geq \mathscr{H}^{2m}[B_{\mathbb{C}^m}(0, 1)] \left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right)^{2m}.$$

Finally, observe that

$$\mathscr{H}^{2m}[B_{\mathbb{C}^m}(0, 1)] = \frac{\pi^m}{m!} = \nu_m[\mathbb{P}_m(\mathbb{C})].$$

Thus, we conclude that

$$\nu_m[V \cap B_{\mathbf{P}}(e_0, \varepsilon)] \geq \nu_m[\mathbb{P}_m(\mathbb{C})]\varepsilon^{2m}(1 - \varepsilon^2).$$

■

The following result immediately follows from Lemma 22 above:

21

Corollary 23 *Let V be a (possibly not equi–dimensional) algebraic projective variety in $\mathbb{P}_N(\mathbb{C})$, and let m be the maximum of the dimensions of its irreducible components. Let $x \in V$ be a point in V and $0 < \varepsilon \leq 1$ a positive real number. Then, the following inequality holds:*

$$\nu_m[V \cap B_{\mathbf{P}}(x, \varepsilon)] \geq \mathcal{C}(V, x)\ \nu_m[\mathbb{P}_m(\mathbb{C})]\varepsilon^{2m}(1 - \varepsilon^2),$$

where $\mathcal{C}(V, x)$ holds for the number of irreducible components of dimension m of V which contain x.

Corollary 24 *Let $V \subseteq \mathbb{P}_N(\mathbb{C})$ be an equi–dimensional algebraic variety of dimension m. Let $0 < \varepsilon < \varepsilon_1$ be two positive real numbers, $\varepsilon < 1$. Assume that $\varepsilon_1 - \varepsilon \leq \frac{\sqrt{2}}{2}$. Then, the following inequality holds for every $z \in V_\varepsilon$,*

$$\frac{\nu_m[B_{\mathbf{P}}(z, \varepsilon_1) \cap V]}{\nu_m[P_m(\mathbb{C})]} \geq \frac{1}{2}(\varepsilon_1 - \varepsilon)^{2m}.$$

Proof.– As $z \in V_\varepsilon$, there exists $y \in V$ such that $d_{\mathbf{P}}(z, y) < \varepsilon$. Hence,

$$B_{\mathbf{P}}(z, \varepsilon_1) \supseteq B_{\mathbf{P}}(y, \varepsilon_1 - \varepsilon).$$

Thus, Lemma 22 implies the following chain of inequalities:

$$\frac{\nu_m[B_{\mathbf{P}}(z, \varepsilon_1) \cap V]}{\nu_m[P_m(\mathbb{C})]} \geq \frac{\nu_m[B_{\mathbf{P}}(y, \varepsilon_1 - \varepsilon) \cap V]}{\nu_m[P_m(\mathbb{C})]} \geq \frac{1}{2}(\varepsilon_1 - \varepsilon)^{2m}.$$

∎

Lemma 25 *Let $V \subseteq \mathbb{P}_N(\mathbb{C})$ be a projective subspace of complex dimension m. Let $0 < \varepsilon < \varepsilon_1 \leq 1$ be two positive real numbers. Then, the following inequality holds for every $z \in V_\varepsilon$,*

$$\frac{\nu_m[B_{\mathbf{P}}(z, \varepsilon_1) \cap V]}{\nu_m[P_m(\mathbb{C})]} \geq (\varepsilon_1^2 - \varepsilon^2)^m \frac{1 - \varepsilon_1^2}{(1 - \varepsilon^2)^m}.$$

Proof.– Let $\mathbb{A}_0^N = \mathbb{P}_N(\mathbb{C}) \setminus \{x_0 = 0\}$ and $\varphi = \varphi_0$ be like in Lemma 21 above. Without loss of generality we may assume that $z = e_0 := (1 : 0 : \cdots : 0) \in V_\varepsilon$. Let $z' \in V$ be a point such that

$$d_{\mathbf{P}}(e_0, V) = d_{\mathbf{P}}(e_0, z') = d < \varepsilon.$$

We may also assume that $\varepsilon_1 < 1$, namely $z' \in \mathbb{A}_0^N \cap V$. As in the proof of Lemma 22 above, we have

$$d_{\mathbb{C}^N}(0, \varphi^{-1}(z')) = \frac{d}{\sqrt{1 - d^2}} \leq \frac{\varepsilon}{\sqrt{1 - \varepsilon^2}}.$$

Moreover,

$$\nu_m[V \cap B_{\mathbf{P}}(e_0, \varepsilon_1)] \geq \mathscr{H}^{2m}\left[\varphi^{-1}(V) \cap B_{\mathbb{C}^N}\left(0, \frac{\varepsilon_1}{\sqrt{1 - \varepsilon_1^2}}\right)\right](1 - \varepsilon_1^2)^{m+1}.$$

Now, observe that $\varphi^{-1}(V) \subseteq \mathbb{C}^N$ is a linear affine subspace . Hence,

$$\|\varphi^{-1}(z')\|_{\mathbb{C}^N} = \frac{d}{\sqrt{1 - d^2}} = d_{\mathbb{C}^N}(0, \varphi^{-1}(V)).$$

Moreover, $\varphi^{-1}(z')$ is orthogonal to the vector space of directions of $\varphi^{-1}(V)$. Namely, for every $x \in \varphi^{-1}(V)$, $x - \varphi^{-1}(z')$ and $\varphi^{-1}(z')$ are orthogonal. Hence, for every $x \in \varphi^{-1}(V)$,

$$\|x\|_{\mathbb{C}^N}^2 = \|x - \varphi^{-1}(z')\|_{\mathbb{C}^N}^2 + \|\varphi^{-1}(z')\|_{\mathbb{C}^N}^2.$$

This obviously implies that

$$\varphi^{-1}(V) \cap B_{\mathbb{C}^N}\left(\varphi^{-1}(z'), \left(\frac{\varepsilon_1^2}{1 - \varepsilon_1^2} - \frac{d^2}{1 - d^2}\right)^{1/2}\right) \subseteq \varphi^{-1}(V) \cap B_{\mathbb{C}^N}\left(0, \frac{\varepsilon_1}{\sqrt{1 - \varepsilon_1^2}}\right).$$

Now, we apply Lemma 19 to conclude the following chain of inequalities:

$$\mathscr{H}^{2m}\left[\varphi^{-1}(V) \cap B_{\mathbb{C}^N}\left(\varphi^{-1}(z'), \left(\frac{\varepsilon_1^2}{1 - \varepsilon_1^2} - \frac{d^2}{1 - d^2}\right)^{1/2}\right)\right] \geq$$

$$\geq \mathscr{H}^{2m}[B_{\mathbb{C}^m}(0,1)]\left(\frac{\varepsilon_1^2}{1 - \varepsilon_1^2} - \frac{d^2}{1 - d^2}\right)^m \geq$$

$$\geq \mathscr{H}^{2m}[B_{\mathbb{C}^m}(0,1)]\left(\frac{\varepsilon_1^2}{1 - \varepsilon_1^2} - \frac{\varepsilon^2}{1 - \varepsilon^2}\right)^m = \mathscr{H}^{2m}[B_{\mathbb{C}^m}(0,1)]\left(\frac{\varepsilon_1^2 - \varepsilon^2}{(1 - \varepsilon_1^2)(1 - \varepsilon^2)}\right)^m.$$

So, we have:

$$\nu_m[V \cap B_{\mathbb{P}}(e_0, \varepsilon_1)] \geq (1 - \varepsilon_1^2)^{m+1} \mathscr{H}^{2m}[B_{\mathbb{C}^m}(0,1)]\left(\frac{\varepsilon_1^2 - \varepsilon^2}{(1 - \varepsilon_1^2)(1 - \varepsilon^2)}\right)^m.$$

Now, $\mathscr{H}^{2m}[B_{\mathbb{C}^m}(0,1)] = \nu_m[\mathbb{P}_m(\mathbb{C})]$. That finishes the proof of the lemma. ∎

3.2 Upper bounds for the volume of a tube in the ambient space.

Now we show a proof of Theorem 1, which is a slight improvement of Theorem 4.2 in [7] (cf. also the article by Renegar [33]). Namely, we state upper and lower bound estimates on the volume of projective tubes about complex projective varieties. The following statement is a technical version of Theorem 1.

Proposition 26 *Let $V \subset \mathbb{P}_N(\mathbb{C})$ be a (possibly singular) projective equi–dimensional variety of dimension $m < N$. Then, the following inequalities hold for every positive real number $\varepsilon \in \mathbb{R}$, $0 < \varepsilon \leq 1$.*

$$\varepsilon^{2(N-m)} \leq \frac{\nu_N[V_\varepsilon]}{\nu_N[\mathbb{P}_N(\mathbb{C})]} \leq C(N,m)\deg(V)\varepsilon^{2(N-m)}.$$

In particular,

$$\frac{\nu_N[V_\varepsilon]}{\nu_N[\mathbb{P}_N(\mathbb{C})]} \leq 2\deg(V)\left(\frac{e\,N\,\varepsilon}{N-m}\right)^{2(N-m)}.$$

Proof.– Let $L \subseteq \mathbb{P}_N(\mathbb{C})$ be a fixed projective subspace of dimension $N - m$. From Corollary 13 we have

$$\nu_m[V_\varepsilon] = \frac{\nu_N[\mathbb{P}_N(\mathbb{C})]}{\nu_{N-m}[\mathbb{P}_{N-m}(\mathbb{C})]} \int_{U \in \mathcal{U}_{N+1}} \nu_{N-m}[V_\varepsilon \cap UL] \, d\mathcal{U}_{N+1}.$$

As V and UL are projective algebraic varieties of respective dimensions m and $N - m$, the Dimension of the Intersection Theorem (cf. [36], [23] for instance) implies

$$V \cap UL \neq \emptyset \qquad \forall \, U \in \mathcal{U}_{N+1}.$$

Moreover, if $z \in V \cap UL$ the following inequality holds:

$$\nu_{N-m}[V_\varepsilon \cap UL] \geq \nu_{N-m}[B_{\mathbf{P}}(z, \varepsilon) \cap UL] = \nu_{N-m}[\mathbb{P}_{N-m}(\mathbb{C})]\varepsilon^{2(N-m)},$$

from which the first inequality of the proposition follows.
For the second inequality, observe that if $\varepsilon > 0$ satisfies

$$\frac{\sqrt{2}}{2} \frac{N - m}{m} \leq \varepsilon \leq 1,$$

then we have

$$C(N, m) \deg(V)\varepsilon^{2(N-m)} \geq 1.$$

In fact, it suffices to see that the following function is always greater than 1 in the interval $[1, N - 1]$:

$$f(x) := 2\frac{N^{2N}}{x^{2x}(N - x)^{2N-2x}} \left(\frac{\sqrt{2}}{2}\frac{N - x}{x}\right)^{2N-2x} = 2\left(\frac{N}{x}\right)^{2N}\frac{1}{2^{N-x}}.$$

Now, $f'(x) \leq 0$ is always negative, and consequently $f(x) \geq f(N - 1) > 1$. The second inequality of the proposition obviously follows in this case.
Assume that $0 < \varepsilon < \min\{1, \frac{\sqrt{2}}{2}\frac{N-m}{m}\}$. Let $\varepsilon_1 > 0$ be another positive real number, $0 < \varepsilon < \varepsilon_1$. We consider the quantity

$$\varphi_V(\varepsilon_1, \varepsilon) = \inf_{z \in V_\varepsilon} \left(\nu_m[B_{\mathbf{P}}(z, \varepsilon_1) \cap V]\right).$$

We will prove that $\varphi_V(\varepsilon_1, \varepsilon) > 0$. Then, we have that:

$$\nu_N[V_\varepsilon] = \int_{V_\varepsilon} 1 \, d\mathbb{P}_N(\mathbb{C}) \leq \int_{z \in V_\varepsilon} \frac{\nu_m[B_{\mathbf{P}}(z, \varepsilon_1) \cap V]}{\varphi_V(\varepsilon_1, \varepsilon)} \, d\mathbb{P}_N(\mathbb{C}) \leq$$

$$\leq \frac{1}{\varphi_V(\varepsilon_1, \varepsilon)} \int_{z \in \mathbb{P}_N(\mathbb{C})} \nu_m[B_{\mathbf{P}}(z, \varepsilon_1) \cap V] \, d\mathbb{P}_N(\mathbb{C}).$$

From Corollary 15, we conclude:

$$\nu_N[V_\varepsilon] \leq \frac{\nu_N[\mathbb{P}_N(\mathbb{C})]}{\varphi_V(\varepsilon_1, \varepsilon)}\nu_m[V]\varepsilon_1^{2N}.$$

Now, from Corollary 14, $\nu_m[V] = \nu_m[\mathbb{P}_m(\mathbb{C})]\deg(V)$. So we conclude:

$$\nu_N[V_\varepsilon] \leq \frac{\nu_N[\mathbb{P}_N(\mathbb{C})]}{\varphi_V(\varepsilon_1, \varepsilon)}\nu_m[\mathbb{P}_m(\mathbb{C})]\deg(V)\varepsilon_1^{2N}. \tag{11}$$

From Corollary 24, the following inequality holds whenever $\varepsilon_1 - \varepsilon \leq \frac{\sqrt{2}}{2}$ and $z \in V_\varepsilon$:

$$\frac{\nu_m[B_{\mathbb{P}}(z, \varepsilon_1) \cap V]}{\nu_m[\mathbb{P}_m(\mathbb{C})]} \geq \frac{1}{2}(\varepsilon_1 - \varepsilon)^{2m}.$$

Thus, whenever $\varepsilon_1 - \varepsilon \leq \frac{\sqrt{2}}{2}$, we have that

$$\varphi_V(\varepsilon_1, \varepsilon) \geq \nu_m[\mathbb{P}_m(\mathbb{C})]\frac{1}{2}(\varepsilon_1 - \varepsilon)^{2m}.$$

Finally, we choose $\varepsilon_1 = \frac{N}{N-m}\varepsilon$. Observe that

$$\varepsilon_1 - \varepsilon = \frac{m}{N-m}\varepsilon < \frac{m}{N-m}\frac{\sqrt{2}}{2}\frac{N-m}{m} = \frac{\sqrt{2}}{2}.$$

From inequality (11) above we conclude

$$\nu_N[V_\varepsilon] \leq \frac{\nu_N[\mathbb{P}_N(\mathbb{C})]\nu_m[\mathbb{P}_m(\mathbb{C})]\deg(V)\left(\frac{N}{N-m}\varepsilon\right)^{2N}}{\nu_m[\mathbb{P}_m(\mathbb{C})]\frac{1}{2}\left(\frac{m}{N-m}\right)^{2m}\varepsilon^{2m}} = \nu_N[\mathbb{P}_N(\mathbb{C})]C(N, m)\deg(V)\varepsilon^{2(N-m)},$$

as wanted. The last inequality of the proposition follows from the next obvious inequality.

$$C(N, m) = 2\left(1 + \frac{N-m}{m}\right)^{2m}\left(\frac{N}{N-m}\right)^{2(N-m)} \leq 2\left(\frac{eN}{N-m}\right)^{2(N-m)}.$$

■

The estimates in Proposition 26 are essentially optimal in the case that $V \subseteq \mathbb{P}_N(\mathbb{C})$ is a linear projective subspace of dimension m. Namely, we have the following estimate.

Proposition 27 *Let $V \subset \mathbb{P}_N(\mathbb{C})$ be a linear subspace of dimension $1 \leq m < N$. Let $0 < \varepsilon$ be a positive real number satisfying*

$$\varepsilon \leq \left(\frac{N-m}{2N}\right)^{1/2}.$$

Then, the following inequalities hold:

$$\binom{N}{m}\varepsilon^{2(N-m)}(1 - \varepsilon^2)^m \leq \frac{\nu_N[V_\varepsilon]}{\nu_N[\mathbb{P}_N(\mathbb{C})]} \leq 6\sqrt{m}\binom{N}{m}\varepsilon^{2(N-m)}(1 - \varepsilon^2)^m.$$

Proof.– The lower bound is in Gray's article [18]. In fact, observe that [18, Cor. 1.3] implies

$$\frac{\nu_N[V_\varepsilon]}{\nu_N[\mathbb{P}_N(\mathbb{C})]} \geq \binom{N}{m}\varepsilon^{2(N-m)}(1-\varepsilon^2)^m.$$

For the upper bound, we follow essentially the same steps as in the proof of Proposition 26, replacing Corollary 24 by Lemma 25. Namely, given two positive real numbers $0 < \varepsilon < \varepsilon_1 < 1$ we define the function

$$\varphi_V(\varepsilon_1, \varepsilon) = \inf_{z \in V_\varepsilon}(\nu_m[B_\mathbb{P}(z, \varepsilon_1) \cap V]).$$

As in the proof of Proposition 26, we conclude:

$$\nu_N[V_\varepsilon] \leq \frac{\nu_N[\mathbb{P}_N(\mathbb{C})]}{\varphi_V(\varepsilon_1, \varepsilon)}\nu_m[\mathbb{P}_m(\mathbb{C})]\varepsilon_1^{2N},$$

since $\deg(V) = 1$. Also, from Lemma 25 we have

$$\varphi_V(\varepsilon_1, \varepsilon) \geq \nu_m[\mathbb{P}_m(\mathbb{C})](\varepsilon_1^2 - \varepsilon^2)^m\frac{1 - \varepsilon_1^2}{(1-\varepsilon^2)^m}.$$

Thus, we conclude

$$\nu_N[V_\varepsilon] \leq \nu_N[\mathbb{P}_N(\mathbb{C})]\frac{\varepsilon_1^{2N}}{(\varepsilon_1^2 - \varepsilon^2)^m}\frac{(1-\varepsilon^2)^m}{1 - \varepsilon_1^2}.$$

Assume that $\varepsilon \leq \left(\frac{N-m}{2N}\right)^{1/2}$. Then, we choose

$$\varepsilon_1 = \left(\frac{N}{N-m}\right)^{1/2}\varepsilon \leq \frac{\sqrt{2}}{2} < 1$$

and we conclude:

$$\nu_N[V_\varepsilon] \leq \nu_N[\mathbb{P}_N(\mathbb{C})]\frac{N^N}{m^m(N-m)^{N-m}}\varepsilon^{2(N-m)}(1-\varepsilon^2)^m\frac{N-m}{N-m-N\varepsilon^2} \leq$$

$$\leq 2\nu_N[\mathbb{P}_N(\mathbb{C})]\frac{N^N}{m^m(N-m)^{N-m}}\varepsilon^{2(N-m)}(1-\varepsilon^2)^m.$$

The following estimate from [39] finishes the proof:

$$\frac{N^N}{m^m(N-m)^{N-m}} < e^{1/6}\sqrt{2\pi}\frac{\sqrt{m}\sqrt{N-m}}{\sqrt{N}}\binom{N}{m} < 3\sqrt{m}\binom{N}{m}.$$

∎

The following corollary is consequence of Proposition 27.

Corollary 28 *Let V be an equi–dimensional projective variety in $\mathbb{P}_N(\mathbb{C})$, $N > 1$, $\dim(V) = m$, and $z \in \mathbb{P}_N(\mathbb{C})$ any point. Let $0 < \varepsilon \leq 1$ be a positive real number, such that*

$$\varepsilon < \left(\frac{m}{2N}\right)^{1/2}.$$

Then, the following inequality holds for every $1 \leq m \leq N - 1$:

$$\frac{\nu_m[V \cap B_\mathbb{P}(z, \varepsilon)]}{\nu_m[V]} \leq 6\sqrt{N-m}\binom{N}{m}\varepsilon^{2m}(1-\varepsilon^2)^{N-m}.$$

Proof.– Let L be any fixed linear subspace of $\mathbb{P}_N(\mathbb{C})$ of dimension $N - m$. From Corollary 13 we conclude

$$\nu_m[V \cap B_{\mathbf{P}}(z,\varepsilon)] = \nu_m[\mathbb{P}_m] \int_{U \in \mathcal{U}_{N+1}} \sharp(UL \cap V \cap B_{\mathbf{P}}(z,\varepsilon)) \, d\mathcal{U}_{N+1}.$$

Hence, we conclude

$$\nu_m[V \cap B_{\mathbf{P}}(z,\varepsilon)] \leq \deg(V)\nu_m[\mathbb{P}_m]\nu_{\mathcal{U}_{N+1}}[U \in \mathcal{U}_{N+1} : UL \cap V \cap B_{\mathbf{P}}(z,\varepsilon) \neq \emptyset] \leq$$

$$\leq \nu_m[V]\nu_{\mathcal{U}_{N+1}}[U \in \mathcal{U}_{N+1} : UL \cap B_{\mathbf{P}}(z,\varepsilon) \neq \emptyset] =$$

$$= \nu_m[V]\nu_{\mathcal{U}_{N+1}}[U \in \mathcal{U}_{N+1} : L \cap U^* B_{\mathbf{P}}(z,\varepsilon) \neq \emptyset],$$

where U^* holds for the conjugate transpose matrix of an unitary matrix U. The mapping $A \longmapsto A^*$ defines an isometry on $\mathcal{U}_{N+1}$. Hence, we have

$$\nu_{\mathcal{U}_{N+1}}[U \in \mathcal{U}_{N+1} : L \cap U^* B_{\mathbf{P}}(z,\varepsilon) \neq \emptyset] = \nu_{\mathcal{U}_{N+1}}[U \in \mathcal{U}_{N+1} : L \cap B_{\mathbf{P}}(Uz,\varepsilon) \neq \emptyset],$$

since $UB_{\mathbf{P}}(z,\varepsilon) = B_{\mathbf{P}}(Uz,\varepsilon)$. Let $L_\varepsilon \subseteq \mathbb{P}_N(\mathbb{C})$ be the tube of radius ε about the projective subspace L and let $U(z,L,\varepsilon) \subseteq \mathcal{U}_{N+1}$ be the set given by

$$U(z,L,\varepsilon) := \{U \in \mathcal{U}_{N+1} : L \cap B_{\mathbf{P}}(Uz,\varepsilon) \neq \emptyset\} = \{U \in \mathcal{U}_{N+1} : Uz \in L_\varepsilon\}.$$

Hence, we have

$$\nu_m[V \cap B_{\mathbf{P}}(z,\varepsilon)] \leq \nu_m[V] \int_{\mathcal{U}_{N+1}} \chi_{U(z,L,\varepsilon)} \, d\mathcal{U}_{N+1}.$$

Now, Corollary 12 implies:

$$\nu_m[V \cap B_{\mathbf{P}}(z,\varepsilon)] \leq \frac{\nu_m[V]}{\nu_N[\mathbb{P}_N(\mathbb{C})]} \int_{\mathbb{P}_N(\mathbb{C})} \chi_{L_\varepsilon} \, d\mathbb{P}_N(\mathbb{C}) = \frac{\nu_N[L_\varepsilon]}{\nu_N[\mathbb{P}_N(\mathbb{C})]}\nu_m[V].$$

Proposition 27 yields:

$$\frac{\nu_m[V \cap B_{\mathbf{P}}(z,\varepsilon)]}{\nu_m[V]} \leq 6\sqrt{N-m}\binom{N}{m}\varepsilon^{2m}(1-\varepsilon^2)^{N-m}.$$

∎

3.3   Proof of Theorem 18

In order to prove inequality (6) of Theorem 18 we discuss two main cases. If $\frac{m-m'}{N-m'} \leq \varepsilon \leq 1$, the quantity on the right is obviously greater than 1 and the inequality immediately follows. We discuss the upper bound in the case that $\varepsilon < \frac{m-m'}{N-m'} < 1$. Let $\varepsilon_1 > 0$ be a positive real number such that $\varepsilon_1 + \varepsilon < 1$. Then, the following holds for every $z \in \mathbb{P}_N(\mathbb{C})$.

$$d_{\mathbf{P}}(z,V') \geq \varepsilon_1 + \varepsilon \Longrightarrow B_{\mathbf{P}}(z,\varepsilon_1) \cap V'_\varepsilon = \emptyset. \tag{12}$$

As in former statements, let $e_0 := (1 : 0 : \cdots : 0) \in \mathbb{P}_N(\mathbb{C})$ be a fixed projective point and let $L_0 \subseteq \mathbb{P}_N(\mathbb{C})$ be a fixed projective linear subspace of dimension $N - m$ such that $e_0 \in L_0$. From Corollary 13 we conclude

$$\nu_m[V'_\varepsilon \cap V]\nu_{N-m}[B_{\mathbf{P}}(e_0,\varepsilon_1) \cap L_0] =$$

$$\nu_m[\mathbb{P}_m(\mathbb{C})]\nu_{N-m}[\mathbb{P}_{N-m}(\mathbb{C})]\int_{U\in\mathcal{U}_{N+1}}\sharp[B_{\mathbf{P}}(Ue_0,\varepsilon_1)\cap V'_{\varepsilon}\cap V\cap UL_0]\,d\mathcal{U}_{N+1}.$$

Now, observe that

$$\sharp[B_{\mathbf{P}}(Ue_0,\varepsilon_1)\cap V'_{\varepsilon}\cap V\cap UL_0]\leq\sharp[V\cap UL_0]\leq\deg(V).$$

On the other hand, if $Ue_0\notin V'_{\varepsilon_1+\varepsilon}$, then

$$\sharp[B_{\mathbf{P}}(Ue_0,\varepsilon_1)\cap V'_{\varepsilon}\cap V\cap UL_0]=0.$$

Thus, let $A_1\subseteq\mathcal{U}_{N+1}$ be the subset given by

$$A_1:=\{U\in\mathcal{U}_{N+1}:Ue_0\in V'_{\varepsilon_1+\varepsilon}\}.$$

We conclude that

$$\nu_m[V'_{\varepsilon}\cap V]\nu_{N-m}[B_{\mathbf{P}}(e_0,\varepsilon_1)\cap L_0]\leq\nu_m[\mathbb{P}_m(\mathbb{C})]\nu_{N-m}[\mathbb{P}_{N-m}(\mathbb{C})]\int_{A_1}\deg(V)\,d\mathcal{U}_{N+1}.$$

From Corollary 12 we have

$$\int_{A_1}\deg(V)\,d\mathcal{U}_{N+1}=\frac{1}{\nu_N[\mathbb{P}_N(\mathbb{C})]}\int_{z\in V'_{\varepsilon_1+\varepsilon}}\deg(V)\,d\mathbb{P}_N(\mathbb{C})=\frac{\deg(V)\nu_N[V'_{\varepsilon_1+\varepsilon}]}{\nu_N[\mathbb{P}_N(\mathbb{C})]}.$$

Thus, we have

$$\nu_m[V'_{\varepsilon}\cap V]\nu_{N-m}[B_{\mathbf{P}}(e_0,\varepsilon_1)\cap L_0]\leq\frac{\nu_m[\mathbb{P}_m(\mathbb{C})]\nu_{N-m}[\mathbb{P}_{N-m}(\mathbb{C})]}{\nu_N[\mathbb{P}_N(\mathbb{C})]}\deg(V)\nu_N[V'_{\varepsilon_1+\varepsilon}].$$

Moreover, the following equality holds:

$$\nu_{N-m}[B_{\mathbf{P}}(e_0,\varepsilon_1)\cap L_0]=\nu_{N-m}[\mathbb{P}_{N-m}(\mathbb{C})]\varepsilon_1^{2(N-m)}.$$

Thus, we conclude

$$\nu_m[V'_{\varepsilon}\cap V]\varepsilon_1^{2(N-m)}\leq\frac{\nu_m[V]}{\nu_N[\mathbb{P}_N(\mathbb{C})]}\nu_N[V'_{\varepsilon_1+\varepsilon}].$$

From Proposition 26, we obtain

$$\frac{\nu_m[V'_{\varepsilon}\cap V]}{\nu_m[V]}\leq C(N,m')\deg(V')\frac{(\varepsilon_1+\varepsilon)^{2(N-m')}}{\varepsilon_1^{2(N-m)}}.$$

Now, choose $\varepsilon_1=\frac{N-m}{m-m'}\varepsilon$ to conclude

$$\frac{\nu_m[V'_{\varepsilon}\cap V]}{\nu_m[V]}\leq C(N,m,m')\deg(V')\varepsilon^{2(m-m')}.$$

As for the proof of inequality (7), let $L\subseteq\mathbb{P}_N(\mathbb{C})$ be a projective linear subspace of dimension $N-m'$. From Corollary 13 we have

$$\nu_m[V'_{\varepsilon}\cap V]=\frac{\nu_m[\mathbb{P}_m(\mathbb{C})]}{\nu_{m-m'}[\mathbb{P}_{m-m'}(\mathbb{C})]}\int_{U\in\mathcal{U}_{N+1}}\nu_{m-m'}[V'_{\varepsilon}\cap V\cap UL]\,d\mathcal{U}_{N+1}.$$

Now, observe that $V' \cap UL \neq \emptyset$ for every $U \in \mathcal{U}_{N+1}$, and if $z \in V' \cap UL$ we have that:

$$\nu_{m-m'}[V'_\varepsilon \cap V \cap UL] \geq \nu_{m-m'}[B_{\mathbb{P}}(z, \varepsilon) \cap (V \cap UL)].$$

From Lemma 9, there is a dense residual subset $W \subseteq \mathcal{U}_{N+1}$ such that for every $U \in W$, $V \cap UL$ is a projective variety of dimension $m - m'$. Hence, Lemma 22 implies the following inequality:

$$\nu_{m-m'}[B_{\mathbb{P}}(z, \varepsilon) \cap (V \cap UL)] \geq \frac{1}{2}\nu_{m-m'}[\mathbb{P}_{m-m'}(\mathbb{C})]\varepsilon^{2(m-m')},$$

and the claim follows. ∎

4 The Condition Number of Linear Algebra.

In this section we apply Proposition 26 and Theorem 18 to prove Corollary 29, which is a more general version of Corollary 4.

Just to fix the notations, let $n_1, n_2 \in \mathbb{N}$ be two positive integer numbers and let $\mathcal{M}_{n_1 \times n_2}(\mathbb{C})$ be the space of $n_1 \times n_2$ complex matrices. From the natural identification $\mathcal{M}_{n_1 \times n_2}(\mathbb{C}) \equiv \mathbb{C}^{n_1 n_2}$, we also have that $\mathbb{P}(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})) \equiv \mathbb{P}(\mathbb{C}^{n_1 n_2}) = \mathbb{P}_{n_1 n_2 - 1}(\mathbb{C})$. Thus, we can consider $\mathbb{P}(\mathcal{M}_{n_1 \times n_2}(\mathbb{C}))$ endowed with the natural Riemannian structure of $\mathbb{P}_{n_1 n_2 - 1}(\mathbb{C})$. From now on, n_1 and n_2 are considered fixed natural numbers such that $n_1 \geq n_2 \geq 2$. The results for $2 \leq n_1 \leq n_2$ are totally symmetrical.

Let $A \in \mathbb{P}(\mathcal{M}_{n_1 \times n_2}(\mathbb{C}))$ be a projective matrix such that $rank(A) = n_2$. Then, the condition number of A is given by the following formula:

$$\kappa_D(A) := \|A\|_F \|A^\dagger\|_2,$$

where $A^\dagger$ stands for the Moore–Penrose inverse of A. Recall that given a projective singular matrix $A \in \mathbb{P}(\mathcal{M}_{n_1 \times n_2}(\mathbb{C}))$ such that $rank(A) = n_2 - 1$, the generalized condition number of A is also defined to be

$$\kappa_D^{n_2-1}(A) := \|A\|_F \|A^\dagger\|_2.$$

Inside the proofs of this section, for simplicity of notation we do not distinguish between a projective matrix $A \in \mathbb{P}(\mathcal{M}_{n_1 \times n_2}(\mathbb{C}))$ and any representant of it. Both elements are simply denoted by A.

Let $\Sigma^{n_2-1} \subseteq \mathbb{P}(\mathcal{M}_{n_1 \times n_2}(\mathbb{C}))$ be the algebraic variety of matrices of rank at most $n_2 - 1$. Namely,

$$\Sigma^{n_2-1} := \{A \in \mathbb{P}(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})) : rank(A) \leq n_2 - 1\}.$$

The main result of this Section is the following one.

Corollary 29 *With the notations and assumptions as above, the following inequality holds:*

$$\frac{\nu_{\dim(\Sigma^{n_2-1})}[A \in \Sigma^{n_2-1} : \kappa_D^{n_2-1}(A) > \frac{1}{\varepsilon}]}{\nu_{\dim(\Sigma^{n_2-1})}[\Sigma^{n_2-1}]} \leq (e \, n_1^2 n_2^3 \varepsilon)^{2(n_1-n_2+3)}. \tag{13}$$

Moreover, in the case that $n_1 = n_2 = n$, the following equality holds:

$$\frac{\nu_{\dim(\Sigma^{n-1})}[A \in \Sigma^{n-1} : \kappa_D^{n-1}(A) > \frac{1}{\varepsilon}]}{\nu_{\dim(\Sigma^{n-1})}[\Sigma^{n-1}]} \leq \frac{7}{10}(n^{10/3}\varepsilon)^6. \tag{14}$$

4.1 Technical statements.

In this subsection we state some technical results to prove Corollary 29. We also recall some properties of the generalized condition number. Let $\mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right)$ be the projective space of complex $n_1 \times n_2$ matrices. For every positive integer r, $1 \leq r \leq n_2$, we denote by Σ^r the algebraic variety of all the complex matrices of rank at most r. Namely,

$$\Sigma^r := \{A \in \mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right) : rank(A) \leq r\}.$$

The first part of the following Proposition is [2, Prop. 1.1]. The equality on the degree can be read in [22, pp. 243-244], or in [14, p. 261].

Proposition 30 *For every positive integer number $1 \leq r \leq n_2$, the set Σ^r is an irreducible projective variety of $\mathbb{P}_{n_1 n_2 - 1}(\mathbb{C})$ of codimension $(n_2 - r)(n_1 - r)$. Moreover,*

$$\deg(\Sigma^r) = \prod_{i=0}^{n_2-r-1} \frac{(n_1 + i)! \ i!}{(r + i)! \ (n_1 - r + i)!}.$$

An immediate consequence is the following corollary.

Corollary 31 *The following equality on the degree of Σ^r holds.*

$$\deg(\Sigma^r) = \prod_{i=1}^{n_1-r} \prod_{j=1}^{n_2-r} \frac{r + i + j - 1}{i + j - 1}.$$

In particular,

$$\deg(\Sigma^r) \leq \binom{n_1}{r}^{n_2-r}, \qquad \deg(\Sigma^r) \leq (r + 1)^{(n_2-r)(n_1-r)}.$$

William Kahan, G.W. Stewart and J. Sun have studied the condition numbers for singular matrices. We refer to [27] and [40] for general background on this topic. However, we recall some basic concepts and results on these numbers. Recall that given any matrix $A \in \mathcal{M}_{n_1 \times n_2}(\mathbb{C})$, there exists a Singular Value Decomposition (SVD) of A,

$$A = U \begin{pmatrix} D \\ 0 \end{pmatrix} V^*. \tag{15}$$

Namely, $A = U \begin{pmatrix} D \\ 0 \end{pmatrix} V^*$ where:

- The matrices $U \in \mathcal{U}_{n_1}$ and $V \in \mathcal{U}_{n_2}$ are unitary matrices of respective sizes n_1 and n_2, and V^* holds for the transpose conjugate of V.

- The matrix $D := Diag(\sigma_1, \cdots, \sigma_{n_2}) \in \mathcal{M}_{n_2}(\mathbb{C})$ is the matrix of singular values of A, $\sigma_1 \geq \cdots \geq \sigma_{n_2} \geq 0$.

- The expression $\begin{pmatrix} D \\ 0 \end{pmatrix} \in \mathcal{M}_{n_1 \times n_2}(\mathbb{C})$ holds for the matrix of n_1 rows and n_2 columns obtained by adding to D a zero matrix of size $(n_1 - n_2) \times n_2$.

Definition 32 (Generalized Condition Number) *Let $A \in \mathcal{M}_{n_1 \times n_2}(\mathbb{C})$ be any matrix. We consider a SVD of A,*

$$A = U \begin{pmatrix} D \\ 0 \end{pmatrix} V^*, \qquad D := Diag(\sigma_1, \cdots, \sigma_{n_2}).$$

For every natural number r, $2 \leq r \leq n_2$, we define the following quantity:

$$\kappa_D^r(A) := \frac{\|A\|_F}{\sqrt{\sigma_r^2 + \cdots + \sigma_{n_2}^2}},$$

where $\|A\|_F := \sqrt{\sigma_1^2 + \cdots + \sigma_{n_2}^2}$ stands the Frobenius norm of A.
This definition is also valid for the projective space of matrices, $\mathbb{P}(\mathcal{M}_{n_1 \times n_2}(\mathbb{C}))$, in the sense that it is does not change under multiplication by a scalar.

In the case that $n_1 = n_2 = n$, the generalized condition number κ_D^n we have defined turns to be the usual condition number for square matrices, $\kappa_D(A) := \|A\|_F \|A^{-1}\|_2$, $A \in \mathcal{M}_n(\mathbb{C})$.

Lemma 33 *The generalized condition number $\kappa_D^r(A)$ of a matrix $A \in \mathcal{M}_{n_1 \times n_2}(\mathbb{C})$ such that $rank(A) = r$ satisfies the following equality:*

$$\kappa_D^r(A) = \|A\|_F \|A^\dagger\|_2,$$

where $A^\dagger$ holds for the Moore-Penrose inverse of A.

Proof.– In fact, if $D = diag(\sigma_1, \cdots, \sigma_r, 0, \ldots, 0)$, $\sigma_1 \geq \ldots \geq \sigma_r > 0$, then $A^\dagger$ is given by the following formula (see [40, pp. 102-104] for details).

$$A^\dagger = V \begin{pmatrix} \sigma_1^{-1} & & & & & 0 & \cdots & 0 \\ & \ddots & & & & & & \\ & & \sigma_r^{-1} & & & \vdots & & \vdots \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & 0 & \cdots & 0 \end{pmatrix} U^* \in \mathcal{M}_{n_2 \times n_1}(\mathbb{C}).$$

So, the equality $\|A^\dagger\|_2 = \sigma_r^{-1}$ immediately follows. ∎

The following result remarks the importance of the generalized condition number as a measure of the stability of the Moore–Penrose inverse of a given matrix under small perturbations. It is an immediate consequence of the Corollary 3.10 in [40, p. 145].

Proposition 34 *Let $A, A' \in \mathcal{M}_{n_1 \times n_2}(\mathbb{C})$ be two matrices of equal rank r. Then, the following inequality holds:*

$$\frac{\|A^\dagger - (A')^\dagger\|_F}{\|(A')^\dagger\|_2} \leq \sqrt{2} \, \kappa_D^r(A) \frac{\|A - A'\|_F}{\|A\|_F}.$$

Now let us recall the Singular Value Decomposition Theorem.

Theorem 35 (Singular Value Decomposition) *Let L and L' be two linear subspaces of $\mathbb{C}^n$ of dimension m. Then, there are orthonormal frames $\{v_1, \ldots, v_m\}$ of L and $\{w_1, \ldots, w_m\}$ of L', and real numbers $1 \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ such that:*

$$< v_i, w_j >= \lambda_i \delta_{ij}.$$

There are different definitions for the distance between two subspaces of the same dimension. The following one is widely accepted (c.f. for example [17, p. 76]).

Definition 36 *Let $L_{\mathbb{R}}, L'_{\mathbb{R}}$ be two real linear subspaces of $\mathbb{R}^n$, of equal real dimension m. Then, we define the projective distance between $L_{\mathbb{R}}$ and $L'_{\mathbb{R}}$ as follows:*

$$dist(L_{\mathbb{R}}, L'_{\mathbb{R}}) = \|\pi_{L_{\mathbb{R}}} - \pi_{L'_{\mathbb{R}}}\|_2,$$

where $\pi_{L_{\mathbb{R}}}$ (resp. $\pi_{L'_{\mathbb{R}}}$) is the orthogonal projection onto $L_{\mathbb{R}}$ (resp. $L'_{\mathbb{R}}$), and $\|\pi_{L_{\mathbb{R}}} - \pi_{L'_{\mathbb{R}}}\|_2$ is the norm of this map as a linear operator.
The distance between two complex subspaces $L, L' \subseteq \mathbb{C}^n$ of equal dimension m is defined the same way:
$$dist(L, L') = \|\pi_L - \pi_{L'}\|_2.$$

Remark 37 *Some properties of this distance may be read in [17] and [46]. We cite two of them:*

- *Let θ be the largest principal angle (in the sense of [17, p. 603]) between L and L'. Then, the following equality holds:*

$$dist(L, L') = \sin \theta.$$

- *If $\dim(L) = \dim(L') = 1$, then $dist(L, L') = d_{\mathbb{P}}(L, L')$ where $d_{\mathbb{P}}(L, L')$ is the projective distance between the projective points defined by L and L'.*

The following theorem relates the generalized condition number κ_D^r to the stability of the solutions of (possibly singular) square systems under perturbations. We guess it has been proved elsewhere but we have not found an appropriate reference to cite.

Proposition 38 *Let $A, A' \in \mathcal{M}_n(\mathbb{C})$ be two square matrices, $rank(A) = rank(A') = r$. Let L and L' be the complex subspaces of dimension $m = n - r$ which are the respective kernels of A and A'. Namely:*

$$L := \{\underline{x} \in \mathbb{C}^n : A\underline{x} = 0\} \qquad L' := \{\underline{x} \in \mathbb{C}^n : A'\underline{x} = 0\}.$$

Then, the following inequality holds:

$$dist(L, L') \leq \kappa_D^r(A) \frac{\|A' - A\|_2}{\|A\|_F}.$$

Proof.– Let $\{v_1, \ldots, v_m\}$, $\{w_1, \ldots, w_m\}$, and $1 \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ be like in Theorem 35, spanning L and L' respectively. The characterization of $dist(L, L')$ as the sinus of the largest principal angle between L and L' reads:

$$dist(L, L') = \sqrt{1 - \lambda_m^2}.$$

On the other hand, the following equality holds:

$$\kappa_D^r(A) \frac{\|A' - A\|_2}{\|A\|_F} = \frac{\|A' - A\|_2}{\sigma_r},$$

where σ_r holds for the smallest non–zero singular value of A.
So, it suffices to prove the following inequality:

$$\sqrt{1 - \lambda_m^2} \leq \frac{\|A' - A\|_2}{\sigma_r}.$$

Let the reader observe that the following equality holds:

$$L^{\perp} = \{w \in \mathbb{C}^n : V^* w \in < e_1, \ldots, e_r >\},$$

where $A = UDV^*$ is the SVD of A, and $< e_1, \ldots, e_r >$ is the subspace of $\mathbb{C}^n$ spanned by the first r vectors of the canonical basis. As a consequence, we observe that for every vector $w \in L^{\perp}$, the following equality holds:

$$A^{\dagger} A w = V \begin{pmatrix} Id_r & 0 \\ 0 & 0 \end{pmatrix} V^* w = w.$$

So, the following inequalities hold for every vector $w \in L^{\perp}$:

$$\|w\|_2 = \|A^{\dagger} A w\|_2 \leq \|A^{\dagger}\|_2 \|Aw\|_2, \Longrightarrow \|Aw\|_2 \geq \frac{\|w\|_2}{\|A^{\dagger}\|_2}.$$

First, suppose that $\lambda_m = 0$. Then, $w_m \in L^{\perp}$. So, we have:

$$\|(A' - A)w_m\|_2 = \|Aw_m\|_2 \geq \frac{1}{\|A^{\dagger}\|_2}, \Longrightarrow \|(A' - A)\|_2 \geq \frac{1}{\|A^{\dagger}\|_2}.$$

From Lemma 33, we conclude that $\frac{1}{\|A^{\dagger}\|_2} = \sigma_r$. So, in this case we have:

$$\sqrt{1 - \lambda_m^2} = 1 \leq \|(A' - A)\|_2 \|A^{\dagger}\|_2 = \frac{\|A' - A\|_2}{\sigma_r},$$

and the theorem follows in the case $\lambda_m = 0$.
Now, suppose that $\lambda_m \neq 0$. Let $w'_m = \frac{w_m}{\lambda_m}$. Then, we have the following equality:

$$< v_m, w'_m - v_m >_2 = \frac{1}{\lambda_m} < v_m, w_m >_2 - \|v_m\|_2^2 = 1 - 1 = 0.$$

We define $\delta w = w'_m - v_m$, and $\delta A = A' - A$. We have the following chain of equalities:

$$\frac{\|\delta w\|_2^2}{\|w'_m\|_2^2} = \lambda_m^2 < w'_m - v_m, w'_m - v_m >_2 =$$

$$\lambda_m^2 < w_m', w_m' >_2 - \lambda_m^2 < w_m', v_m > - \lambda_m^2 < v_m, w_m' - v_m >_2 = 1 - \lambda_m^2,$$

and consequently:

$$\frac{\|\delta w\|_2}{\|w_m'\|_2} = \sqrt{1 - \lambda_m^2}.$$

So, it suffices to prove that:

$$\|\delta A\|_2 \geq \frac{\sigma_r \|\delta w\|_2}{\|w_m'\|_2}.$$

Now, we have that $\delta A w_m' + A \delta w = (A + \delta A)(v_m + \delta w) = A' w_m' = 0$, and consequently:

$$\delta A \frac{w_m'}{\|w_m'\|_2} = \frac{-A \delta w}{\|w_m'\|_2},$$

Hence,

$$\|\delta A\|_2 \geq \frac{\|A \delta w\|_2}{\|w_m'\|_2}.$$

So, to finish the proof we must check that:

$$\|A \delta w\|_2 \geq \sigma_r \|\delta w\|_2.$$

Now, observe that $\delta w \in L^\perp$. So, the following inequality holds:

$$\|A \delta w\|_2 \geq \frac{1}{\|A^\dagger\|_2} \|\delta w\|_2.$$

From Lemma 33, $\frac{1}{\|A^\dagger\|_2} = \sigma_r$ and the theorem follows. ∎

The following theorem is usually attributed to Eckart and Young. A brief history on this result with references to Schmidt and Mirsky can be read in [40, p. 210]. It is an immediate consequence of Theorem 4.18 in [40, p. 208].

Theorem 39 (Schmidt–Mirsky–Eckart–Young) *Let A be a matrix in $\mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right)$. Let $2 \leq r \leq n_2$ be a natural number. Then, the following holds:*

$$d_{\mathbf{P}}\left(A, \Sigma^{r-1}\right) = \frac{1}{\kappa_D^r(A)}.$$

Proof.– Theorem [40, p. 208] is the affine version of the theorem. Namely, for any affine matrix $A \in \mathcal{M}_{n_1 \times n_2}(\mathbb{C})$, the following equality holds:

$$\min_{\substack{rank(A') \leq r - 1 \\ A' \in \mathcal{M}_{n_1 \times n_2}(\mathbb{C})}} \|A' - A\|_F = \sqrt{\sigma_r^2 + \cdots + \sigma_{n_2}^2},$$

where $\sigma_r, \ldots, \sigma_{n_2}$ hold for the last singular values of A. To achieve the projective version of this result, we choose a representant A such that $\|A\|_F = \sigma_1^2 + \cdots + \sigma_{n_2}^2 = 1$. Consider the SVD of A, $A = U\binom{D}{0}V^*$. Consider the matrix $D' = Diag(\sigma_1, \ldots, \sigma_{r-1}, 0, \ldots, 0)$. Then, the matrix $A' = U\binom{D'}{0}V^*$ satisfies:

$$| < A', A >_2 | = \sigma_1^2 + \ldots + \sigma_{r-1}^2 \in \mathbb{R}, \qquad rank(A') = r - 1, \qquad \|A'\|_F^2 = \sigma_1^2 + \ldots + \sigma_{r-1}^2.$$

Then, the following chain of equalities holds:

$$d_{\mathbb{P}}\left(A, A'\right) = \sqrt{1 - \frac{(\sigma_1^2 + \ldots + \sigma_{r-1}^2)^2}{\sigma_1^2 + \ldots + \sigma_{r-1}^2}} = \sqrt{\sigma_r^2 + \cdots + \sigma_{n_2}^2} = \frac{1}{\kappa_D^r(A)}.$$

Now, let $A' \in \mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right)$ be any projective matrix such that $rank(A') \leq r - 1$. We can choose a representant of A' such that:

$$\|A'\|_F = 1 - (\sigma_r^2 + \cdots + \sigma_{n_2}^2), \qquad < A', A >_2 \in \mathbb{R}^{0,+}.$$

Then, $\sigma_r^2 + \cdots + \sigma_{n_2}^2 \leq \|A' - A\|_F^2 = < A' - A, A' - A >_2 = 2 - (\sigma_r^2 + \cdots + \sigma_{n_2}^2) - 2 < A', A >_2$, and the following chain of equalities holds:

$$| < A, A' >_2 | \leq \frac{2 - 2(\sigma_r^2 + \cdots + \sigma_{n_2}^2)}{2} = 1 - (\sigma_r^2 + \cdots + \sigma_{n_2}^2).$$

So, the following chain of inequalities holds:

$$d_{\mathbb{P}}\left(A, A'\right) = \sqrt{1 - \frac{| < A', A >_2 |^2}{\|A\|_F^2 \|A'\|_F^2}} \geq \sqrt{1 - \frac{(1 - \sigma_r^2 + \cdots + \sigma_{n_2}^2)^2}{1 - \sigma_r^2 + \cdots + \sigma_{n_2}^2}} = \frac{1}{\kappa_D^r(A)}.$$

That finishes the proof of the lemma. ∎

The following corollaries bound the distribution of κ_D^r in different subspaces of $\mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right)$.

Corollary 40 *For every positive integer number $r \in \mathbb{N}$, $2 \leq r \leq n_2$, and for every positive real number $0 < \varepsilon < 1$, the probability that a random projective matrix $A \in \mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right)$ has a generalized condition number $\kappa_D^r(A)$ greater than $\frac{1}{\varepsilon}$ is bounded by the following formula:*

$$\frac{\nu_{n_1 n_2 - 1}[A \in \mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right) : \kappa_D^r(A) > \frac{1}{\varepsilon}]}{\nu_{n_1 n_2 - 1}[\mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right)]} \leq 2 \left[\frac{e\,(n_1 n_2 - 1)\sqrt{r}}{(n_1 - r + 1)(n_2 - r + 1)} \varepsilon \right]^{2(n_1 - r + 1)(n_2 - r + 1)}.$$

Moreover, in the case that $n_1 = n_2 = n$ and $r = n - 1$, the following inequality holds:

$$\frac{\nu_{n^2 - 1}[A \in \mathbb{P}\left(\mathcal{M}_n(\mathbb{C})\right) : \kappa_D^{n-1}(A) > \frac{1}{\varepsilon}]}{\nu_{n^2 - 1}[\mathbb{P}\left(\mathcal{M}_n(\mathbb{C})\right)]} \leq \frac{1}{6} \left(\frac{e\,n^{5/2}\,\varepsilon}{4} \right)^8.$$

Proof.– From Theorem 39, the following equality holds:

$$\frac{\nu_{n_1 n_2 - 1}[A \in \mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right) : \kappa_D^r(A) > \frac{1}{\varepsilon}]}{\nu_{n_1 n_2 - 1}[\mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right)]} = \frac{\nu_{n_1 n_2 - 1}[A \in \mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right) : d_{\mathbb{P}}\left(A, \Sigma^{r-1}\right) < \varepsilon]}{\nu_{n_1 n_2 - 1}[\mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right)]}.$$

Proposition 26 immediately yields a bound for this quantity. From Corollary 31 we know the dimension and the degree of Σ^{r-1}. In the particular case that $n_1 = n_2 = n$ and $r = n - 1$, we use the sharp bound of Proposition 26 to obtain the inequality of the corollary. ∎

Corollary 41 *With the notations above, the following inequality holds:*

$$\frac{\nu_{r(n_2+n_1)-r^2-1}[A \in \Sigma^r : \kappa_D^r(A) > \frac{1}{\varepsilon}]}{\nu_{r(n_2+n_1)-r^2-1}[\Sigma^r]} \leq \deg(\Sigma^{r-1})D(n_1, n_2, r)\varepsilon^{2(n_2+n_1-2r+1)},$$

where

$$D(n_1, n_2, r) := C(n_1 n_2 - 1, r(n_2 + n_1) - r^2 - 1, (r-1)(n_2 + n_1) - (r-1)^2 - 1)$$

and $C(N, m, m')$ is as in Theorem 18 for every three positive integer numbers $N > m > m' \in \mathbb{N}$.

Proof.– From Theorem 39, the following equality holds:

$$\frac{\nu_{r(n_2+n_1)-r^2-1}[A \in \Sigma^r : \kappa_D^r(A) > \frac{1}{\varepsilon}]}{\nu_{r(n_2+n_1)-r^2-1}[\Sigma^r]} = \frac{\nu_{r(n_2+n_1)-r^2-1}[A \in \Sigma^r : d_{\mathbf{P}}(A, \Sigma^{r-1}) < \varepsilon]}{\nu_{r(n_2+n_1)-r^2-1}[\Sigma^r]}.$$

Theorem 18 yields a bound for this quantity. The expressions for the dimension and degree of Σ^{r-1} and Σ^r are known from Corollary 31. ∎

Remark 42 *For fixed n_2, n_1 and r the bounds we obtain become much better than those stated in the general results.*
With the same technique we can also bound the probability distribution of the generalized condition number $\kappa_D^{r'}$ in Σ^r for every possible integer values of n_1, n_2, r, r' such that $1 \leq r' \leq r \leq n_2 \leq n_1$.

4.2 Proof of Corollary 29.

We apply Corollary 41 to the case that $r = n_2 - 1$. The constant appearing in Corollary 41 turns to be:

$$C(n_1 n_2 - 1, n_1 n_2 - n_1 + n_2 - 2, n_1 n_2 - 2n_1 + 2n_2 - 5) \leq$$

$$\leq 2\left(\frac{e\, n_1 n_2}{2n_1 - 2n_2 + 4}\right)^{4n_1-4n_2+8}(2e)^{2n_1-2n_2+6} \leq \left(\frac{2e^3}{16}n_1^2 n_2^2\right)^{2n_1-2n_2+6} <$$

$$< \left(e\, n_1^2 n_2^2\right)^{2n_1-2n_2+6}.$$

Moreover, the degree of Σ^{n_2-2} is specified in Proposition 30:

$$\deg(\Sigma^{n_2-2}) = \binom{n_1}{n_2-2}\binom{n_1+1}{n_2-1}\frac{1}{n_1-n_2+3} \leq n_2^{2(n_1-n_2+2)}.$$

Equation (13) in Corollary 29 follows. As for equation (14), observe that in the case that $n_1 = n_2 = n$ are equal,

$$C(n^2 - 1, n^2 - 2, n^2 - 5)\deg(\Sigma^{n-2}) = \frac{1}{2}C(n^2 - 1, n^2 - 5)C(4, 1)\prod_{i=0}^{1}\frac{(n+i)!i!}{(n-2+i)!(i+2)!} =$$

$$2\left(\frac{(n^2-1)^{n^2-1}}{(n^2-5)^{n^2-5}4^4}\frac{4^4}{3^3}\right)^2\frac{n!(n+1)!}{(n-2)!(n-1)!2!3!}=$$

$$=2\left(\left(1+\frac{4}{n^2-5}\right)^{n^2-5}\frac{(n^2-1)^4}{3^3}\right)^2\frac{n^2(n+1)(n-1)}{12}\leq$$

$$2\frac{e^8}{3^6}\frac{n^{20}}{12}\leq\frac{7}{10}n^{20},$$

and the corollary follows. ∎

4.3   The expected value for the Condition Number

In this subsection, we obtain upper bounds for the expected value of the generalized condition number from the probability distributions above, and we prove Corollary 44, which is a technical version of Corollary 5 at the Introduction. We will use the following simple result, which may be a well–known fact in Probability Theory.

Lemma 43 *Let X be a positive real valued random variable such that for every positive real number $t > 0$*

$$Prob[X > t] < ct^{-\alpha},$$

where $Prob[\cdot]$ holds for Probability, and $c > 0, \alpha > 1$ are some positive constants. Then, the following inequality holds:

$$E[X]\leq c^{\frac{1}{\alpha}}\frac{\alpha}{\alpha-1}.$$

Proof.– We use the following equality, which is a well–known fact from Probability Theory.

$$E[X]=\int_0^\infty Prob[X>t]\,dt,$$

Then, observe that for every positive real number $s > 0$,

$$E[X]=\int_0^\infty Prob[X>t]\,dt\leq s+c\int_s^\infty t^{-\alpha}\,dt=s+c\frac{s^{1-\alpha}}{\alpha-1}.$$

Let $s := c^{\frac{1}{\alpha}}$, and the lemma follows. ∎

Corollary 44 *The expected value of κ_D^{n-1} in the space Σ^{n-1} satisfies:*

$$E_{\Sigma^{n-1}}[\kappa_D^{n-1}]\leq c_1\,n^{10/3},$$

where $c_1 := \frac{6}{5}\left(\frac{7}{10}\right)^{1/6}\leq 1.14$ is this positive constant. Moreover, the expected value of κ_D^{n-1} in the whole space $\mathbb{P}\left(\mathcal{M}_n(\mathbb{C})\right)$ satistifes

$$E_{\mathcal{M}_n(\mathbb{C})}[\kappa_D^{n-1}]\leq c_2\,n^{5/2},$$

where $c_2 := \frac{2\,e}{7}\frac{1}{6^{1/8}}\leq 0.621$ is this positive constant.

Proof.– From Corollary 29, we know that

$$\frac{\nu_{\dim(\Sigma^{n-1})}[A \in \Sigma^{n-1} : \kappa_D^{n-1}(A) > \frac{1}{\varepsilon}]}{\nu_{\dim(\Sigma^{n-1})}[\Sigma^{n-1}]} \leq \frac{7}{10}(n^{10/3}\varepsilon)^6.$$

Hence, for every positive real number $t > 0$, the probability that a randomly chosen singular matrix $A \in \Sigma^{n-1}$ satisfies $\kappa_D^{n-1}(A) > t$ is at most

$$\frac{\nu_{\dim(\Sigma^{n-1})}[A \in \Sigma^{n-1} : \kappa_D^{n-1}(A) > \frac{1}{1/t}]}{\nu_{\dim(\Sigma^{n-1})}[\Sigma^{n-1}]} \leq \frac{7}{10}n^{20}\frac{1}{t^6}.$$

The first estimation of the corollary follows from Lemma 43 above. As for the second one, from Corollary 40 we know that the probability that a randomly chosen matrix $A \in \mathbb{P}(\mathcal{M}_n(\mathbb{C}))$ satisfies $\kappa_D^{n-1}(A) > t$ is at most

$$\frac{\nu_{n^2-1}[A \in \mathbb{P}(\mathcal{M}_n(\mathbb{C})) : \kappa_D^r(A) > \frac{1}{1/t}]}{\nu_{n^2-1}[\mathbb{P}(\mathcal{M}_n(\mathbb{C}))]} \leq \frac{1}{6}\left[\frac{e\, n^{5/2}}{4}\right]^8 \frac{1}{t^8}.$$

The corollary follows from Lemma 43.

■

4.4 Some other applications.

Corollary 45 *Let $1 < n \in \mathbb{N}$ be a natural number, and let $\mathcal{SIM}_n(\mathbb{C}) \subseteq \mathcal{M}_n(\mathbb{C})$ be the set of symmetric matrices of size n. Then $\mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))$ is a complex projective space of dimension $\frac{n(n+1)}{2} - 1$. Moreover, the following inequality holds:*

$$\frac{\nu_{\frac{n^2+n}{2}-1}[A \in \mathbb{P}(\mathcal{SIM}_n(\mathbb{C})) : \kappa_D^n(A) > \frac{1}{\varepsilon}]}{\nu_{\frac{n^2+n}{2}-1}[\mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))]} \leq 2\left[e\left(\frac{n^2+n}{2}-1\right)\sqrt{n}\,\varepsilon\right]^2.$$

Proof.– From Theorem 39, the following equality holds:

$$\frac{\nu_{\frac{n^2+n}{2}-1}[A \in \mathbb{P}(\mathcal{SIM}_n(\mathbb{C})) : \kappa_D^n(A) > \frac{1}{\varepsilon}]}{\nu_{\frac{n^2+n}{2}-1}[\mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))]} = \frac{\nu_{\frac{n^2+n}{2}-1}[A \in \mathbb{P}(\mathcal{SIM}_n(\mathbb{C})) : d_{\mathbb{P}}(A, \Sigma^{n-1}) < \varepsilon]}{\nu_{\frac{n^2+n}{2}-1}[\mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))]}.$$

Observe that this is not enough to achieve the proof of the corollary. We prove the following formula:

$$\frac{\nu_{\frac{n^2+n}{2}-1}[A \in \mathbb{P}(\mathcal{SIM}_n(\mathbb{C})) : d_{\mathbb{P}}(A, \Sigma^{n-1}) < \varepsilon]}{\nu_{\frac{n^2+n}{2}-1}[\mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))]} =$$

$$\frac{\nu_{\frac{n^2+n}{2}-1}[A \in \mathbb{P}(\mathcal{SIM}_n(\mathbb{C})) : d_{\mathbb{P}}(A, \Sigma^{n-1} \cap \mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))) < \varepsilon]}{\nu_{\frac{n^2+n}{2}-1}[\mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))]}. \tag{16}$$

First, observe that it suffices to prove equality (16) for the set of symmetric matrices such that the have all the singular values distinct and non–zero. In fact, the complementary of

this set is a zero-measure subset of $\mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))$ and does not affect to the estimates on the volume.

Let $A \in \mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))$ be a symmetric matrix. Let $A = UDV^*$ be its SVD, and suppose that the singular values of A, $\sigma_1, \ldots, \sigma_n$, are all distinct and non–zero. Suppose $A = U_1 DV_1^*$, $A = U_2 DV_2^*$ are two SVDs of A. The following equalities hold:

$$U_1 DV_1^* = U_2 DV_2^*, \qquad U_1 D^2 V_1^* = U_2 D^2 V_2^*.$$

Given any matrix $A' \in \mathbb{P}(\mathcal{M}_n(\mathbb{C}))$, and given two natural numbers $1 \leq i, j \leq n$, we denote by $(A')_{ij}$ the corresponding entry of the matrix A'. Then, the following equalities hold:

$$(U_2^* U_1)_{ij} = (V_2^* V_1)_{ij} \frac{\sigma_i}{\sigma_j}, \qquad (U_2^* U_1)_{ij} = (V_2^* V_1)_{ij} \frac{\sigma_i^2}{\sigma_j^2}, \qquad i, j = 1 \ldots n.$$

From the fact that $\sigma_1, \ldots, \sigma_n$ are all distinct and non–zero, we deduce that

$$(U_2^* U_1)_{ij} = 0 \qquad \text{if } i \neq j.$$

So, $U_2^* U_1$ is a diagonal matrix, and the same can be said of $V_1^* V_2$. Now, let $D' = Diag(\sigma_1, \ldots, \sigma_{n-1}, 0)$ be the matrix obtained by replacing the last element of D by 0. As we have seen in the proof of Theorem 39, the following equality holds:

$$d_{\mathbf{P}}(A, \Sigma^{n-1}) = d_{\mathbf{P}}(A, UD'V^*).$$

So, to prove equation (16) we must check that $UD'V^* \in \mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))$. From the fact that A is symmetric, we deduce that:

$$UDV^* = (V^*)^t D(U)^t.$$

This implies that $V^t U$ and $V^*(U^*)^t$ are diagonal matrices, and they commute with D and D'. Moreover, $V^*(U^*)^t = U^*(V^*)^t$ and $V^t U V^*(U^*)^t = V^t U U^*(V^*)^t = Id_n$. As a consequence, the following chain of equalities holds:

$$UD'V^* = (V^*)^t V^t U D'V^*(U^*)^t U^t = (V^*)^t D'V^t U V^*(U^*)^t U^t = (V^*)^t D'U^t.$$

This proves that $UD'V^* \in \mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))$ and equation (16) follows. From Proposition 26 we deduce the bound for the right hand term of equation (16), provided that $\Sigma^{n-1} \cap \mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))$ is a projective subvariety of $\mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))$ of codimension 1 and degree bounded by the Bézout Inequality:

$$\deg(\Sigma^{n-1} \cap \mathbb{P}(\mathcal{SIM}_n(\mathbb{C}))) \leq \deg(\Sigma^{n-1}) = n.$$

$\blacksquare$

Corollary 46 *Let $\mathcal{B}_{ij}(\mathbb{C}) \subseteq \mathcal{M}_{n_1 \times n_2}(\mathbb{C})$ be the set of matrices A of the following shape:*

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1 \in \mathcal{M}_{i \times j}(\mathbb{C}), A_2 \in \mathcal{M}_{(n_1-i)\times(n_2-j)}(\mathbb{C})$. *That is,*

$$\mathcal{B}_{ij}(\mathbb{C}) \equiv \mathcal{M}_{ij}(\mathbb{C}) \oplus \mathcal{M}_{(n_1-i)\times(n_2-j)}(\mathbb{C})$$

can be identified with the direct sum of $\mathcal{M}_{ij}(\mathbb{C})$ *and* $\mathcal{M}_{(n_1-i)\times(n_2-j)}(\mathbb{C})$. *Then* $\mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))$ *is a complex projective space of dimension* $ij + (n_1 - i)(n_2 - j) - 1$, *and the following inequality holds:*

$$\frac{\nu_{ij+(n_1-i)(n_2-j)-1}[A \in \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C})) : \kappa_D^{n_2}(A) > \frac{1}{\varepsilon}]}{\nu_{ij+(n_1-i)(n_2-j)-1}[\mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))]} \leq$$

$$\leq 2\left[\frac{e(ij + (n_1 - i)(n_2 - j) - 1)}{n_1 - n_2 + 1}\sqrt{n_2}\,\varepsilon\right]^{2(n_1-n_2+1)}.$$

Proof.– From Theorem 39, the following equality holds:

$$\frac{\nu_{ij+(n_1-i)(n_2-j)-1}[A \in \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C})) : \kappa_D^{n_2}(A) > \frac{1}{\varepsilon}]}{\nu_{ij+(n_1-i)(n_2-j)-1}[\mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))]} =$$

$$= \frac{\nu_{ij+(n_1-i)(n_2-j)-1}[A \in \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C})) : d_{\mathbb{P}}(A, \Sigma^{n_2-1}) < \varepsilon]}{\nu_{ij+(n_1-i)(n_2-j)-1}[\mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))]}.$$

Observe that this is not enough to achieve the proof of the corollary. We prove the following formula:

$$\frac{\nu_{ij+(n_1-i)(n_2-j)-1}[A \in \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C})) : d_{\mathbb{P}}(A, \Sigma^{n_2-1}) < \varepsilon]}{\nu_{ij+(n_1-i)(n_2-j)-1}[\mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))]} =$$

$$\frac{\nu_{ij+(n_1-i)(n_2-j)-1}[A \in \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C})) : d_{\mathbb{P}}(A, \Sigma^{n_2-1} \cap \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))) < \varepsilon]}{\nu_{ij+(n_1-i)(n_2-j)-1}[\mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))]}. \tag{17}$$

In fact, let $A \in \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))$. Let $A' \in \Sigma^{n_2-1}$ be a singular matrix such that $d_{\mathbb{P}}(A, \Sigma^{n_2-1}) = d_{\mathbb{P}}(A, A')$. From the expression of A' (see Theorem 39) it is obvious that $A' \in \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))$ and equation (17) follows. Now, from Proposition 26 we obtain a bound for the right hand term in equation (17), provided that $\Sigma^{n_2-1} \cap \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))$ is a projective subvariety of $\mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))$ of codimension $n_1 - n_2 + 1$ and degree bounded by the Bézout Inequality:

$$\deg(\Sigma^{n_2-1} \cap \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))) \leq \deg(\Sigma^{n_2-1}) \leq n_2^{n_1-n_2+1}.$$

We obtain the following inequality:

$$\frac{\nu_{ij+(n_1-i)(n_2-j)-1}[A \in \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C})) : d_{\mathbb{P}}(A, \Sigma^{n_2-1} \cap \mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))) < \varepsilon]}{\nu_{ij+(n_1-i)(n_2-j)-1}[\mathbb{P}(\mathcal{B}_{ij}(\mathbb{C}))]} \leq$$

$$2\left[\frac{e(ij + (n_1 - i)(n_2 - j) - 1)}{n_1 - n_2 + 1}\sqrt{n_2}\,\varepsilon\right]^{2(n_1-n_2+1)}.$$

■

The reader may observe that Corollaries in this Section are particular cases of the more general statement that follows:

Theorem 47 *Let r be a positive integer number, $2 \leq r \leq n_2$. Let $\mathcal{C} \subseteq \mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right)$ be an equi–dimensional algebraic variety of dimension m. Suppose that there exists an equi–dimensional algebraic variety $\mathcal{C}' \subseteq \mathbb{P}\left(\mathcal{M}_{n_1 \times n_2}(\mathbb{C})\right)$ of dimension $m' < m$ such that for every projective matrix $A \in \mathcal{C}$ the following property holds:*

$$d_{\mathbb{P}}\left(A, \Sigma^{r-1}\right) = d_{\mathbb{P}}\left(A, \mathcal{C}'\right).$$

Then, the following inequality holds:

$$\frac{\nu_m[\{A \in \mathcal{C} : \kappa_D^r > \varepsilon^{-1}\}]}{\nu_m[\mathcal{C}]} \leq C(n_1 n_2 - 1, m, m') \deg(\mathcal{C}') \varepsilon^{2(m-m')},$$

where $C(n_1 n_2 - 1, m, m')$ is the constant defined in Section 3.

Proof.– From Theorem 39, the following equality holds:

$$\frac{\nu_m[\{A \in \mathcal{C} : \kappa_D^r > \varepsilon^{-1}\}]}{\nu_m[\mathcal{C}]} = \frac{\nu_m[\{A \in \mathcal{C} : d_{\mathbb{P}}\left(A, \Sigma^{r-1}\right) < \varepsilon\}]}{\nu_m[\mathcal{C}]}.$$

Thus,

$$\frac{\nu_m[\{A \in \mathcal{C} : \kappa_D^r > \varepsilon^{-1}\}]}{\nu_m[\mathcal{C}]} = \frac{\nu_m[\{A \in \mathcal{C} : d_{\mathbb{P}}\left(A, \mathcal{C}'\right) < \varepsilon\}]}{\nu_m[\mathcal{C}]},$$

and the claim follows from Theorem 18. ∎

Acknowledgements

References

[1] L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and real computation*. Springer-Verlag, New York, 1998.

[2] W. Bruns and U. Vetter. *Determinantal rings*, vol. 1327 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.

[3] D. Castro, J. L. Montaña, L. M. Pardo, and J. San Martín. The distribution of condition numbers of rational data of bounded bit length. *Found. Comput. Math.*, 2 (2002), 1–52.

[4] K.-K. Choi. On the distribution of points in projective space of bounded height. *Trans. Amer. Math. Soc.*, 352 (2000), 1071–1111.

[5] M. Demazure. *Catastrophes et bifurcations*. Paris:Ellipses, Paris, 1989.

[6] J. W. Demmel. The geometry of ill-conditioning. *J. Complexity*, 3 (1987), 201–229.

[7] J. W. Demmel. The probability that a numerical analysis problem is difficult. *Math. Comp.*, 50 (1988), 449–480.

[8] C. Eckart and G. Young. The approximation of one matrix by another of lower rank. *Psychometrika*, 1 (1936), 211–218.

[9] A. Edelman. Eigenvalues and condition numbers of random matrices. *SIAM J. Matrix Anal. Appl.*, 9 (1988), 543–560.

[10] A. Edelman. On the distribution of a scaled condition number. *Math. Comp.*, 58 (1992), 185–190.

[11] H. Federer. Some theorems on integral currents. *Trans. Amer. Math. Soc.*, 117 (1965), 43–67.

[12] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[13] F. J. Flaherty. The volume of a tube in complex projective space. *Illinois J. Math.*, 16 (1972), 627–638.

[14] W. Fulton. *Intersection theory*, vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.

[15] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, 1987.

[16] M. Giusti, J. Heintz, K. Hägele, J. E. Morais, L. M. Pardo, and J. L. Montaña. Lower bounds for Diophantine approximations. *J. Pure Appl. Algebra*, 117/118 (1997), 277–317.

[17] G. H. Golub and C. F. Van Loan. *Matrix computations*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, third edition, 1996.

[18] A. Gray. Volumes of tubes about complex submanifolds of complex projective space. *Trans. Amer. Math. Soc.*, 291 (1985), 437–449.

[19] A. Gray. *Tubes*, vol. 221 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, second edition, 2004.

[20] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, vol. 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999.

[21] K. Hägele, J. E. Morais, L. M. Pardo, and M. Sombra. On the intrinsic complexity of the arithmetic Nullstellensatz. *J. Pure Appl. Algebra*, 146 (2000), 103–183.

[22] J. Harris. *Algebraic geometry*, vol. 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992.

[23] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977.

[24] J. Heintz. Definability and fast quantifier elimination in algebraically closed fields. *Theoret. Comput. Sci.*, 24 (1983), 239–277.

[25] N. J. Higham. *Accuracy and stability of numerical algorithms*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2002.

[26] R. Howard. The kinematic formula in Riemannian homogeneous spaces. *Mem. Amer. Math. Soc.*, 106 (1993), vi+69.

[27] W. Kahan. Huge Generalized Inverses of Rank-Deficient Matrices, Unpublished Manuscript.

[28] T. Krick and L. M. Pardo. A computational method for Diophantine approximation. In *Algorithms in algebraic geometry and applications (Santander, 1994)*, vol. 143 of *Progr. Math.*, 193–253. Birkhäuser, Basel, 1996.

[29] L. Mirsky. Results and problems in the theory of doubly-stochastic matrices. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 1 (1962/1963), 319–334.

[30] F. Morgan. *Geometric measure theory*. Academic Press Inc., Boston, MA, 1988.

[31] D. Mumford. *Algebraic geometry. I*. Springer-Verlag, Berlin, 1976.

[32] J. von Neumann and H. H. Goldstine. Numerical inverting of matrices of high order. *Bull. Amer. Math. Soc.*, 53 (1947), 1021–1099.

[33] J. Renegar. On the efficiency of Newton's method in approximating all zeros of a system of complex polynomials. *Math. Oper. Res.*, 12 (1987), 121–148.

[34] L. A. Santaló. *Integral geometry and geometric probability*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976.

[35] E. Schmidt. Zur Theorie der linearen und nichtlineinaren Integralgleichungen. I Tiel. Entwicklung willkürlichen Funktionen nach System vorgeschriebener. *Mahtematische Annalen*, 63 (1907), 433–476.

[36] I. R. Shafarevich. *Basic algebraic geometry. 1*. Springer-Verlag, Berlin, second edition, 1994.

[37] I. R. Shafarevich. *Basic algebraic geometry. 2*. Springer-Verlag, Berlin, second edition, 1994.

[38] S. Smale. On the efficiency of algorithms of analysis. *Bull. Amer. Math. Soc. (N.S.)*, 13 (1985), 87–121.

[39] P. Stănică. Good lower and upper bounds on binomial coefficients. *JIPAM. J. Inequal. Pure Appl. Math.*, 2 (2001), Article 30, 5 pp. (electronic).

[40] G. W. Stewart and J. G. Sun. *Matrix perturbation theory*. Computer Science and Scientific Computing. Academic Press Inc., Boston, MA, 1990.

[41] G. Stolzenberg. *Volumes, limits, and extensions of analytic varieties*. Lecture Notes in Mathematics, No. 19. Springer-Verlag, Berlin, 1966.

[42] E. Study. Kurzeste Wege in complexen Gebiete. *Math. annalen*, 60 (1905), 321–377.

[43] L. N. Trefethen and D. Bau, III. *Numerical linear algebra*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

[44] A. M. Turing. Rounding-off errors in matrix processes. *Quart. J. Mech. Appl. Math.*, 1 (1948), 287–308.

[45] W. Vogel. *Lectures on results on Bezout's theorem*, vol. 74 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Published for the Tata Institute of Fundamental Research, Bombay, 1984.

[46] A. Weinstein. Almost invariant submanifolds for compact group actions. *J. Eur. Math. Soc. (JEMS)*, 2 (2000), 53–86.

[47] A. Weyl. On the volume of tubes. *Amer. J. Math*, 61 (1939), 461–472.

[48] J. H. Wilkinson. *The algebraic eigenvalue problem*. Clarendon Press, Oxford, 1965.

[49] R. A. Wolf. The volume of tubes in complex projective space. *Trans. Amer. Math. Soc.*, 157 (1971), 347–371.