Convexity properties of the condition number II [†]

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This paper is dedicated to Steve Smale, on his 80th birthday.

Abstract

In our previous paper [2], we studied the condition metric in the space of maximal rank $n \times m$ matrices. Here, we show that this condition metric induces a Lipschitz-Riemann structure on that space. After investigating geodesics in such a nonsmooth structure, we show

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that the inverse of the smallest singular value of a matrix is a log-convex function along geodesics (Theorem 1).

We also show that a similar result holds for the solution variety of linear systems (Theorem 31).

Some of our intermediate results, such as Theorem 12, on the second covariant derivative or *Hessian* of a function with symmetries on a manifold, and Theorem 29 on piecewise self-convex functions, are of independent interest.

Those results were motivated by our investigations on the complexity of path-following algorithms for solving polynomial systems.

1 Introduction

Let two integers $1 \leq n \leq m$ be given and let us consider the space of matrices $\mathbb{K}^{n\times m}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, equipped with the Frobenius inner product

$$\langle M, N\rangle_F = \text{trace } (N^*M) = \sum_{i,j} m_{ij}\overline{n_{ij}}.$$

We denote by

$$\sigma_1(A) \geq \ldots \geq \sigma_{n-1}(A) \geq \sigma_n(A) \geq 0$$

the singular values of a matrix $A \in \mathbb{K}^{n\times m}$, by $\mathbb{GL}_{n,m}$ the space of matrices $A \in \mathbb{K}^{n\times m}$ with maximal rank, that is rank $A = n$ or, equivalently, $\sigma_n(A) > 0$, and by $\mathcal{N}$ the set of singular (or rank deficient) matrices:

$$\mathcal{N} = \mathbb{K}^{n\times m} \setminus \mathbb{GL}_{n,m} = \left\{A \in \mathbb{K}^{n\times m} \ : \ \sigma_n(A) = 0\right\}.$$

The distance of a matrix $A \in \mathbb{K}^{n\times m}$ from $\mathcal{N}$ is given by its smallest singular value:

$$d_F(A, \mathcal{N}) = \min_{S\in\mathcal{N}} \|A - S\|_F = \sigma_n(A).$$

Consider now the problem of connecting two matrices with the shortest possible path staying, as much as possible, away from the set of singular matrices. We realize this objective by considering an absolutely continuous path $A(t)$, $a \leq t \leq b$, with given endpoints (say $A(a) = A$ and $A(b) = B$) which minimizes its **condition length** defined by

$$L_\kappa = \int_a^b \left\|\frac{dA(t)}{dt}\right\|_F \sigma_n(A(t))^{-1}dt.$$

We call **minimizing condition path** an absolutely continuous path which minimizes this integral in the set of absolutely continuous paths with the same end-points. We define a **minimizing condition geodesic** as a minimizing condition path parametrized by the condition arc length, that is when

$$\left\| \frac{dA(t)}{dt} \right\|_F \sigma_n(A(t))^{-1} = 1 \text{ a. e.}$$

A **condition geodesic** is an absolutely continuous path which is locally a minimizing condition geodesic. This concept of geodesic is related to the Riemannian structure defined on $\mathbb{GL}_{n,m}$ by:

$$\langle M, N \rangle_{\kappa,A} = \sigma_n(A)^{-2} \text{Re} \langle M, N \rangle_F .$$

We call it the **condition Riemann structure** on $\mathbb{GL}_{n,m}$.

Our objective is to investigate the properties of the smallest singular value $\sigma_n(A(t))$ along a condition geodesic. Our main result says:

Theorem 1. *For any condition geodesic $t \to A(t)$ in $\mathbb{GL}_{n,m}$, the map $t \to \log\left(\sigma_n^{-2}(A(t))\right)$ is convex.*

This theorem extends our main result in [2]. In that paper, the same theorem is proven for those condition geodesic arcs contained in the open subset

$$\mathbb{GL}_{n,m}^{>} = \{A \in \mathbb{GL}_{n,m} \ : \ \sigma_{n-1}(A) > \sigma_n(A)\}$$

that is when the smallest singular value $\sigma_n(A)$ is simple. The reason for this restriction is easy to explain. The smallest singular value $\sigma_n(A)$ is smooth in $\mathbb{GL}_{n,m}^{>}$, and, in that case, we can use the toolbox of Riemannian geometry. But it is only locally Lipschitz in $\mathbb{GL}_{n,m}$; for this reason we call the condition structure in $\mathbb{GL}_{n,m}$ a **Lipschitz-Riemannian structure**.

Motivation: Let us now say a word about our motivations. The today classical papers [28], [29], and [30] by Shub and Smale relate complexity bounds for homotopy methods to solve polynomial systems to the condition number of the encountered problems along the considered homotopy path. Ill-conditioned problems slow the algorithm and increase its complexity. For this reason it is natural to consider paths which avoid ill-posed problems, and, at the same time, are as short as possible. The condition metric has been designed to construct such paths. It has been introduced by Shub in [27], then studied by Beltrán and Shub in [4] in spaces of polynomial equations (see also [12, 3, 22].) When we started to work in this project, we expected to reduce the more general problem of finding good homotopy paths for non-linear systems to the 'linear' case. Unfortunately this seems to be a harder problem, to be pursued later.

The case of linear maps (and related spaces) appears in Beltrán-Dedieu-Malajovich-Shub [2] and Boito-Dedieu [7].

In the linear case, it is rather a remarkable fact that the inverse of the squared distance to singular matrices $\sigma_n^{-2}(A(t))$ is log-convex along the condition geodesics. So, in particular, the maximum of $\log(\sigma_n(A(t))^{-2})$ and the maximum of $\sigma_n(A(t))^{-1}$ along such paths is necessarily obtained at its endpoints and the condition geodesics stay away from singular matrices.

This is clearly not true in the usual metric, since straight lines can get arbitrarily close to the variety of degenerate matrices. This suggests the following application:

If we consider a condition path connecting a given $A \in \mathbb{GL}_{n,m}$ to (for example) $I_{n,m}\|A\|_F/\sqrt{n}$ ($I_{n,m}(i,j) = 1$ if $i = j$ and 0 otherwise), for any matrix $A(t)$ in this path, according to Theorem 1, one has

$$\frac{\sqrt{n}}{\|A\|_F} \leq \sigma_n(A(t))^{-1} \leq \sigma_n(A)^{-1}.$$

We think this property may help to find good preconditioners to solve linear systems.

There are other motivations. Convexity of the distance or similar function to the ill-posed problems may play a role in optimization. Witness for example the role played by the barrier function in linear programming theory. Two of us will be expanding on this theme in a forthcoming paper.

Outline of the paper

The condition number is not of class $\mathcal{C}^1$, hence we cannot apply the usual Riemannian geometry to the condition metric. In **Section 2**, we introduce Lipschitz- Riemann structures and develop the basic results, that allow us to do differential geometry in the non-smooth case. Using nonsmooth analysis techniques, we prove that any condition geodesic is $\mathcal{C}^1$ with a locally Lipschitz derivative (Theorem 3). Such techniques are already present in Boito-Dedieu [7].

In **Section 3** we develop an important tool for proving self-convexity, allowing a more systematic use of the symmetries. (A symmetry is an isometry of a manifold that leaves a function invariant). Theorem 12 gives a simplified computation of the Hessian when there is a Lie group of symmetries. This theorem may be of independent interest. It is so natural we would not be surprised if it is already known, but we have not found it anywhere. We were led to this theorem sometime after a conversation with John Lott on Hessians and Riemannian submersions while he was visiting the University of Toronto.

The strategy for proving the main theorem is to decompose the space of matrices in a finite union of smooth manifolds, so that in each of them the metric is smooth. In **Section 4** we produce this decomposition, we study the group of symmetries of the condition number and then, using Theorem 12, we establish self-convexity on each piece.

In **Section 5**, we prove a result that may be of independent interest, Theorem 29: piecing together convexity results on restrictions of the Lipschitz-Riemann structure to a union of submanifolds of varying dimensions, where the structure is smooth, to obtain a global result.

In **Section 6**, we use all these tools to finish the proof of Theorem 1. We use the same tools in **Section 7** to state and prove Theorem 31 about self-convexity in the solution variety

$$\mathcal{W} = \{(A, x) \in \mathbb{GL}_{n,n+1} \times \mathbb{P}(\mathbb{K}^n) : Ax = 0\} \ .$$

Above, the notation $\mathbb{P}(\mathbb{E})$ denotes the projectivization of a linear space $\mathbb{E}$. Namely, it is the space (manifold) of real or complex lines in $\mathbb{E}$ passing through the origin. For instance, $\mathbb{P}(\mathbb{R}^3)$ is the classical projective plane, that can also be obtained by identifying antipodal points of the sphere $\mathbb{S}^2$.

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2 Geodesics in Lipschitz-Riemann structures, and self-convexity

2.1 Lipschitz-Riemann structures

Most textbooks of Riemannian geometry define a Riemannian structure on a smooth manifold $\mathcal{M}$ as a scalar product $\langle \cdot, \cdot \rangle_x$ on each tangent space $T_x\mathcal{M}$, depending smoothly on x. Here we drop the smoothness hypothesis.

Definition 2. A **Lipschitz-Riemann structure** on a $\mathcal{C}^2$ manifold $\mathcal{M}$ is a scalar product $\langle \cdot, \cdot \rangle_x$ at each $T_x\mathcal{M}$, such that its coefficients are locally Lipschitz functions of x. Also, let $\|u\|_x = \sqrt{\langle u, u \rangle_x}$ be the associated norm in $T_x\mathcal{M}$.

The **length** of an absolutely continuous path $x(t) \in \mathcal{M}$, $a \le t \le b$, is defined as the integral

$$L(x, a, b) = \int_a^b \|\dot{x}(\tau)\|_{x(\tau)} d\tau,$$

where $\dot{x}(t)$ denotes the derivative with respect to t. Its **arc length** is given by the map

$$t \in [a, b] \to L(x, a, t) \in [0, L(x, a, b)].$$

The **distance** $d(a, b)$ between two points $a, b \in \mathcal{M}$ is the infimum of all the lengths of the paths containing a and b in their image. We call **minimizing path** an absolutely continuous path such that $L(x, a, b) = d(a, b)$.

It is usual in differential geometry textbooks to construct geodesics as solutions of a certain second order differential equation, the *geodesic differential equation*. Unfortunately, the coefficients of this equation are given by a formula in terms of the partial derivatives of the metric coefficients. In a Lipschitz-Riemann structure, those coefficients are assumed to be Lipschitz, not necessarily differentiable functions. Also, it turns out that minimizing paths are not necessarily smooth.

We define a **minimizing geodesic** as a minimizing path parametrized by arc length, that is when

$$\|\dot{x}(t)\|_{x(t)} = 1 \text{ a. e.}$$

A path in $\mathcal{M}$ parametrized by arc length is a **geodesic** when it is locally a minimizing geodesic.

The main result of this section is the following:

Theorem 3. *Any geodesic for a Lipschitz-Riemann structure belongs to the class $\mathcal{C}^{1+Lip}$ that is $\mathcal{C}^1$ with a locally Lipschitz derivative.*

This theorem is proved in section 2.4, it extends a similar result by Charles Pugh [24] who proves the existence of locally minimizing $\mathcal{C}^{1+Lip}$ geodesics. His argument is based on a smooth approximation of the Lipschitz structure where the classical toolbox of Riemannian geometry applies, followed by a *passage à la limite*.

Using different techniques we prove here this regularity assumption for all geodesics.

An immediate consequence of [8, Cor .VIII-4 p.126] is that $\mathcal{C}^{1+Lip} = W^{2,\infty}$ the Sobolev space of maps f with $f'' \in L^\infty$.

2.2 Existence of geodesics in a Lipschitz-Riemann structure

Existence of minimizing geodesics with given endpoints may be deduced from the Hopf-Rinow Theorem. Because we cannot assume the smoothness of geodesics, we refer to Gromov's version of this theorem [18, Th.1.10]. A metric space (X, d) is a **path metric space** if the distance between each pair of points equals the infimum of the lengths of curves joining the points.

Theorem 4. *If (X, d) is a complete, locally compact path metric space, then*

- *Each bounded, closed subset is compact,*
- *Each pair of points can be joined by a minimizing geodesic.*

Two examples of such spaces are given by Boito-Dedieu [7] for linear maps (X is one of the connected components of $\mathbb{GL}_{n,m}$ equipped with the condition structure), and by Shub [27] when X is the solution variety associated with the homogeneous polynomial system solving problem equipped with the corresponding condition structure.

2.3 Lipschitz-Riemann structures in $\mathbb{R}^k$, generalized gradients and the problem of Bolza

An important example of Lipschitz-Riemann structure is given by an open set $\Omega \subset \mathbb{R}^k$ equipped with the scalar product

$$\langle u, v \rangle_x = v^T H(x) u$$

where H is a locally Lipschitz map from Ω into the set of positive definite $n \times n$ matrices.

A minimizing geodesic $x(t) \in \Omega$, $a \leq t \leq b$, minimizes the integral

$$\int_a^b \sqrt{\dot{y}(t)^T H(y(t)) \dot{y}(t)} dt$$

in the set of absolutely continuous paths $y(t)$ with endpoints $y(a) = x(a)$, and $y(b) = x(b)$. This is an instance of the Bolza problem.

For a smooth integrand L, a local solution $x(t)$ of the Bolza problem

$$\inf \int_a^b L(y(t), \dot{y}(t)) dt, \tag{2.1}$$

where the infimum is taken in the set of a.c. paths with given endpoints, satisfies the Euler-Lagrange differential equation

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t)) + \frac{\partial L}{\partial x}(x(t), \dot{x}(t)) = 0 \quad a.e. \tag{2.2}$$

In our context, it is possible to differentiate $L(x, \dot{x}) = \sqrt{\dot{x}^T H(x) \dot{x}}$ with respect to the second argument by ordinary differential calculus:

$$\frac{\partial}{\partial \dot{x}} L(x, \dot{x}) : y \mapsto \frac{1}{L(x, \dot{x})} \dot{x}^T H(x) y.$$

If we avoid $\dot{x} = 0$ (which will be the case), we deduce that L is smooth in the variable $\dot{x}$ and locally Lipschitz in the variable x. For this reason we replace the classical geodesic differential equation by a generalized version of the Euler-Lagrange equation (2.2) based on generalized gradients.

Let $f : \Omega \subset \mathbb{R}^k \to \mathbb{R}$ be a locally Lipschitz function defined on an open set. Its **one-sided directional derivative** at $x \in \Omega$ in the direction $d \in \mathbb{R}^k$ is defined as

$$f'(x, d) = \lim_{t \to 0_+} \frac{f(x + td) - f(x)}{t}.$$

The **generalized directional derivative in Clarke's sense** of f at $x \in \Omega$ in the direction d is defined as

$$f^o(x, d) = \limsup_{\substack{y \to x \\ t \to 0_+}} \frac{f(y + td) - f(y)}{t}$$

and the **generalized gradient** of f at x is the nonempty compact subset of $\mathbb{R}^k$ given by

$$\partial f(x) = \left\{ s \in \mathbb{R}^k \ : \ \langle s, d \rangle \leq f^o(x, d) \text{ for all } d \in \mathbb{R}^k \right\}.$$

It turns out that the generalized gradient is always a convex set. When $f \in \mathcal{C}^1(\Omega)$ the generalized gradient is just the usual one: $\partial f(x) = \{\nabla f(x)\}$. The generalized directional derivative is related to the gradient via the equality

$$f^o(x, d) = \max_{s \in \partial f(x)} \langle s, d \rangle .$$

We say that f is **regular at** x when the two directional derivatives exist and are equal:

$$f^o(x, d) = f'(x, d) \text{ for any } d \in \mathbb{R}^k.$$

When f is defined on a $\mathcal{C}^1$ manifold $\mathcal{M}$, we say that f is **regular at** $m \in \mathcal{M}$ when its composition with a local chart at m gives a regular map in the usual meaning.

Good references for this topic is Clarke [11] or Schirotzek [26].

For the problem of Bolza described above the counterpart of the Euler-Lagrange equation is given by the following result (see [11] Theorem 4.4.3, and [10]).

Theorem 5. *Let x solve the Bolza problem (2.1) in the case in which $L(x, \dot{x})$ is a locally Lipschitz map and suppose that $\dot{x}$ is essentially bounded. Then there is an absolutely continuous map p such that*

$$\dot{p}(t) \in \partial_x L(x(t), \dot{x}(t)) \text{ and } p(t) \in \partial_{\dot{x}} L(x(t), \dot{x}(t)) \quad a.e.$$

2.4 Proof of Theorem 3

Since Theorem 3 is of local nature, it suffices to prove it locally in $\mathbb{R}^k$. Once this is done, take a local chart and transfer the Lipschitz-Riemann structure of $\mathcal{M}$ to an open set $\Omega \subset \mathbb{R}^k$ where the theorem is already proved. Therefore, let us show the theorem in $\mathbb{R}^k$.

By definition, a geodesic is a locally minimizing geodesic. Thus, it suffices to establish the theorem in this case.

A minimizing geodesic $x(t) \in \Omega$, $a \leq t \leq b$, is parametrized by arc length so that

$$\dot{x}(t)^T H(x(t)) \dot{x}(t) = 1 \quad a.e.,$$

Thus, $\dot{x}(t)$ is $\neq 0$ and essentially bounded:

$$\dot{x} \in L^\infty\left([a,b], \mathbb{R}^k\right).$$

Moreover, x minimizes the integral

$$\int_a^b \sqrt{\dot{y}(t)^T H(y(t)) \dot{y}(t)}dt$$

in the set of absolutely continuous paths with endpoints $y(a) = x(a)$, and $y(b) = x(b)$. Thus, according to Theorem 5, there is an absolutely continuous arc p such that

$$\dot{p}(t) \in \partial_x \sqrt{\dot{x}(t)^T H(x(t))\dot{x}(t)}, \tag{2.3}$$

$$p(t) \in \partial_{\dot{x}} \sqrt{\dot{x}(t)^T H(x(t))\dot{x}(t)} \tag{2.4}$$

for almost all $t \in [a,b]$. Since our integrand is smooth in the $\dot{x}$ variable we may write (2.4)

$$p(t) = \frac{H(x(t))\dot{x}(t)}{\sqrt{\dot{x}(t)^T H(x(t))\dot{x}(t)}} = H(x(t))\dot{x}(t).$$

Thus, $\dot{x}(t) = H(x(t))^{-1}p(t)$ is absolutely continuous and $x(t)$ possesses a.e. a second derivative $\ddot{x}(t) \in L^1([a,b], \mathbb{R}^k)$.

We now have to show that this second derivative is essentially bounded. This comes from (2.3). Since $\sqrt{\cdot}$ is a smooth function we get from Proposition 2.3.3 and Theorem 2.3.9 of Clarke's book [11] that

$$\partial_x \sqrt{\dot{x}^T H(x)\dot{x}} \subset \frac{\dot{x}^T \partial H(x)\dot{x}}{2\sqrt{\dot{x}^T H(x)\dot{x}}} = \frac{1}{2}\dot{x}^T \partial H(x)\dot{x},$$

with

$$\dot{x}^T \partial H(x)\dot{x} = \sum_{i,j} \dot{x}_i \dot{x}_j \partial h_{ij}(x).$$

Equation (2.3) implies

$$\dot{p}(t) \in \frac{1}{2}\dot{x}(t)^T \partial H(x(t))\dot{x}(t) \text{ a.e.}$$

From the hypothesis, the functions $h_{ij}(x)$ are locally Lipschitz. Their generalized gradients are compact convex sets in $\mathbb{R}^k$. The union of all these sets along the path $x(t)$ gives us a bounded set. Since the curve $\dot{x}(t)$ is continuous, we deduce from these considerations, that $\dot{p}(t)$ is bounded a.e. Thus $p(t)$ is Lipschitz, and $\dot{x}(t) = H(x(t))^{-1}p(t)$ is also Lipschitz. The second derivative $\ddot{x}(t)$ is thus bounded by the Lipschitz constant of $\dot{x}(t)$, and we are done.

Remark 6. The previous lines give the following properties for a geodesic x in Ω: $x \in \mathcal{C}^{1+Lip}$, $\dot{x}^T H(x)\dot{x} = 1$, and

$$\frac{d}{dt}(H(x)\dot{x}) \in \frac{1}{2}\dot{x}^T \partial H(x)\dot{x} = \frac{1}{2}\sum_{i,j} \dot{x}_i \dot{x}_j \partial h_{ij}(x) \text{ a.e.}$$

The initial value problem, and even the boundary value problem associated with this second order differential inclusion, may have many solutions. Examples are given in [7]. Moreover, solutions are not necessarily locally minimizing geodesics and geodesics are not necessarily unique.

2.5 Conformal Lipschitz-Riemann structure

The example of a Lipschitz-Riemann structure which motivates this paper is given by the condition structure on $\mathbb{GL}_{n,m}$. It is obtained in multiplying the Frobenius scalar product by the locally Lipschitz function σ_n^{-2}. Let us put it in a more general setting.

Definition 7. Let $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ be a $\mathcal{C}^2$ Riemannian manifold, and let $\alpha : \mathcal{M} \to \mathbb{R}$ be a locally Lipschitz function with positive values. Let $\mathcal{M}_\kappa$ be the manifold $\mathcal{M}$ with the new metric

$$\langle\cdot,\cdot\rangle_{\kappa,x} = \alpha(x)\langle\cdot,\cdot\rangle_x$$

called α-Riemann structure. When α is the square of the (unscaled) condition number, i.e. $\alpha(A) = \|A^\dagger\|_2^2 = \sigma_n^{-2}$, this is also called the **condition Riemann structure** or simply the **condition structure**. We say that α is **self-convex** when $\log \alpha(\gamma(t))$ is convex for any geodesic γ in $\mathcal{M}_\kappa$.

We denote by L (respectively L_κ) the length of a curve γ in the $\mathcal{M}$-structure (respectively in the $\mathcal{M}_\kappa$-structure). We will speak of **length** or **condition length**, and also of **distance** or **condition distance**, **geodesics** or **condition geodesics** and so on.

Examples of self-convex maps are given in [2] where this concept is introduced for the first time.

Using this definition Theorem 1 above reads

$$\alpha(A) = \sigma_n(A)^{-2} \text{ is self-convex in } \mathbb{GL}_{n,m}.$$

3 Self-convexity in the smooth case and the computation of Hessians.

3.1 Self-convexity in the smooth case

Self convexity in the smooth case was studied in our previous paper [2] in this journal. We refer the reader to Section 2 of [2] for basic definitions regarding convexity and geodesic convexity. A snapshot of the main features of self-convexity in the smooth case follows.

We denote by D the Levi-Civita connection and by $D_X T$ the **covariant derivative** of a tensor T in the direction given by a vector field X. Recall that if we assume geodesic coordinates in the neighborhood of a point p, then $(D_X T)_p$ is the same as the ordinary directional (or Lie) derivative. The covariant derivative is coordinate independent, in the sense that $D_X T$ is a tensor.

If f is a smooth enough function, then its derivative with respect to a vector field is denoted by $X(f)$, so that $X(f)(p) = Df(p)X(p) = \langle \nabla f(p), X(p) \rangle_p$. The second covariant derivative of a function f (sometimes also known as the *Hessian*) is defined by

$$D^2 f(X,Y) = D(Df)(X,Y) = D_X(Y(f)) - (D_X Y)(f) \tag{3.1}$$

where X and Y are smooth vector fields. The operator above is symmetric, in the sense that $D^2 f(X,Y) = D^2 f(Y,X)$ (see e.g. [5, p.305])

When $\alpha : \mathcal{M} \to \mathbb{R}$ is $\mathcal{C}^2$, self-convexity of α is equivalent to the second covariant derivative of $\log(\alpha)$ being positive semi-definite in the α-condition Riemann structure (see [32] Chap. 3, Theorem 6.2). Note that the second covariant derivative of a map $\mathcal{M} \to \mathbb{R}$ is different in $\mathcal{M}$ and in $\mathcal{M}_\kappa$. We denote them respectively by D^2 or D^2_κ. Self-convexity of α is equivalent to $D^2_\kappa \log(\alpha)$ being positive semi-definite.

Proposition 2 of [2] is

Proposition 8. *For a function $\alpha : \mathcal{M} \to \mathbb{R}$ of class $\mathcal{C}^2$ with positive values self-convexity is equivalent to*

$$2\alpha(x) D^2\alpha(x)(\dot{x},\dot{x}) + \|D\alpha(x)\|_x^2 \|\dot{x}\|_x^2 - 4(D\alpha(x)\dot{x})^2 \geq 0 \tag{3.2}$$

for all $x \in \mathcal{M}$ and for all vector $\dot{x} \in T_x\mathcal{M}$, the tangent space at x.

3.2 Self-convexity in a product space

Proposition 2 of [2] has an immediate corollary which can be useful. Suppose $\mathcal{N}$ is another $\mathcal{C}^2$ Riemannian manifold. Give $\mathcal{M}\times\mathcal{N}$ the product metric. Let $\pi : \mathcal{M}\times\mathcal{N} \to \mathcal{M}$ be the projection on the first factor and $\hat{\alpha} : \mathcal{M}\times\mathcal{N} \to \mathbb{R}$ be the composition $\hat{\alpha} = \alpha \circ \pi$.

Proposition 9. *Let α be of class $\mathcal{C}^2$ in $\mathcal{M}$. Then, α is self-convex in $\mathcal{M}$ if and only if $\hat{\alpha}$ is self-convex in $\mathcal{M}\times\mathcal{N}$.*

We thank an anonymous referee for pointing out the *if* part of this Proposition and simplifying the proof.

Proof. We prove first the only if part. Let $(x, y) \in \mathcal{M}\times\mathcal{N}$ and assume normal (geodesic) coordinates in a neighborhood of $x \in \mathcal{M}$. Also, assume normal coordinates around $y \in \mathcal{N}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{N}}$.

We claim that this defines a system of normal coordinates in $\mathcal{M}\times\mathcal{N}$. This can be seen from the fact that the exponential map in a product manifold $\mathcal{M}\times\mathcal{N}$ is the partitionning of the exponential mappings of $\mathcal{M}$ and $\mathcal{N}$. However, we give a direct proof below.

Let g_{ij} and Γ_{ij}^k denote respectively the coefficients of the first fundamental form $\langle\cdot,\cdot\rangle_{x,y}$ and the Christoffel symbols. By construction, $g_{ij}(x, y) = \delta_{ij}$. Also, it is easy to see that for all indexes i, j, k,

$$\Gamma_{ij}^k(x, y) = 0\ .$$

Indeed, if indices (i, j, k) correspond to the same component $\mathcal{M}$ or $\mathcal{N}$ this follows from the choice of normal coordinates in each component. Otherwise, say that i, j correspond to coordinates in $\mathcal{M}$ and k to coordinates in $\mathcal{N}$. Then $g_{ik} \equiv g_{jk} \equiv 0$ and furthermore,

$$\frac{\partial}{\partial u^k} g_{ij}(x, y) = 0.$$

Thus $\Gamma_{ikj}(x, y) = 0$ for all indexes i, j, k. This implies that $\Gamma_{ij}^k(x, y) = 0$ as well. Thus we have a normal system of coordinates around $(x, y) \in \mathcal{M}\times\mathcal{N}$.

In that system of coordinates,

$$D^2_{\mathcal{M}\times\mathcal{N}}\hat{\alpha}(x, y) = \begin{bmatrix} D^2_{\mathcal{N}}\alpha(x) & 0 \\ 0 & 0 \end{bmatrix}$$

From the block structure of the second covariant derivative above, it is clear that $D^2_{\mathcal{M}\times\mathcal{N}}\hat{\alpha}(x,y)$ is positive semi-definite if and only if $D^2_{\mathcal{N}}\alpha(x)$ is positive definite.

□

We have raised the question in the introduction of whether self-convexity of the condition number holds for the condition Riemann structure on the solution variety considered in [27]. The theorems proven in this paper apply to the case of linear systems, but with the use of Proposition 9 they give us some information on polynomial systems almost for free.

Let $\mathbf{d} = (d_1, \ldots, d_n)$. Consider the vector space

$$\mathcal{P}_{\mathbf{d},0} = \{(f_1, \ldots, f_n) : f_i \in \mathbb{C}[x_1, \ldots, x_n] \text{ with } \deg f_i = d_i \text{ and } f_i(0) = 0\}.$$

An important point is that self-convexity is well-defined for Riemannian manifolds. Therefore, if we want to speak of self-convexity in $\mathcal{P}_{\mathbf{d},0}$, we need to make it into an inner product vector space. We will follow [6] and assume the unitarily invariant metric in the space of degree d_i polynomials. This is the same as the metric for symmetric d_i-tensors. Then we define the product metric for $\mathcal{P}_{\mathbf{d}}$ and it is inherited by the subspace $\mathcal{P}_{\mathbf{d},0}$. In more precise terms: if $f_i(x) = \sum_{1\le|a|\le d_i} f_{ia} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $g_i(x) = \sum_{1\le|a|\le d_i} g_{ia} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ then we set

$$\langle f, g \rangle = \sum_{i=1}^{n} \sum_{1\le|a|\le d_i} \frac{f_{ia}\bar{g}_{ia}}{\binom{d_i}{a}}$$

with

$$\binom{d_i}{a} = \frac{d_i!}{a_1! a_2! \ldots a_n! (d_i - |a|)!}.$$

This vector space splits as $\mathcal{P}_{\mathbf{d},0} = \mathcal{L}_0 \oplus (\text{H.O.T.})_0$ where $\mathcal{L}_0$ are linear and $(\text{H.O.T.})_0$ are higher order polynomials vanishing at 0. Those two spaces are orthogonal. The inner product for linear polynomials is

$$\begin{aligned}\langle Ax, Bx \rangle = \sum_{i=1}^{n} \frac{1}{d_i} \sum_{j=1}^{n} A_{ij}\bar{B}_{ij} = \\ = \operatorname{tr}\left(\left(\begin{bmatrix} 1/\sqrt{d_1} & & \\ & \ddots & \\ & & 1/\sqrt{d_n} \end{bmatrix} B \right)^* \left(\begin{bmatrix} 1/\sqrt{d_1} & & \\ & \ddots & \\ & & 1/\sqrt{d_n} \end{bmatrix} A \right) \right).\end{aligned}$$

The unscaled[6, Prop.5 p.228], normalized[6, p.233] condition number is defined, for $f \in \mathcal{P}_{\mathbf{d},0}$, by

$$\mu(f,0) = \left\| Df(0)^{-1} \begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix} \right\|_2 = \sigma_n^{-1} \left(\begin{bmatrix} 1/\sqrt{d_1} & & \\ & \ddots & \\ & & 1/\sqrt{d_n} \end{bmatrix} Df(0) \right) .$$

The right-hand term is the (unscaled) condition number for $\mathcal{L}_0$. It coincides with the unscaled condition number for matrices, which is the topic of this paper.

Proposition 10. *μ is self-convex in its domain of definition $\mathcal{P}_{\mathbf{d},0} \setminus \Sigma$, where $\Sigma = \{\mathbf{f} \in \mathcal{P}_{\mathbf{d},0} : D\mathbf{f}(0)$ is degenerate$\}$.*

Proof. Immediate from Proposition 9 and Theorem 1. □

3.3 Computation of the Hessian

When analyzing the convexity properties of $\sigma_n(A)$, we first note that this function is invariant through unitary changes of coordinates, namely

$$\sigma_n(A) = \sigma_n(UAV^*)$$

for unitary matrices $U \in \mathbb{U}_n$, and $V \in \mathbb{U}_m$ (resp. orthogonal matrices $U \in \mathbb{O}_n$, and $V \in \mathbb{O}_m$). Let us consider this situation in a general framework.

A *Lie group* is a group that is also a smooth manifold, and such that the group operations (multiplication and inversion) are smooth. We say that a Lie group G acts (smoothly) on a manifold $\mathcal{M}$ if there is a smooth map $\ell : G \times \mathcal{M} \to \mathcal{M}$ with

$$\ell((g_1 g_2), p) = \ell(g_1, \ell(g_2, p)) \text{ and } \ell(1, p) = p \ .$$

In the example above, $G = \mathbb{U}_n \times \mathbb{U}_m$ acts on $\mathbb{GL}_{n,m}$ by $\ell((U,V),p) = UpV^*$. For simplicity, we may write $g(p)$ for $\ell(g,p)$ and assimilate g to the mapping $p \to g(p) = \ell(g,p)$.

An **isometry** of $\mathcal{M}$ is a diffeomorphism of $\mathcal{M}$ that preserves Riemannian distance. We say that the Lie group G acts by isometries when for all g, the corresponding map $g : p \to g(p)$ is an isometry of $\mathcal{M}$.

Definition 11. Let $\alpha : \mathcal{M} \to \mathbb{R}$. A **group of symmetries** of α is a Lie group, acting smoothly by isometries on $\mathcal{M}$, and leaving α invariant (that is, $\alpha(g(p)) = \alpha(p)$ for all $g \in G$ and $p \in \mathcal{M}$.

Let 1 be the unit of the group G. We will denote by $\mathfrak{g}$ the Lie algebra of G and by $\exp : \mathfrak{g} \simeq T_1 G \to G$ the exponential function (See, for instance, [20]). For instance, if $G = \mathbb{U}_n$, then 1 is the $n \times n$ identity matrix, and $\mathfrak{g}$ is the algebra of skew-Hermitian matrices. Moreover, $\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$.

Note that it may happen (for instance, if G is a discrete group) that $\mathfrak{g} = \{0\}$ and hence $T_p G_p = \{0\}$.

Given $p \in \mathcal{M}$, $G(p) = \{g(p) : g \in G\}$ will denote the G-orbit of p. The orbit $G(p)$ is a manifold [20, Cor 2.19]. If the group G is compact, the orbit is then an embedded submanifold of $\mathcal{M}$. In any case, $T_p G(p)$ will denote the tangent space of the orbit $G(p)$ at p, as a subspace of $T_p\mathcal{M}$. It can also be described as the set of all

$$\frac{d}{dt}(\exp(ta)(p)) \mid_{t=0},$$

for $a \in \mathfrak{g}$, the Lie algebra of G.

For instance, when $G = \mathbb{U}_n \times \mathbb{U}_m$, then $\mathfrak{g}$ is $A_n \times A_m$ (the skew-Hermitian matrices) and $\exp(ta)$ is the usual matrix exponential:

$$\exp(t(a_1, a_2))(p) = \left(I + ta_1 + \frac{t^2}{2}a_1^2 + \cdots\right) p \left(I + ta_2 + \frac{t^2}{2}a_2^2 + \cdots\right)^*$$

Theorem 12. *Let $\mathcal{M}$ be a smooth Riemannian manifold. Let $\alpha : \mathcal{M} \to \mathbb{R}$ be of class $\mathcal{C}^2$, and let G be a group of symmetries of α. Let $p \in \mathcal{M}$. Let $w = b + k \in T_p\mathcal{M}$ where $k \in T_p G(p)$, $b \perp T_p G(p)$. Let the vector field K be the infinitesimal generator associated with some element a in the Lie Algebra $\mathfrak{g}$ of G, where $k = \frac{d}{dt}(\exp(ta)(p)) \mid_{t=0}$. Namely,*

$$K(q) = \frac{d}{dt}(exp(ta)q) \mid_{t=0}, \quad q \in \mathcal{M}.$$

Let $\phi_t(q) = \phi(t, q)$ be the flow of grad α*, defined for $t \in (-\varepsilon, \varepsilon)$ and q close enough to p. Let B be a smooth vector field in $\mathcal{M}$ such that $B(\phi_t(p)) = D\phi_t(p)b$ where D denotes the usual derivative applied to the diffeormorphism $\phi_t : \mathcal{M} \to \mathcal{M}$. Then, the following equality holds:*

$$D^2\alpha(p)(w, w) =$$

$$D^2\alpha(p)(b,b) + \frac{1}{2}\langle \text{grad } (\|K\|^2)(p), \text{grad } \alpha(p)\rangle_p + \text{grad } \alpha(\langle B, K\rangle)(p).$$

Above, grad $\alpha(\langle B, K\rangle)(p) = \langle \text{grad } \alpha(p), \text{grad } (\langle B, K\rangle_p)\rangle_p$ is the directional derivative of $\langle B, K\rangle$ with respect to grad α.

Let us recall from (3.1) the intrinsic definition of the second covariant derivative or *Hessian*.

$$D^2\alpha(p)(v,w) = X(Y(\alpha))_p - (D_X Y)(\alpha)_p,$$

where X, Y are vector fields, $X(p) = v$, $Y(p) = w$, and D is the Levi-Civita connection. Also, $[X, Y]$ is the **Lie bracket** of two vector fields X and Y. It is defined for any α of class $\mathcal{C}^2$ by

$$[X, Y](\alpha) = X(Y(\alpha)) - Y(X(\alpha)).$$

It turns out that this is a first order differential operator, hence $[X, Y]$ is a vector field.

Another useful identity relating the Lie bracket and the Levi-Civita connection is:

$$[X, Y] = D_X Y - D_Y X \tag{3.3}$$

The proof of Theorem 12 is a consequence of the two following lemmas:

Lemma 13. *For any vector field X on M, we have*

$$2D^2\alpha(X, K) = \text{grad } \alpha(\langle X, K\rangle) - \langle [\text{grad } \alpha, X], K\rangle.$$

Moreover,

$$D^2\alpha(p)(k,k) = \frac{1}{2}\langle \text{grad } (\|K\|^2)(p), \text{grad } \alpha(p)\rangle_p, \tag{3.4}$$

Proof. We recall that for vector fields X, Y, Z,

$$2\langle D_X Y, Z\rangle = X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) + \tag{3.5}$$

$$\langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle - \langle [Y, Z], X\rangle.$$

Note that $K(p) = k$ and $K(q) \in T_q G(q)$ for $q \in \mathcal{M}$. As α is G-invariant,

$$K(\alpha) = \langle K, \text{grad } \alpha\rangle = 0. \tag{3.6}$$

Moreover, the one-parameter group generated by K consists of global isometries, thus K is a Killing vector field, which implies that for any pair of vector fields X, Y,

$$\langle D_Y K, X\rangle + \langle D_X K, Y\rangle = 0, \text{ or equivalently using (3.5)}$$

$$K(\langle Y, X\rangle) + \langle [Y, K], X\rangle + \langle [X, K], Y\rangle = 0. \tag{3.7}$$

We can now compute

$$2D^2\alpha(X, K) = 2X(K(\alpha)) - 2(D_X K)(\alpha) = -2\langle D_X K, \text{grad } \alpha\rangle =$$

$$-X(\langle K, \text{grad } \alpha\rangle) - K(\langle X, \text{grad } \alpha\rangle) + \text{grad } \alpha(\langle X, K\rangle)$$
$$-\langle [X, K], \text{grad } \alpha\rangle - \langle [\text{grad } \alpha, X], K\rangle + \langle [K, \text{grad } \alpha], X\rangle.$$

From (3.7) we know that

$$-K(\langle X, \text{grad } \alpha\rangle) - \langle [X, K], \text{grad } \alpha\rangle + \langle [K, \text{grad } \alpha], X\rangle = 0.$$

Using $\langle \text{grad } \alpha, K\rangle = 0$, we conclude

$$2D^2\alpha(X, K) = \text{grad } \alpha(\langle X, K\rangle) - \langle [\text{grad } \alpha, X], K\rangle,$$

which proves the first assertion.

When $X = K$, the second term above vanishes: using (3.3),

$$\begin{aligned}
\langle [K, \text{grad } \alpha], K\rangle &= \langle D_K \text{grad } \alpha, K\rangle - \langle D_{\text{grad } \alpha} K, K\rangle \\
&= \langle D_K \text{grad } \alpha, K\rangle + \langle \text{grad } \alpha, D_K K\rangle \\
&= K(\langle K, \text{grad } \alpha\rangle) \\
&= 0.
\end{aligned}$$

Equation (3.4) follows.

□

Lemma 14.

$$2D^2\alpha(p)(k, b) = \text{grad } \alpha(\langle B, K\rangle)(p).$$

Proof. By continuity of the formulas in the lemma, we can assume that $k \neq 0$ and that $b, \text{grad } \alpha(p)$ are lineary independent. Let N_0 be a codimension 2 submanifold of $\mathcal{M}$ with p in its interior. Assume that $b \in T_p\mathrm{N}_0$, k is orthogonal to $T_p\mathrm{N}_0$, and $\text{grad } \alpha(p) \notin T_pN_0$.

Let $\mathrm{N} = \cup\phi_t(\mathrm{N}_0)$ with ϕ_t the flow associated with grad α and where the union is taken in a small interval around $t = 0$. N is a codimension 1 submanifold. For small ε, the integral curve of grad α is thus contained in N, and for $q = \phi_t(p)$, we have $B(q) = D\phi_t(p)b \in T_q\mathrm{N}$. Both grad α and B are tangent to N by construction. By Frobenius Theorem, $[B, \text{grad } \alpha]$ is again tangent to N. In particular, $[\text{grad } \alpha, B](p) \in T_p\mathrm{N}$, and hence $\langle[\text{grad } \alpha, B], K\rangle(p) = 0$. From Lemma 13,

$$2D^2\alpha(B, K) = \text{grad } \alpha(\langle B, K\rangle) - \langle[\text{grad } \alpha, B], K\rangle = \text{grad } \alpha(\langle B, K\rangle)$$

at p as wanted. □

Proof of Theorem 12. The second covariant derivative is a symmetric bilinear form. Thus,

$$D^2\alpha(p)(v, v) = D^2\alpha(p)(b, b) + D^2\alpha(p)(k, k) + 2D^2\alpha(p)(b, k).$$

Theorem 12 follows from lemmas 13 and 14. □

Corollary 15. *Assume that for every $p \in \mathcal{M}$:*

- $D^2_\kappa \log(\alpha)(p)$ *is positive semi-definite in* $(T_pG(p))^\perp$,
- *For* $b \in T_p\mathcal{M}$, $b \perp T_pG(p)$, *we have that* $D\phi_t(p)b \perp T_{\phi_t(p)}G(\phi_t(p))$. *Here,* $\phi_t(q) = \phi(t, q)$ *is the flow of* grad α, *defined for* $t \in (-\varepsilon, \varepsilon)$ *and* q *close enough to* p.
- *For every* $a \in \mathfrak{g}$, *the associated vector field* $K(q) = \frac{d}{dt}(exp(ta)q)\mid_{t=0}$, $q \in M$, *satisfies*

$$\alpha D(\|K\|^2)(\text{grad } \alpha) + \|K\|^2\|\text{grad } \alpha\|^2 \geq 0.$$

Then, α *is self-convex in* $\mathcal{M}$.

Proof. α is self-convex if and only if $D^2_\kappa \log(\alpha)$ is positive semi-definite. Now, let $v = b + k \in \mathcal{M}$. According to Theorem 12,

$$D^2_\kappa \log(\alpha)(p)(v,v) = D^2_\kappa \log(\alpha)(p)(b,b) +$$

$$\frac{1}{2}\langle \text{grad}_\kappa((\|K\|_\kappa)^2)(p), \text{grad}_\kappa \log(\alpha)(p)\rangle_{\kappa,p} + \text{grad}_{\kappa,p}\alpha(\langle B,K\rangle_\kappa)(p),$$

where K is as defined in Theorem 12 and B is a vector field such that $B(\phi_t(p)) = D\phi_t(p)b$. Note that $\text{grad}_\kappa \alpha(\langle B,K\rangle_\kappa)$ depends only on the value of B, and K along the integral curve $\phi_t(p)$. Moreover,

$$\langle B,K\rangle_\kappa(\phi_t(p)) = \alpha(\phi_t(p))\langle B(\phi_t(p)), K(\phi_t(p))\rangle =$$

$$\alpha(\phi_t(p))\langle D\phi_t(p)(b), K(\phi_t(p))\rangle = 0,$$

from the second item in the hypotheses of our corollary. Thus, we have

$$D^2_\kappa \log(\alpha)(p)(v,v) =$$

$$D^2_\kappa \log(\alpha)(p)(b,b) + \frac{1}{2}\langle \text{grad}_\kappa((\|K\|_\kappa)^2)(p), \text{grad}_\kappa \log(\alpha)(p)\rangle_{\kappa,p}.$$

This quantity has to be non-negative for every v or, equivalently,

- $D^2_\kappa \log\alpha(p)$ has to be positive semi-definite in $(T_pG(p))^\perp$, and
- $\langle \text{grad}_\kappa((\|K\|_\kappa)^2)(p), \text{grad}_\kappa \log(\alpha)(p)\rangle_{\kappa,p} \geq 0$ for every vector field K, $K(q) = \frac{d}{dt}(exp(ta)q)\|_{t=0}$ where $a \in \mathfrak{g}$.

The second of these two items can be re-written using the original Riemannian structure $\langle\cdot,\cdot\rangle$. Note that

$$(\|K\|_\kappa)^2 = \alpha\|K\|^2,$$

$$\text{grad}_\kappa((\|K\|_\kappa)^2) = \frac{1}{\alpha}\text{grad }(\alpha\|K\|^2) = \text{grad }\|K\|^2 + \|K\|^2\frac{\text{grad }\alpha}{\alpha},$$

$$\text{grad}_\kappa \log(\alpha) = \frac{1}{\alpha}\text{grad }(\log\alpha) = \frac{\text{grad }\alpha}{\alpha^2}.$$

Thus,

$$\begin{aligned}\langle \text{grad }_\kappa((\|K\|_\kappa)^2), \text{grad}_\kappa \log(\alpha)\rangle_\kappa &= \langle \text{grad }\|K\|^2 + \|K\|^2\frac{\text{grad }\alpha}{\alpha}, \frac{\text{grad }\alpha}{\alpha^2}\rangle \\ &= \frac{1}{\alpha^3}\left(\alpha D(\|K\|^2)(\text{grad }\alpha) + \|K\|^2\|\text{grad }\alpha\|^2\right).\end{aligned}$$

The corollary follows. □

4 Self-convexity in spaces of matrices

Let $u \leq n$ and $(k) = (k_1, \ldots, k_u) \in \mathbb{N}^u$ such that $k_1 + \cdots + k_u = n$. We define $\mathcal{P}_{(k)}$ as the set of matrices $A \in \mathbb{GL}_{n,m}$ with u distinct singular values

$$\sigma_1(A) > \cdots > \sigma_u(A) > 0,$$

$\sigma_i(A)$ having the multiplicity k_i. Such a matrix has a singular value decomposition $A = UDV^*$ with $U \in \mathbb{U}_n$, $V \in \mathbb{U}_m$ and $D \in \mathbb{GL}_{n,m}$ with

$$D = \text{diag}\ \left(\overbrace{\sigma_1, \ldots, \sigma_1}^{k_1}, \ldots, \overbrace{\sigma_u, \ldots, \sigma_u}^{k_u} \right) = \text{diag}\ (\sigma_1 I_{k_1}, \ldots, \sigma_u I_{k_u}).$$

Above, $\mathbb{U}_n$ is the group of unitary $n \times n$ matrices. If $\mathbb{K} = \mathbb{R}$, it should be replaced by the group of orthogonal $n \times n$ matrices.

We also let

$$\mathcal{D}_{(k)} = \left\{ D \in \mathcal{P}_{(k)}\ :\ D = \text{diag}\ (\sigma_1 I_{k_1}, \ldots, \sigma_u I_{k_u}),\ \sigma_1 > \cdots > \sigma_u \right\}.$$

Notice that the singular values $\sigma_1 > \cdots > \sigma_u$ can vary within each $\mathcal{P}_{(k)}$ or each $\mathcal{D}(k)$.

Proposition 16. *$\mathcal{P}_{(k)}$ is a real smooth embedded submanifold of $\mathbb{GL}_{n,m}$. Its real codimension is*

- $k_1^2 + \cdots + k_u^2 - u$ *if* $\mathbb{K} = \mathbb{C}$.
- $\frac{1}{2}(n + k_1^2 + \cdots + k_u^2) - u$ *if* $\mathbb{K} = \mathbb{R}$.

The tangent space to $\mathcal{P}_{(k)}$ at a matrix

$$D = \begin{pmatrix} \sigma_1 I_{k_1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_u I_{k_u} & 0 & \cdots & 0 \end{pmatrix}$$

is the set of matrices

$$\begin{pmatrix} \lambda_1 I_{k_1} + A_1 & * & * & * & \cdots & * \\ * & \ddots & * & * & \cdots & * \\ * & * & \lambda_u I_{k_u} + A_u & * & \cdots & * \end{pmatrix}$$

where $A_1, \ldots, A_u$ are skew-symmetric matrices of respective sizes $k_1, \ldots, k_u$, $\lambda_1, \ldots, \lambda_u \in \mathbb{R}$, and the other entries are complex numbers (real, if $\mathbb{K} = \mathbb{R}$). Moreover, for any $i = 1, \ldots, u$, $\sigma_i : \mathcal{P}_{(k)} \to \mathbb{R}$ is a smooth function.

Proof. To prove that $\mathcal{P}_{(k)}$ is a real smooth embedded submanifold of $\mathbb{GL}_{n,m}$ we use Lemma 33 (see the appendix). We take $G = \mathbb{U}_n \times \mathbb{U}_m$, $\mathcal{M} = \mathbb{GL}_{n,m}$, and $\mathcal{D} = \mathcal{D}_{(k)}$. The group action of G on $\mathcal{M}$ is given by

$$(U, V, X) \in G \times \mathbb{GL}_{n,m} \to UXV^* \in \mathbb{GL}_{n,m}.$$

Under this action, the image of $\mathcal{D}_{(k)}$ is $\mathcal{P}_{(k)}$. Define the equivalence relation $\mathcal{R}$ in $\mathbb{U}_n \times \mathbb{U}_m \times \mathcal{D}_{(k)}$ by

$$(U, V, D)\mathcal{R}(U', V', D') \text{ if and only if } UDV^* = U'D'V'^*.$$

Since D is diagonal this is equivalent to

$$D' = D, \quad U' = UM, \quad V' = VM_W,$$

where M and M_W are unitary block-diagonal matrices

$$M = \text{diag }(U_1, \ldots, U_u), \quad M_W = \text{diag }(U_1, \ldots, U_u, W),$$

with $U_i \in \mathbb{U}_{k_i}$, and $W \in \mathbb{U}_{m-n}$. Note that the set $\mathcal{I}_{(k)}$ of such pairs (M, M_W) is the isotropy group of any $D \in \mathcal{D}_{(k)}$. Also, the relation $\mathcal{R}$ is invariant under left $\mathbb{U}_n \times \mathbb{U}_m$ action, namely:

$$(U, V, D)\mathcal{R}(U', V', D') \Leftrightarrow (QU, RV, D)\mathcal{R}(QU', RV', D')$$

for any $(Q, R) \in \mathbb{U}_n \times \mathbb{U}_m$.

It is easy to see that the graph of this equivalence relation, that is the set of pairs $((U, V, D), (UM, VM_W, D))$, with U, V, D, M, and W as before, is a closed submanifold in $(G \times \mathcal{D}_{(k)}) \times (G \times \mathcal{D}_{(k)})$. Indeed, this graph is the image of the diffeomorphic embedding

$$\begin{array}{ccc} G \times \mathcal{D}_{(k)} \times \mathbb{U}_{(k)} \times (U_{(k)} \times \mathbb{U}_{m-n}) & \to & (G \times \mathcal{D}_{(k)}) \times (G \times \mathcal{D}_{(k)}) \\ ((U, V), D, M, M_W) & \mapsto & ((U, V, D), (UM, VM_W, D)) \end{array}$$

($\mathbb{U}_{(k)} = \mathbb{U}_{k_1} \otimes \cdots \otimes \mathbb{U}_{k_u}$ are the unitary block-diagonal matrices).

Thus the quotient space $(G \times \mathcal{D}_{(k)})/\mathcal{R}$ is equipped with a unique manifold structure making π (the canonical surjection) a submersion.

Let us define

$$i : (G \times \mathcal{D}_{(k)})/\mathcal{R} \to \mathbb{GL}_{n,m}, \quad i(\pi(U, V, D)) = UDV^*.$$

The injectivity of i follows by construction of $\mathcal{R}$: elements of $(G\times\mathcal{D}_{(k)})/\mathcal{R}$ are represented non-uniquely by elements $(U,V,D)\in(G\times\mathcal{D}_{(k)})$. Two of those elements (say (U,V,D) and (U',V',D')) represent the same equivalence class if and only if $UDV^* = U'D'V'^*$.

We still have to check that this map is an immersion. For any $(\dot U,\dot V,\dot D)$ in the tangent space $T_{(U,V,D)}G\times\mathcal{D}_{(k)}$ we have

$$D(i\circ\pi)(U,V,D)(\dot U,\dot V,\dot D) = \dot U D V^* + U\dot D V^* + UD\dot V^* = U(AD+\dot D-DB)V^*$$

with $\dot U = UA$, $\dot V = VB$, A and B skew-symmetric matrices of respective size n and m. When $AD+\dot D-DB=0$, we obtain, via an easy computation,

$$\dot D = 0,\quad A = \mathrm{diag}\,(A_1,\ldots,A_u),\quad B = \mathrm{diag}\,(A_1,\ldots,A_u,C),$$

where A_i and C are skew-symmetric matrices of respective sizes k_i and $m-n$. Thus $(\dot U,\dot V,\dot D)=(UA,VB,0)$ is tangent to the fiber of π in $G\times\mathcal{D}_{(k)}$ above $\pi(U,V,D)$ so that $D\pi(U,V,D)(\dot U,\dot V,\dot D)=0$. In other words

$$Di(\pi(U,V,D))(D\pi(U,V,D)(\dot U,\dot V,\dot D)) = 0 \Longrightarrow D\pi(U,V,D)(\dot U,\dot V,\dot D)=0$$

that is $Di(\pi(U,V,D))$ is injective.

The last point to check to apply Lemma 33 is the continuity of the inverse of i. Suppose that $X_p\to X$ with $X_p,X\in\mathrm{Im}\; i=\mathcal{P}_{(k)}$. We can write them $X_p = U_pD_pV_p^*$ and $X=UDV^*$. Let (U_{p_q},V_{p_q}) be a subsequence which converges to $(\tilde U,\tilde V)$ (G is compact). Since $X_{p_q}\to X$ we have $D_{p_q}\to\tilde U^*X\tilde V=\tilde D$, and $\tilde U\tilde D\tilde V^* = UDV^*$. Now we consider the sequence $\tilde U^*X_p\tilde V$. It is a convergent sequence, hence it has a unique limit $\tilde D$ and $(\tilde U,\tilde V,\tilde D)\mathcal{R}(U,V,D)$. Thus, $\pi(\tilde U^*U_p,\tilde V^*V_p,D_p)$ converges to $\pi(I,I,D)$. By left $\mathbb{U}_n\times U_m$ action, we conclude that $\pi(U_p,V_p,D_p)$ converges to $\pi(U,V,D)$ as required.

Thus, the hypothesis of Lemma 33 is satisfied and $\mathcal{P}_{(k)}$ is a real smooth embedded submanifold of $\mathbb{GL}_{n,m}$.

The computation of its dimension is easy: it is given by the difference of the dimension of $G\times\mathcal{D}_{(k)}$ and the dimension of the fiber above any point in the quotient space, that is

$$\dim\mathbb{U}_n+\dim\mathbb{U}_m+u-\dim\mathbb{U}_{k_1}-\ldots-\dim\mathbb{U}_{k_u}-\dim\mathbb{U}_{m-n}.$$

The tangent space $T_D\mathcal{P}_{(k)}$, $D=\mathrm{diag}\,(\sigma_1 I_{k_1},\ldots,\sigma_u I_{k_u})$, is the image of the tangent space $T_{(I_n,I_m,D)}G\times\mathcal{D}_{(k)}$ by the derivative $D(i\circ\pi)(I_n,I_m,D)$. It is the set of matrices $AD + \dot{D} - DB$ with $\dot{D} = \text{diag}\,(\lambda_1 I_{k_1}, \ldots, \lambda_u I_{k_u})$, A and B skew symmetric of sizes n and m. They all have the type described in Proposition 16 and this space of matrices has the right dimension.

Let us prove the smoothness of the map $X \in \mathcal{P}_{(k)} \to \sigma_i(X) \in \mathbb{R}$. Since the map $(U, V, D) \in G \times \mathcal{D}_{(k)} \to \sigma_i(D)$ is smooth, and constant in the equivalence classes, the map $\pi(U, V, D) \in (G \times \mathcal{D}_{(k)})/\mathcal{R} \to \sigma_i(D) = \sigma_i(UDV^*)$ is also smooth. Thus the map $X = UDV^* \in \mathcal{P}_{(k)} \to \sigma_i(X)$ is smooth as the composition of the previous map by i^{-1}. □

Lemma 17. *Let I be an open interval. Let $(\gamma(t))_{t \in I}$ be a smooth path in $\mathcal{P}_{(k)}$. Then, there are smooth paths $U(t) \in \mathbb{U}_n$, $V(t) \in U_m$ and $\Sigma(t) \in \mathcal{D}_{(k)}$ so that*

$$\gamma(t) = U(t)\Sigma(t)V(t)^* \tag{4.1}$$

for all $t \in I$.

We give two quite different proofs of this result.

Proof. Consider the mapping $\pi : \mathbb{U}_n \times \mathcal{D}_{(k)} \times \mathbb{U}_m \to \mathcal{P}_{(k)}$ sending (U, D, V) to UDV^*. Note that π is surjective. We claim that it is also a submersion: by unitary invariance, we may assume that $U = I_n$, $V = I_m$. Then, for skew-symmetric matrices A, B of respective sizes n, m, we have:

$$D\pi(I, D, V)(A, \dot{D}, B) = AD + \dot{D} + DB.$$

It is a simple exercise to check that one can get any matrix in the tangent space $T_D\mathcal{P}_{(k)}$, computed in Proposition 16, by choosing appropriate A, B. Thus, $D\pi(I, D, I)$ is surjective, and π is a submersion.

Finally, we claim that π is also a proper map (i.e. the preimage of a compact set is a compact set): let $K \subseteq \mathcal{P}_{(k)}$ be a compact subset. The mapping sending a matrix to its (ordered) singular values is continuous, and hence the set of singular values of matrices in K is the continuous image of a compact set, thus a compact set, call it $K' \subseteq \mathcal{D}_{(k)}$. Thus, $\pi^{-1}(K)$ is a closed (because π is continuous) subset of the compact set $\mathbb{U}_n \times K' \times \mathbb{U}_m$, thus a compact set. This proves that π is proper.

A theorem by Ehresmann [15] (see [25, Th. 5.1] for a general version on a more modern framework) says that, under these hypotheses, π is actually a locally trivial fibration which implies that it defines a fiberbundle. Hence, π has the homotopy lifting property and in particular any path in $\mathcal{P}_{(k)}$ can be smoothly lifted to a path in $\mathcal{U}_n \times \mathcal{D}_{(k)} \times \mathcal{U}_m$ as wanted.

□

As an alternative, we have:

Proof. We will show that $U(t)$, $V(t)$ and $\Sigma(t)$ are solutions of a certain differential equation on the manifold $\mathbb{U}_n \times \mathbb{U}_m \times \mathcal{D}_{(k)}$. An important fact to be used below is that $T_I\mathbb{U}_n$ is the space of skew-hermitian matrices. In the real case, $T_I\mathbb{O}_n$ is the space of skew-symmetric matrices. Let us assume for a while that (4.1) admits a solution. Differentiating (4.1) with respect to t, we obtain after a few trivial manipulations that

$$U(t)^*\dot{\gamma}(t)V(t) = U(t)^*\dot{U}(t)\Sigma(t) - \Sigma(t)V(t)^*\dot{V}(t) + \dot{\Sigma}(t).$$

For shortness, let $M(t) = U(t)^*\dot{\gamma}(t)V(t)$, $A(t) = U(t)^*\dot{U}(t) \in T_I\mathbb{U}_n$ and $B(t) = V(t)^*\dot{V}(t) \in T_I\mathbb{U}_m$. We have now:

$$M(t) = A(t)\Sigma(t) - \Sigma(t)B(t) + \dot{\Sigma}(t).$$

Using block notation, we obtain for $i < j$ that

$$M_{ij}(t) = \sigma_j(t)A_{ij}(t) - \sigma_i(t)B_{ij}(t).$$

The equation for block $M_{ji}(t)$ reads:

$$M_{ji}(t) = \sigma_i(t)A_{ji}(t) - \sigma_j(t)B_{ji}(t).$$

Transposing,

$$M_{ji}(t)^* = -\sigma_i(t)A_{ij}(t) + \sigma_j(t)B_{ij}(t).$$

We obtain therefore

$$\left\{ \begin{array}{lcl} A_{ij} & = & \frac{1}{\sigma_j^2-\sigma_i^2}\left(\sigma_j M_{ij}(t) + \sigma_i M_{ji}(t)^*\right) \\ B_{ij} & = & \frac{1}{\sigma_j^2-\sigma_i^2}\left(\sigma_i M_{ij}(t) + \sigma_j M_{ji}(t)^*\right) \end{array} \right. \tag{4.2}$$

The blocks in the diagonal (that is, $i = j$) are of the form

$$M_{ii}(t) = \sigma_i(A_{ii} - B_{ii}) + \dot{\sigma}_i I_{k_i},$$

hence we can solve by setting

$$A_{ii} = -B_{ii} = \frac{1}{2\sigma_i}(M_{ii}(t) - \dot{\sigma}_i I_{k_i}). \tag{4.3}$$

Equations (4.2)-(4.3) are a system of smooth non-autonomous ordinary differential equations in variables $U \in \mathbb{U}_n$, $V \in \mathbb{U}_m$ and $\Sigma \in \mathcal{D}_{(k)}$. The Lipschitz condition holds. Hence, for every $t_0 \in I$, there are $\epsilon > 0$ and local solutions $U(t)$, $V(t)$ and $\Sigma(t)$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$, solving (4.1).

In order to show the existence of a global solution on all the interval, we need to check that as $t \to t_0 + \epsilon$, the solution converges to a limit in $\mathbb{U}_n \times \mathbb{U}_m \times \mathcal{D}_{(k)}$. The convergence of $U(t)$ and $V(t)$ follows from compactness of the unitary group. Because $\gamma(t_0 + \epsilon) \in \mathcal{P}_{(k)}$,

$$\lim_{t \to t_0 + \epsilon} \Sigma(t) \in \mathcal{D}_{(k)}.$$

Hence, the solution $(U(t), V(t), \Sigma(t))$ can be extended to an interval that is open and closed in I, hence to all I. □

Let $\alpha : \mathbb{GL}_{n,m}$ be defined by $\alpha(A) = \sigma_n(A)^{-2}$. We also denote by $\alpha = \sigma_u^{-2}$ its restriction to $\mathcal{P}_{(k)}$ or to $\mathcal{D}_{(k)}$. We first consider the case of diagonal matrices, then we prove self-convexity of α in $\mathcal{P}_{(k)}$.

Lemma 18. *Let $\mathcal{P}_{(k)}$ be equipped with the condition metric structure*

$$\langle \cdot, \cdot \rangle_\kappa = \sigma_u^{-2} \mathrm{Re} \langle \cdot, \cdot \rangle_F .$$

1. *If $\Sigma_1, \Sigma_2 \in \mathcal{D}_{(k)}$, then any minimizing condition geodesic in $\mathcal{P}_{(k)}$ joining Σ_1 and Σ_2 lies in $\mathcal{D}_{(k)}$,*

2. *The set $\mathcal{D}_{(k)}$ is a totally geodesic submanifold of $\mathcal{P}_{(k)}$ for the condition metric, namely, every geodesic in $\mathcal{D}_{(k)}$ for the induced structure is also a geodesic in $\mathcal{P}_{(k)}$, or equivalently:*

3. *If $\Sigma \in \mathcal{D}_{(k)}$ and $\dot{\Sigma} \in T_\Sigma \mathcal{D}_{(k)}$, then the unique geodesic in $\mathcal{P}_{(k)}$ through Σ with tangent vector $\dot{\Sigma}$ at Σ, remains in $\mathcal{D}_{(k)}$.*

Moreover, $\alpha = \sigma_u^{-2}$ is log-convex in $\mathcal{D}_{(k)}$.

Proof. According to Proposition 16, $\mathcal{P}_{(k)}$ is a smooth Riemannian manifold for the condition structure.

Let $\gamma(t)$, $0 \le t \le T$, be a minimizing condition geodesic with endpoints Σ_1 and $\Sigma_2 \in \mathcal{D}_{(k)}$. Let $\gamma(t) = U_t \Sigma_t V_t^*$ be a singular value decomposition of $\gamma(t)$, choosen as in Lemma 17. Let $\sigma_u(t)$ be the smallest singular value of $\gamma(t)$. It suffices to see that $L_\kappa(\Sigma) \le L_\kappa(\gamma)$ that is

$$\int_0^T \|\dot{\Sigma}_t\|_F \sigma_u(t)^{-1} dt \le \int_0^T \|\dot{\gamma}_t\|_F \sigma_u(t)^{-1} dt.$$

Since

$$\dot{\gamma}_t = \dot{U}_t \Sigma_t V_t^* + U_t \dot{\Sigma}_t V_t^* + U_t \Sigma_t \dot{V}_t^*,$$

with $\dot{U}_t = U_t A_t$, $\dot{V}_t = V_t B_t$, A_t and B_t skew-symmetric, we see that

$$\|\dot{\gamma}_t\|_F^2 = \|A_t\Sigma_t + \dot{\Sigma}_t - \Sigma_t B_t\|_F^2 = \|\dot{\Sigma}_t\|_F^2 + \|A_t\Sigma_t - \Sigma_t B_t\|_F^2 \geq \|\dot{\Sigma}_t\|_F^2$$

because the diagonal terms in $\dot{\Sigma}_t$ are real numbers and those of $A_t\Sigma_t - \Sigma_t B_t$ are purely imaginary when $\mathbb{K} = \mathbb{C}$ and vanish when $\mathbb{K} = \mathbb{R}$. When γ_t does not belong to $\mathcal{D}_{(k)}$, then the inequality above is strict.

The second assertion is an easy consequence of the first one. The third assertion is another classical characterization of totally geodesic submanifolds, see [23] Chapter 4, Proposition 13 or Theorem 5.

Finally, for log-convexity of $\alpha(X) = \sigma_u(X)^{-2}$, using [2] Proposition 3, it suffices to see that for $\Sigma \in \mathcal{D}_{(k)}$ and $\dot{\Sigma} \in T_\Sigma \mathcal{D}_{(k)}$,

$$2\|\dot{\Sigma}\|^2 \|D\sigma_u(\Sigma)\|^2 \geq D^2\sigma_u^2(\Sigma)(\dot{\Sigma}, \dot{\Sigma}), \tag{4.4}$$

where the second derivative is computed in the Frobenius metric structure. Now,

$$D(\sigma_u)(\Sigma)(\dot{\Sigma}) = \sigma_u(\dot{\Sigma}).$$

Thus

$$\left\| D(\sigma_u)(\Sigma)(\dot{\Sigma}) \right\|^2$$

is maximized for the 'unit vector' (in block representation)

$$\dot{\Sigma} = \frac{1}{\sqrt{k_u}} \begin{bmatrix} 0_{k_1} & & & \\ & \cdots & & \\ & & 0_{k_{u-1}} & \\ & & & I_{k_u} \end{bmatrix}.$$

We deduce that

$$\|D\sigma_u(\Sigma)\|^2 = \frac{1}{k_u},$$

hence

$$2\|\dot{\Sigma}\|^2 \|D\sigma_u(\Sigma)\|^2 = \frac{2\|\dot{\Sigma}\|^2}{k_u} \geq 2\sigma_u(\Sigma).$$

The right-hand-side of (4.4) is precisely

$$D^2\sigma_u^2(\Sigma)(\dot{\Sigma}, \dot{\Sigma}) = D(2\sigma_u(\Sigma)\sigma_u(\dot{\Sigma}))(\dot{\Sigma}) = 2\sigma_u(\dot{\Sigma})^2,$$

and equation (4.4) follows. □

Proposition 19. *The map $\alpha = \sigma_u^{-2}$ is self-convex in $\mathcal{P}_{(k)}$.*

Proof. By unitary invariance, we may choose as initial point a matrix $\Sigma \in \mathcal{D}_{(k)}$ with ordered distinct diagonal entries $\sigma_1 > \ldots > \sigma_u > 0$. We use Corollary 15, with the group G being $\mathbb{U}_n \times \mathbb{U}_m$ and the action

$$\begin{array}{ccc} \mathbb{U}_n \times \mathbb{U}_m \times \mathcal{P}_{(k)} & \longrightarrow & \mathcal{P}_{(k)} \\ ((U,V),A) & \mapsto & UAV^*. \end{array}$$

The Lie algebra of G is the set $\mathcal{A}_n \times \mathcal{A}_m$ where $\mathcal{A}_k$ is the set of $k \times k$ skew-symmetric matrices.

We write $G(L)$ for the G-orbit of a point $L \in \mathcal{P}_{(k)}$. In our case, this is the manifold of all ULV^* with $U \in \mathbb{U}_n$, $V \in \mathbb{U}_m$. The tangent space to the Lie group action at L is the tangent manifold $T_L G(L) \subseteq T_L \mathcal{P}_{(k)}$.

First, we note that for any $L \in \mathcal{D}_{(k)}$, we have

$$(T_L G(L))^\perp = T_L \left\{ULV^* : U \in \mathbb{U}_n, V \in \mathbb{U}_m\right\}^\perp =$$
$$\left\{B_1 L + L B_2^* : (B_1, B_2) \in \mathcal{A}_n \times \mathcal{A}_m\right\}^\perp .$$

Let us denote by S this last set. We claim that $S = \mathcal{D}_{(k)}$. Indeed, $\mathcal{D}_{(k)} \subseteq S$, because the diagonal of any matrix of the form $B_1 L + L B_2^*$ is purely imaginary and hence orthogonal to $\mathcal{D}_{(k)}$. The other inclusion is easily checked by a dimensional argument: The dimension of $\mathcal{D}_{(k)}$ is u and the dimension of S is

$$\dim(\mathcal{P}_{(k)}) - \dim \left\{B_1 L + L B_2^* : (B_1, B_2) \in \mathcal{A}_n \times \mathcal{A}_m\right\},$$

that is $\dim(\mathcal{P}_{(k)})$ minus the dimension of the orbit of L under the action of $\mathbb{U}_n \times \mathbb{U}_m$. We have computed these two quantities in Proposition 16, and we immediately conclude that $\dim(S) = u$, for both $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$. Thus, for all $L \in \mathcal{D}_{(k)}$,

$$(T_L G(L))^\perp = \mathcal{D}_{(k)}.$$

We now check the three conditions of Corollary 15.

- $D^2_\kappa \log(\alpha)(\Sigma)$ is positive semi-definite in $(T_\Sigma G(\Sigma))^\perp$: let

  $$\dot{\Sigma} \in (T_\Sigma G(\Sigma))^\perp = T_\Sigma \mathcal{D}_{(k)},$$

  and let γ be a condition geodesic in $\mathcal{D}_{(k)}$ such that $\gamma(0) = \Sigma$, $\dot{\gamma}(0) = \dot{\Sigma}$. We have to check that

  $$\frac{d^2}{dt^2} \log \alpha(\gamma(t)) \mid_{t=0} \geq 0.$$

  This is true as α is log-convex in $\mathcal{D}_{(k)}$ from Lemma 18.

- We have to check that for small enough t, and for
$$\dot{\Sigma} \in (T_\Sigma G(\Sigma))^\perp = T_\Sigma \mathcal{D}_{(k)},$$
$D\phi_t(D)\dot{\Sigma}$ belongs to
$$T_{\phi_t(\Sigma)} G(\Sigma)^\perp = T_\Sigma \mathcal{D}_{(k)},$$
where ϕ_t is the flow of $\mathrm{grad}_\kappa \alpha$. In our case, ϕ_t can be computed exactly. Indeed,
$$\mathrm{grad}_\kappa \alpha = \frac{1}{\alpha} \mathrm{grad}\ \alpha = -\frac{2}{k_u \sigma_u} E,$$
where
$$E = \mathrm{diag}\ (0, \ldots, 0, \overbrace{1, \ldots, 1}^{k_u}).$$
Thus, grad α preserves the diagonal form, and $\phi_t(\Sigma) \in \mathcal{D}_{(k)}$ is a diagonal matrix, for every t while defined. Thus, $D\phi_t(\Sigma)(\dot{\Sigma})$ is again a diagonal matrix, for every diagonal matrix $\dot{\Sigma}$. This proves that the second condition of Corollary 15 applies to our case.

- For $(B_1, B_2) \in \mathcal{A}_n \times \mathcal{A}_m$, the vector field K on $\mathbb{GL}_{n,m}$ generated by (B_1, B_2) is
$$K(A) = \frac{d}{dt} \left(e^{tB_1} A e^{tB_2^*} \right) |_{t=0} = B_1 A + A B_2^*.$$
Note that
  1. K^* as a linear operator on $\mathbb{GL}_{n,m}$ satisfies $K^*(A) = B_1^* A + A B_2$.
  2. $\|K(\Sigma)\|^2 = \|B_1 \Sigma + \Sigma B_2^*\|^2$,
  3. For $w \in T_\Sigma \mathcal{P}_{(k)}$, $D(\|K\|^2)(\Sigma) w = 2\mathrm{Re}\langle K^* K(\Sigma), w\rangle = 2\mathrm{Re}\langle B_1 B_1^* \Sigma + \Sigma B_2 B_2^* - 2 B_1 \Sigma B_2^*, w\rangle$.
  4. grad $\alpha(\Sigma) = -\frac{2}{k_u \sigma_u^3} E$ where $E = \mathrm{diag}\ (0, \ldots, 0, \overbrace{1, \ldots, 1}^{k_u})$.

Thus,
$$\alpha(\Sigma) D(\|K\|^2)(\Sigma)(\mathrm{grad}\ \alpha(\Sigma)) + \|K(\Sigma)\|^2 \|\mathrm{grad}\ \alpha(\Sigma)\|^2 =$$

$$\frac{4}{k_u\sigma_u^6}\left(-\sigma_u \mathrm{Re}\langle B_1B_1^*\Sigma+\Sigma B_2B_2^*-2B_1\Sigma B_2^*, E\rangle+\|B_1\Sigma-\Sigma B_2\|^2\right).$$

Hence, it suffices to see that $J\geq 0$ where

$$J=\|B_1\Sigma-\Sigma B_2\|^2-\sigma_u\mathrm{Re}\langle B_1B_1^*\Sigma+\Sigma B_2B_2^*-2B_1\Sigma B_2^*, E\rangle.$$

Expanding this expression and writing $\Sigma'=\Sigma^*-\sigma_uE^*$, we have

$$J=\mathrm{Re}\left(\text{trace }(B_1B_1^*\Sigma\Sigma')+\text{trace }(\Sigma'\Sigma B_2B_2^*)-2\text{trace }(B_1\Sigma B_2^*\Sigma')\right),$$

which by Lemma 20 below is a non-negative quantity. The proposition follows.

□

Lemma 20. *Let $\Sigma=$ diag $(\sigma_1I_{k_1},\ldots,\sigma_{u-1}I_{k_{u-1}},\sigma_uI_{k_u})\in\mathbb{GL}_{n,m}$ and $\Sigma'=$ diag $(\sigma_1I_{k_1},\ldots,\sigma_{u-1}I_{k_{u-1}},0I_{k_u})\in\mathbb{GL}_{m,n}$. Then, for any skew-symmetric matrices B,C of respective sizes n,m, we have:*

$$\mathrm{Re}\left(\text{trace }(BB^*\Sigma\Sigma')+\text{trace }(\Sigma'\Sigma CC^*)-2\text{ trace }(B\Sigma C^*\Sigma')\right)\geq 0.$$

Proof. We denote

$$J=\mathrm{Re}\left(\text{trace }(BB^*\Sigma\Sigma')+\text{trace }(\Sigma'\Sigma CC^*)-2\text{ trace }(B\Sigma C^*\Sigma')\right).$$

Write

$$\Sigma=\begin{pmatrix}L & 0 & 0\\ 0 & \sigma_uI_{k_u} & 0\end{pmatrix},\quad \Sigma'=\begin{pmatrix}L & 0\\ 0 & 0\\ 0 & 0\end{pmatrix},$$

and let us write B,C by blocks,

$$B=\begin{pmatrix}B_1 & B_2\\ -B_2^* & B_4\end{pmatrix},\quad C=\begin{pmatrix}C_1 & C_2 & C_3\\ -C_2^* & C_4 & C_5\\ -C_3^* & -C_5^* & C_6\end{pmatrix}$$

where B_1,C_1 are of the size of L and B_4,C_4 are of the size of I_{k_u}. Then,

$$\begin{aligned}\text{trace }(BB^*\Sigma\Sigma') &= \text{trace }((B_1B_1^*+B_2B_2^*)L^2),\\ \text{trace }(\Sigma'\Sigma CC^*) &= \text{trace }(L^2(C_1C_1^*+C_2C_2^*+C_3C_3^*)),\\ \text{trace }(B\Sigma C^*\Sigma') &= \text{trace }(B_1LC_1^*L+\sigma_uB_2C_2^*L).\end{aligned}$$

Thus,

$$J \geq \mathrm{Re}\left(\mathrm{trace}\ ((B_1B_1^* + C_1C_1^*)L^2 - 2B_1LC_1^*L)\right) +$$
$$\mathrm{Re}\left(\mathrm{trace}\ ((B_2B_2^* + C_2C_2^* + C_3C_3^*)L^2 - 2\sigma_u B_2C_2^*L)\right).$$

We will prove that these two terms are non-negative. For the first one, note that

$$\mathrm{Re}\left(\mathrm{trace}\ ((B_1B_1^* + C_1C_1^*)L^2 - 2B_1LC_1^*L)\right) = \|B_1L - LC_1\|^2 \geq 0.$$

For the second one, we check that for every l, $1 \leq l \leq n-k_u$ the l-th diagonal entry of the matrix $(B_2B_2^* + C_2C_2^* + C_3C_3^*)L^2 - 2\sigma_u B_2C_2^*L$ has a positive real part. Indeed, if we denote by $v \in \mathbb{K}^{k_u}$ the l-th row of B_2 by $w \in \mathbb{K}^{k_u}$ the l-th row of C_2 and by x the l-th row of C_3, we have

$$\mathrm{Re}\left((B_2B_2^* + C_2C_2^* + C_3C_3^*)L^2 - 2\sigma_u B_2C_2^*L\right)_{l,l} =$$

$$\sigma_l^2\left((\|v\|^2 + \|w\|^2 + \|x\|^2) - 2\frac{\sigma_u}{\sigma_l}\mathrm{Re}\langle v, w\rangle\right) \geq \sigma_l^2\|v - w\|^2 \geq 0$$

as $\sigma_u < \sigma_l$. This finishes the proof of Lemma 20 and hence of Proposition 19. □

5 Puting pieces together

Before stating the main result of this section we have to introduce the following machinery:

5.1 Second symmetric derivatives

In the case of Lipschitz-Riemann structures, the mappings we want to consider are not necessarily $\mathcal{C}^2$ and, to study their convexity properties, an approach based on the usual covariant second derivative is insufficient. We will use instead the second symmetric upper derivative.

Let $U \subseteq \mathbb{R}^k$ be an open set and $\phi : U \to \mathbb{R}$ be any function. The **second symmetric upper derivative** of ϕ at $x \in U$ in the direction $v \in \mathbb{R}^k$ is

$$\overline{\mathcal{SD}^2}\phi(x; v) = \limsup_{h \mapsto 0} \frac{\phi(x + hv) + \phi(x - hv) - 2\phi(x)}{h^2}$$

which is allowed to be $\pm\infty$. If $U \subseteq \mathbb{R}$ is an interval, we simply write $\overline{\mathcal{SD}}^2\phi(x)$ for $\overline{\mathcal{SD}}^2\phi(x;1)$.

It is well-known that a continuous function ϕ on an interval is convex if and only if $\overline{\mathcal{SD}}^2\phi(x) \geq 0$ for all x (see for example [31] Theorem 5.29). There is a stronger result due to Burkill [9] Theorem 1.1 (see also [31] Corollary 5.31) which uses a weaker hypothesis:

Theorem 21 (Burkill)**.** *Let $\phi :]a,b[\to \mathbb{R}$ be a continuous function such that $\overline{\mathcal{SD}}^2\phi(x) \geq 0$ for almost all $x \in]a,b[$, and assume that $\overline{\mathcal{SD}}^2\phi(x) > -\infty$ for $x \in]a,b[$. Then, ϕ is a convex function.*

Theorem 21 will allow us to assemble the pieces where convexity is proven in Proposition 19 to prove our main results (Theorems 1 and 31). We proceed a little more generally as the result may be of interest in other circumstances. Let $\mathcal{M}$ be a k-dimensional $\mathcal{C}^2$ manifold (not necessarily having a Riemannian structure).

Definition 22. Let $\alpha : \mathcal{M} \to \mathbb{R}$. We say that $\overline{\mathcal{SD}}^2\alpha$ **is bounded from** $-\infty$ (denoted $\overline{\mathcal{SD}}^2\alpha > -\infty$) if for every $x \in \mathcal{M}$ there is an open neighborhood $U_x \subseteq \mathcal{M}$ and a coordinate chart $\varphi_x : U_x \to \mathbb{R}^k$, $\varphi_x(x) = 0$ such that

$$\overline{\mathcal{SD}}^2(\alpha \circ \varphi_x^{-1})(0; v) > -\infty$$

for every $v \in \mathbb{R}^k$.

The following lemma is a consequence of Definition 22.

Lemma 23. *Let $\mathcal{M}$ be a $\mathcal{C}^2$ manifold, let $\alpha : \mathcal{M} \to]0,\infty[$ be a locally Lipschitz mapping. Then, $\overline{\mathcal{SD}}^2\alpha > -\infty$ if and only if, for any function $\phi :]0,\infty[\to \mathbb{R}$ of class $\mathcal{C}^2$, $\overline{\mathcal{SD}}^2(\phi \circ \alpha) > -\infty$. In particular, $\overline{\mathcal{SD}}^2\alpha > -\infty$ if and only if $\overline{\mathcal{SD}}^2(\log \circ \alpha) > -\infty$.*

Proof. The *if* part is trivial (just make $\phi(t) = t$). In order to prove the *only if* part, we assume that $\overline{\mathcal{SD}}^2\alpha > -\infty$. Let $x \in \mathcal{M}$ and let $\varphi_x : U_x \to \mathbb{R}^k$ be a coordinate chart such that $\varphi_x(x) = 0$ and $\overline{\mathcal{SD}}^2(\alpha \circ \varphi_x^{-1})(0; v) > -\infty$ for each $v \in \mathbb{R}^k$. There is a sequence $h_p \to 0$ such that

$$\lim_{p\to\infty} \frac{\alpha(\varphi_x^{-1}(h_p v)) + \alpha(\varphi_x^{-1}(-h_p v)) - 2\alpha(x)}{h_p^2} = C > -\infty.$$

Let us define $H_p = \alpha(\varphi_x^{-1}(h_p v)) - \alpha(x)$, and similarly $K_p = \alpha(\varphi_x^{-1}(-h_p v)) - \alpha(x)$. By Taylor's formula we get

$$\phi(\alpha(\varphi_x^{-1}(h_p v))) = \phi(\alpha(x)) + \phi'(\alpha(x))H_p + \phi''(\alpha(x))\frac{H_p^2}{2} + o(H_p^2),$$

and similarly

$$\phi(\alpha(\varphi_x^{-1}(-h_p v))) = \phi(\alpha(x)) + \phi'(\alpha(x))K_p + \phi''(\alpha(x))\frac{K_p^2}{2} + o(K_p^2),$$

so that

$$\frac{\phi(\alpha(\varphi_x^{-1}(h_p v))) + \phi(\alpha(\varphi_x^{-1}(-h_p v))) - 2\phi(\alpha(x)}{h_p^2} =$$

$$\phi'(\alpha(x))\frac{H_p + K_p}{h_p^2} + \phi''(\alpha(x))\frac{H_p^2 + K_p^2}{2h_p^2} + \frac{o(H_p^2) + o(K_p^2)}{h_p^2}.$$

Notice that $\lim_{p\to\infty} \frac{H_p+K_p}{h_p^2} = C$. Since $h \to \alpha(\varphi_x^{-1}(hv))$ is Lipschitz in a neighborhood of 0 we have, for a suitable constant $D > 0$, $H_p^2 \leq Dh_p^2$ and $K_p^2 \leq Dh_p^2$. Thus, taking the lim sup as $p \to \infty$ gives $\overline{\mathcal{SD}}^2\phi(\alpha \circ \varphi_x^{-1})(0, v) \geq C + D > -\infty$ and we are done. □

5.2 Projecting geodesics on submanifolds : the Euclidean case

The following technical lemma, interesting by itself, is a consequence of Lebesgue's Density Theorem.

Lemma 24. *For any locally integrable function f defined in $\mathbb{R}$ with values in $\mathbb{R}^n$, let $x \in \mathbb{R}$ be a point where f is locally integrable. This means that $F'(x) = f(x)$ where F denotes an antiderivative of f. Then*

$$\lim_{\varepsilon\to 0} \frac{2}{\varepsilon^2} \int_x^{x+\varepsilon} (y - x)f(y)dy = f(x).$$

Proof. Notice that, by Lebesgue's differentiation theorem, an antiderivative F of f exists a. e. and it is absoluttly continuous. Suppose that $F(x) = 0$. Let us define

$$h(y) = \begin{cases} F(y)/(y - x) & \text{if } y \neq x, \\ f(x) & \text{if } y = x, \end{cases}$$

so that h is a continuous function and $F(y) = (y-x)h(y)$ for any y. Integrating by parts gives

$$\int_x^{x+\varepsilon} (y-x)f(y)dy = \varepsilon F(x+\varepsilon) - \int_x^{x+\varepsilon} F(y)dy$$

so that

$$\frac{2}{\varepsilon^2}\int_x^{x+\varepsilon} (y-x)f(y)dy = 2\frac{F(x+\varepsilon)-F(0)}{\varepsilon} - \frac{2}{\varepsilon^2}\int_x^{x+\varepsilon} (y-x)h(y)dy.$$

Since h is continuous, by the Mean Value Theorem, there exists $\zeta \in [x, x+\varepsilon]$ such that

$$\frac{2}{\varepsilon^2}\int_x^{x+\varepsilon} (y-x)h(y)dy = \frac{2h(\zeta)}{\varepsilon^2}\int_x^{x+\varepsilon} (y-x)dy = h(\zeta) \to h(x) = f(x)$$

as $\varepsilon \to 0$. On the other hand

$$\lim_{\varepsilon\to 0} 2\frac{F(x+\varepsilon)-F(x)}{\varepsilon} = 2f(x).$$

Thus

$$\lim_{\varepsilon\to 0} \frac{2}{\varepsilon^2}\int_x^{x+\varepsilon} (y-x)f(y)dy = 2f(x) - f(x) = f(x)$$

and we are done. □

Our aim is now to see how close are a geodesic in a Lipschitz-Riemannian manifold and a geodesic in a submanifold when they have the same tangent at a given point. Let us start to study a simple case.

Let us consider the Lipschitz-Riemann structure defined on an open, k-dimensional set $\Omega \subset \mathbb{R}^k$ containing 0 by the scalar product $\langle u, v\rangle_x = v^T H(x) u$ (see section 2.3).

1. The matrix $H(0)$ is supposed to have the following block structure

$$H(0) = \begin{pmatrix} H_p(0) & 0 \\ 0 & H_{k-p}(0) \end{pmatrix}.$$

We also suppose that (see section 2.3)

2. The entries $h_{ij}(x)$ of $H(x)$ are regular at $x = 0$,

The set $\Omega_p = \Omega \cap (\mathbb{R}^p \times \{0\})$ is a submanifold in Ω. We suppose that

3. H_p is $\mathcal{C}^2$ in Ω_p,

so that Ω_p is in fact a smooth $\mathcal{C}^2$ Riemannian manifold for the induced $H-$structure. Let us now consider a vector $a \in \mathbb{R}^p \times \{0\}$ and three parametrized curves denoted by x, x_p, and y defined in a neighborhood of 0 in $\mathbb{R}$, and such that:

4. $x(0) = x_p(0) = y(0) = 0$,
5. $\dot{x}(0) = \dot{x}_p(0) = \dot{y}(0) = a$,
6. x is a geodesic in $\mathbb{R}^k$ for the H-structure,
7. x_p is its orthogonal projection onto $\mathbb{R}^p \times \{0\}$,
8. y is a geodesic in $\mathbb{R}^p \times \{0\}$ for the induced structure.

According to Theorem 3, x has regularity $\mathcal{C}^{1+Lip}$ so that its second derivative exists a.e. We suppose here that

9. The second derivative $\ddot{x}(t)$ is defined at $t = 0$, and

$$\frac{d}{dt} |_{t=0} \left(H(x(t))\dot{x}(t)\right) \in \frac{1}{2} \sum_{i,j} \dot{x}_i(0)\dot{x}_j(0)\partial h_{ij}(x(0)).$$

In this context we have:

Lemma 25. *Under the hypotheses* 1 *to* 9 *above, the curves* x_p *and* y *have a contact of order* 2 *at* 0*:* $x_p(s) = y(s) + o(s^2)$.

Proof. By hypothesis 3,

$$y \in \mathcal{C}^2. \tag{5.1}$$

Hypothesis 8 says that y is a geodesic. Because geodesics are parametrized by arc length,

$$\dot{y}^T(s)H_p(y(s))\dot{y}(s) = 1. \tag{5.2}$$

The Euler-Lagrange equation for geodesics in is now

$$\frac{d}{ds}(H_p(y(s))\dot{y}(s)) = \frac{1}{2} \sum_{i,j} \dot{y}_i(s)\dot{y}_j(s)\text{grad } h_{p,ij}(y(s)), \tag{5.3}$$

where $h_{p,ij}$ is h_{ij}, seen as a function of $x_1, \ldots, x_p$. The differential system (5.1-5.3) actually defines $y(s)$ as a curve in Ω_p, in function of the initial condition $(y(0), \dot{y}(0))$.

Moreover, thanks to hypothesis 9 we have:

$$H(x(0))\ddot{x}(0) + \frac{d}{dt}_{|t=0}(H(x(t)))\dot{x}(0) \in \frac{1}{2}\sum_{i,j=1}^{k} \dot{x}_i(0)\dot{x}_j(0)\partial h_{ij}(x(0)).$$

When we project it onto $\mathbb{R}^p$ we get, with $x(t) = \begin{pmatrix} x_p(t) \\ x_{k-p}(t) \end{pmatrix}$,

$$H_p(0)\ddot{x}_p(0) + \frac{d}{dt}_{|t=0}(H_p(x_p(t)))\dot{x}_p(0) \in \frac{1}{2}\sum_{i,j=1}^{p} \dot{x}_{p,i}(0)\dot{x}_{p,j}(0)\Pi_{\mathbb{R}^p}\partial h_{ij}(x(0)).$$

Since the functions $h_{ij}(x)$ for $i, j = 1 \ldots p$ are regular (hypothesis 2), from Clarke [11] Proposition 2.3.15, we obtain

$$\Pi_{\mathbb{R}^p}\partial h_{ij}(x(0)) = \partial h_{p,ij}(x_p(0)) = \text{grad } h_{p,ij}(x_p(0))$$

so that

$$H_p(0)\ddot{x}_p(0) + \frac{d}{dt}\ |_{t=0}\ (H_p(x_p(t)))\dot{x}_p(0) = \frac{1}{2}\sum_{i,j=1}^{p} \dot{x}_{p,i}(0)\dot{x}_{p,j}(0)\text{grad } h_{p,ij}(x_p(0)),$$

Taking at $t = 0$ the differential equation giving y and noting that $y(0) = x_p(0)$, $\dot{y}(0) = \dot{x}_p(0)$ gives

$$\ddot{y}(0) = \ddot{x}_p(0).$$

We want to prove that $x_p(s) = y(s) + o(s^2)$. According to Taylor's formula with integral remainder, we have

$$x_p(s) - y(s) = x_p(0) - y(0) + s(\dot{x}_p(0) - \dot{y}(0)) + \int_0^s (\ddot{x}_p(\sigma) - \ddot{y}(\sigma))\sigma d\sigma,$$

so that,

$$2\frac{x_p(s) - y(s)}{s^2} = \frac{2}{s^2}\int_0^s (\ddot{x}_p(\sigma) - \ddot{y}(\sigma))\sigma d\sigma.$$

From Lemma 24 and hypothesis 9, the limit of this expression exists at $s = 0$, and it is equal to $\ddot{x}_p(0) - \ddot{y}(0) = 0$. This achieves the proof. □

Proposition 26. *Let $\mathbb{R}^k$ be endowed with the Lipschitz-Riemann structure defined by $\langle u, v\rangle_x = v^T H(x)u$, where the entries $h_{ij}(x)$ of $H(x)$ are regular and $H(x)$ has the block structure*

$$H(x) = \begin{pmatrix} H_p(x) & 0 \\ 0 & H_{k-p}(x) \end{pmatrix},$$

for $x \in \mathbb{R}^k$. Assume that H_p is $\mathcal{C}^2$ for all $x \in \mathbb{R}^p \times \{0\} \subset \mathbb{R}^k$. Let $x : [a, b] \to \mathbb{R}^k$ be a geodesic in $\mathbb{R}^k$ with respect to the Lipschitz-Riemann structure. Then, there exists a zero-measure set $Z \subseteq [a, b]$ such that for $t_0 \in [a, b] \setminus Z$ the following holds:

"If $x(t_0) \in \mathbb{R}^p \times \{0\} \subset \mathbb{R}^k$ and $\dot{x}(t_0) \in \mathbb{R}^p \times \{0\} \subset \mathbb{R}^k$, then the projection $x_p(t) = \pi_{\mathbb{R}^p}(x(t))$ has a contact of order 2 with $y(t)$, the unique geodesic in $\mathbb{R}^p$ with respect to the Lipschitz-Riemann structure H_p with initial conditions $y(t_0) = \pi_{\mathbb{R}^p}(x(t_0))$ and $\dot{y}(t_0) = \pi_{\mathbb{R}^p}(\dot{x}(t_0))$".

Proof. From Remark 6, there exists a zero measure $Z \subseteq [a, b]$ such that for $t_0 \in [a, b] \setminus Z$, $\ddot{x}(t_0)$ exists and

$$\frac{d}{dt}\,|_{t=t_0}\,(H(x(t))\dot{x}(t)) \in \frac{1}{2}\sum_{i,j} \dot{x}_i(t_0)\dot{x}_j(t_0)\partial h_{ij}(x(t_0)).$$

From Lemma 25, for every such t_0, if in addition $x(t_0) \in \mathbb{R}^p \times \{0\}$ and $\dot{x}(t_0) \in \mathbb{R}^p \times \{0\}$, then $x_p(t)$ has a contact of order 2 with $y(t)$ and we are done. □

5.3 Projecting geodesics on submanifolds : the Riemannian case

Our aim, in this section, is to prove another version of Lemma 25 in a different geometric context. Let $\mathcal{M}$ be a $\mathcal{C}^3$ Riemannian manifold with distance d, of dimension k, and let $\mathcal{N}$ be a submanifold of dimension p.

Let us first define the **projection onto** $\mathcal{N}$ (Fig.1). To each $q \in \mathcal{N}$ and to a vector $u \neq 0$ normal to $\mathcal{N}$ at q we associate the geodesic $\gamma_{q,u}$ in $\mathcal{M}$ such that $\gamma_{q,u}(0) = q$ and $\dot{\gamma}_{q,u}(0) = u$. Let $n \in \mathcal{N}$ be given, and let U be an open neighborhood of n such that, for each $m \in U$ there exists a unique geodesic arc $\gamma_{q,u}(t)$, t in an open interval containing 0, contained in U and containing m. Thus U is the union of such geodesic arcs and two of them have always a void intersection. This picture defines a map $K : U \to \mathcal{N}$ by $K(m) = q$

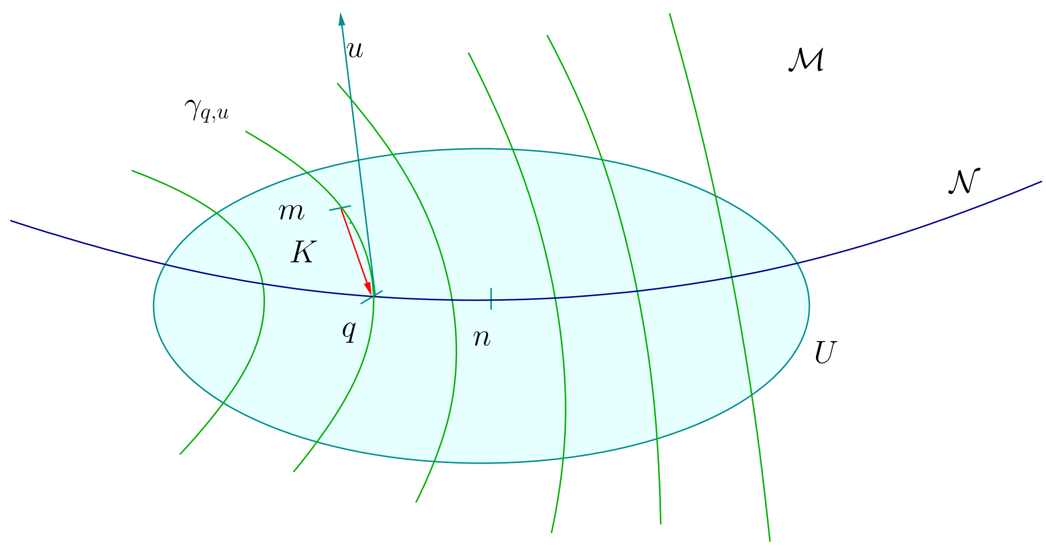

Figure 1: The projection $K : \mathcal{M} \to \mathcal{N}$.

if $m = \gamma_{q,u}(t)$. The map K is the **projection map onto** $\mathcal{N}$. It has the following classical properties:

1. It is defined in the neighborhood U of $n \in \mathcal{N}$,
2. For each $m \in U$, $K(m)$ is the unique point in $\mathcal{M}$ such that

$$\inf_{q \in \mathcal{N}} d(m, q) = d(m, K(m))$$

3. K is $\mathcal{C}^2$.

See Li-Nirenberg [21] or Beltran-Dedieu-Malajovich-Shub [2].

Let $\alpha : \mathcal{M} \to]0, \infty[$ be a locally Lipschitz, regular map (see section 2.3). It defines a conformal Lipschitz-Riemann structure on $\mathcal{M}$ associated with the inner product

$$\langle \cdot, \cdot \rangle_{\alpha,m} = \alpha(m) \langle \cdot, \cdot \rangle_m .$$

We call it the $\alpha-$structure. We suppose that α is $\mathcal{C}^2$ when it is restricted to $\mathcal{N}$ so that $\mathcal{N}$ is $\mathcal{C}^2$ and not only Lipschitz for the induced $\alpha-$structure.

Proposition 27. *Under the hypotheses above, let $\gamma : [a, b] \to \mathcal{M}$ be a geodesic curve in $\mathcal{M}$ for the $\alpha-$structure. Then, there exists a zero-measure set $Z \subseteq [a, b]$ such that for $t_0 \in [a, b] \setminus Z$ the following holds:*

"If $\gamma(t_0) \in \mathcal{N}$ and $\dot\gamma(t_0) \in T_{\gamma(t_0)}\mathcal{N}$, then the projection $\gamma_{\mathcal{N}}(t) = (K \circ \gamma)(t)$ of γ onto $\mathcal{N}$ has a contact of order 2 with $\delta(t)$, the unique geodesic in $\mathcal{N}$ such that $\delta(t_0) = \gamma(t_0)$ and $\dot\delta(t_0) = \dot\gamma(t_0)$".

Remark 28. If $\mathcal{M}$, $\mathcal{N}$ and α are assumed to be smooth then $Z = \emptyset$ in Proposition 27. See for example the proof of Proposition 5.9 in [33].

Proof. The proof consists in a transfer from $\mathcal{M}$ to $\mathbb{R}^k$ where we apply Proposition 27. Let

$$Z = \left\{ t_0 \in [a,b] : \gamma(t_0) \in \mathcal{N}, \dot\gamma(t_0) \in T_{\gamma(t_0)}\mathcal{N} \text{ but } \lim_{t\to t_0} \frac{\gamma_{\mathcal{N}}(t) - \delta(t)}{(t-t_0)^2} \neq 0 \right\}.$$

We have to check that Z is a zero measure set. It suffices to see that for every $t \in (a,b)$ there is an open interval I containing t and such that $I \cap Z$ has zero measure. Without loss of generality, we may assume that $t = 0$. Thus, let $t = 0 \in (a,b)$ and let $n = \gamma(0)$.

Since $\mathcal{M}$ is $\mathcal{C}^3$, the normal bundle to $\mathcal{N}$ is $\mathcal{C}^2$ and there exists a $\mathcal{C}^2$ diffeomorphism $\phi : U \to V \subset \mathbb{R}^k$, where V is an open set containing 0, satisfying

1. $\phi(n) = 0$,
2. $\phi(U \cap \mathcal{N}) = V \cap (\mathbb{R}^p \times \{0\})$,
3. For any $q \in \mathcal{N}$ and any vector $u \neq 0$ normal to $\mathcal{N}$ at q, $\phi(\gamma_{q,u})$ is a straight line in $\mathbb{R}^k$ orthogonal to $\mathbb{R}^p \times \{0\}$.

We make ϕ an isometry in defining on $V \subset \mathbb{R}^k$ a Lipschitz-Riemannian structure by

$$\langle D\phi(m)u, D\phi(m)v \rangle_{\phi(m)} = \alpha(m) \langle u, v \rangle_m$$

for any $m \in U$, and $u, v \in T_m\mathcal{M}$. Let us denote $x = \phi(m)$, $a = D\phi(m)u$, $b = D\phi(m)v$, we also write this scalar product

$$\langle a, b \rangle_x = b^T H(x) a$$

where H is a locally Lipschitz map from V into the $k \times k$ positive definite matrices.

Notice that H is regular because α is regular in $\mathcal{N}$.

Since for every $\hat n \in \mathcal{N} \cap U$,

$$D\phi(\hat n)\left(T_{\hat n}\mathcal{N}\right) = \mathbb{R}^p \times \{0\} \text{ and } D\phi(\hat n)\left((T_{\hat n}\mathcal{N})^\perp \right) = \{0\} \times \mathbb{R}^{k-p},$$

$H(x)$ has the block structure

$$H(x) = \begin{pmatrix} H_p(x) & 0 \\ 0 & H_{k-p}(x) \end{pmatrix}.$$

Since α is $\mathcal{C}^2$ when restricted to $\mathcal{N}$ we have the same regularity for the restriction of H to $\mathbb{R}^p \times \{0\}$.

Since ϕ is an isometry the curves $\phi \circ \gamma$ and $\phi \circ \delta$ are geodesics in $\mathbb{R}^k$ and $\mathbb{R}^p \times \{0\}$ respectively, and, from the definition of ϕ, the orthogonal projection (in the Euclidean meaning) of $\phi \circ \gamma$ onto $\mathbb{R}^p \times \{0\}$ is equal to $\phi \circ \gamma_{\mathcal{N}}$.

Thus, the hypotheses of Proposition 26 are satisfied so that $\phi \circ \gamma_{\mathcal{N}}$ and $\phi \circ \delta$ have an order 2 contact at every t out of a zero measure set Z_0. This gives easily an order 2 contact for $\gamma_{\mathcal{N}}$ and δ at $t \notin Z_0$ in $\mathcal{M}$ in terms of the $\alpha-$distance but also, since $1/\alpha$ is locally Lipschitz, in terms of the initial Riemannian distance. The proposition follows. □

5.4 Arriving to the main theorem

We are now ready to state the main theorem in this section:

Theorem 29 (Piecing together)**.** *$\mathcal{M} = \cup_{i=1}^{\infty} \mathcal{M}_i$ is a $\mathcal{C}^3$ Riemannian manifold, enumerable union of the submanifolds $\mathcal{M}_i$. Let $\alpha : \mathcal{M} \to]0, \infty[$ be a locally Lipschitz mapping. Assume that:*

1. *α is regular,*
2. *For each i, the restriction of α to $\mathcal{M}_i$ is $\mathcal{C}^2$ and self-convex in $\mathcal{M}_i$,*
3. *$\overline{\mathcal{SD}}^2 \alpha > -\infty$.*

Then, α is self-convex in $\mathcal{M}$.

Proof. Once again we add to $\mathcal{M}$ the $\alpha-$structure. If this theorem is false, there exists a geodesic γ in $\mathcal{M}$ for the $\alpha-$structure such that

$$\overline{\mathcal{SD}}^2 \log(\alpha(\gamma(t))) < 0$$

on a positive measure set $P \subset \mathbb{R}$ (Theorem 21 and Lemma 23). Since an enumerable union of zero-measure sets is also a zero-measure set, we can suppose that $P \subset \mathcal{M}_i$ for some i, so that $\gamma(t) \in \mathcal{M}_i$ for every $t \in P$.

According to the Lebesgue Density Theorem, almost all points $t \in P$ are density points, that is

$$\lim_{\varepsilon \to 0} \frac{\text{meas}\,(P \cap [t-\varepsilon, t+\varepsilon])}{2\varepsilon} = 1.$$

We remove the "non-density points" from P to obtain a new set, also called P, with positive measure and only density points. Since $\gamma \in \mathcal{C}^{1+Lip}$ (Theorem 3), the second derivative $\ddot{\gamma}(t)$ exists for almost all t. We also remove from P the zero measure set of Proposition 27.

Let $t \in P$ be given. Since it is a density point of P, we have $s \in P$ for "a lot of points" close to t. Since $\gamma(s) \in \mathcal{M}_i$ for such points, and since γ is $\mathcal{C}^1$, we get

$$\dot{\gamma}(t) \in T_{\gamma(t)}\mathcal{M}_i.$$

Take now the geodesic δ in $\mathcal{M}_i$ for the induced $\alpha-$structure such that $\delta(t) = \gamma(t)$ and $\dot{\delta}(t) = \dot{\gamma}(t)$. As we have removed the zero-measure set of Proposition 27, γ_i and δ have a contact of order 2 at t.

By self-convexity of α in $\mathcal{M}_i$, and since δ is $\mathcal{C}^2$ we get

$$\overline{\mathcal{SD}}^2 \log \circ\alpha \circ \delta(t) = \frac{d^2}{dt^2} \log \circ\alpha \circ \delta(t) \geq 0.$$

Let us now consider

$$\Delta^2(h) = \frac{\log \circ\alpha \circ \gamma(t+h) + \log \circ\alpha \circ \gamma(t-h) - 2\log \circ\alpha \circ \gamma(t)}{h^2}.$$

It is not difficult to prove that t is a density point of

$$Q = \{s = t + h \in P \ : \ t - h \in P\}.$$

Let us denote by γ_i the projection of γ on $\mathcal{M}_i$ (see section 5.3). For the points $s = t + h \in Q$, one has $\gamma(t+h) = \gamma_i(t+h)$, $\gamma(t-h) = \gamma_i(t-h)$, and $\gamma(t) = \gamma_i(t)$, thus

$$\Delta^2(h) = \frac{\log \circ\alpha \circ \gamma_i(t+h) + \log \circ\alpha \circ \gamma_i(t-h) - 2\log \circ\alpha \circ \gamma_i(t)}{h^2}.$$

From the contact of order 2 between γ_i and δ we then conclude,

$$\Delta^2(h) = \frac{\log \circ\alpha \circ \delta(t+h) + \log \circ\alpha \circ \delta(t-h) - 2\log \circ\alpha \circ \delta(t) + o(h^2)}{h^2}.$$

Since δ is $\mathcal{C}^2$, taking the limit as $h \to 0$ gives

$$\lim \Delta^2(h) = \frac{d^2}{dt^2} \log \circ \alpha \circ \delta(t).$$

Since this last expression is nonegative we obtain

$$\overline{\mathcal{SD}}^2 \log(\alpha(\gamma(t))) \geq \lim \Delta^2(h) \geq 0$$

which contradicts our hypothesis $\overline{\mathcal{SD}}^2 \log(\alpha(\gamma(t))) < 0$ on P. □

6 Proof of Theorem 1

Theorem 1 is a consequence of Theorem 29 applied to $\mathcal{M} = \mathbb{GL}_{n,m}$ considered as the union of the submanifolds $\mathcal{P}_{(k)}$ (see section 4) and to the mapping $\alpha(A) = \sigma_n(A)^{-2}$, the inverse of the square of the smallest singular value of $A \in \mathbb{GL}_{n,m}$. According to propositions 16 and 19 we just have to prove that α is a regular map and that $\overline{\mathcal{SD}}^2\alpha > -\infty$. Let us start with this last inequality.

We must prove that for every $A \in \mathbb{GL}_{n,m}$, $B \in \mathbb{K}^{n\times m}$,

$$\overline{\mathcal{SD}}^2\sigma_n^{-2}(A;B) = \limsup_{h\mapsto 0} \frac{\sigma_n^{-2}(A_h) + \sigma_n^{-2}(A_{-h}) - 2\sigma_n^{-2}(A)}{h^2} > -\infty,$$

where $A_h = A + hB$. Now, let $\mathcal{S}_n^+$ be the set of symmetric, positive definite $n \times n$ matrices. Then,

$$\sigma_n^{-2}(A_h) + \sigma_n^{-2}(A_{-h}) = \lambda_n^{-1}(A_hA_h^*) + \lambda_n^{-1}(A_{-h}A_{-h}^*).$$

where, λ_n denotes the smallest eigenvalue. Since, for any $S \in \mathcal{S}_n^+$,

$$\lambda_n(S) = \inf_{u\in\mathbb{R}^n,\ \|u\|=1} u^T S u,$$

it is a concave function of S, and λ_n^{-1} is convex. Thus,

$$\lambda_n^{-1}(A_hA_h^*)+\lambda_n^{-1}(A_{-h}A_{-h}^*) \geq 2\lambda_n^{-1}\left(\frac{A_hA_h^* + A_{-h}A_{-h}^*}{2}\right) = 2\lambda_n^{-1}(AA^*+h^2BB^*).$$

We conclude that

$$\overline{\mathcal{SD}}^2\sigma_n^{-2}(A;B) \geq \limsup_{h\mapsto 0} \frac{2\lambda_n^{-1}(AA^* + h^2BB^*) - 2\lambda_n^{-1}(AA^*)}{h^2}.$$

This last quantity is bounded in absolute value since λ_n^{-1} is locally Lipschitz, so in particular $\overline{\mathcal{SD}^2}\sigma_n^{-2}(A;B) > -\infty$.

To prove that α is regular it suffices to write it as the composition of $\mathcal{C}^1$ maps and of the convex λ_n^{-1} which is also a regular map (see [11] Prop. 2.3.6). This finishes the proof of our Main Theorem 1.

7 The solution variety

As in [2], we are also interested in the log–convexity of $\sigma_n(A)^{-1}$ in the solution variety:

$$\mathcal{W} = \{(A, x) \in \mathbb{GL}_{n,n+1} \times \mathbb{P}(\mathbb{K}^{n+1}) : Ax = 0\}.$$

Remark 30. In [2] we have sometimes taken A to lie in the unit sphere of $\mathbb{K}^{n\times m}$ or even the projective space $\mathbb{P}(\mathbb{K}^{n\times m})$. The interested reader can check [2] for the relations between self-convexity in the various settings.

Theorem 31. *For any condition geodesic $t \to (A(t), x(t))$ in $\mathcal{W}$, the map $t \to \log(\sigma_n^{-2}(A(t)))$ is convex.*

As we have done in the case of $\mathbb{GL}_{n,m}$, we divide the proof in several sections.

7.1 The smooth part of $\mathcal{W}$

Let $u \le n$ and $(k) = (k_1, \dots, k_u) \in \mathbb{N}^u$ such that $k_1 + \dots + k_u = n$. We define $\mathcal{W}_{(k)} = \{(A, x) \in \mathcal{W} : A \in \mathcal{P}_{(k)}\}$.

Proposition 32. *For any choice of (k), the set $\mathcal{W}_{(k)}$ is a smooth submanifold of $\mathcal{W}$, σ_u is a smooth function and $\alpha = \sigma_u^{-2}$ is self–convex in $\mathcal{W}_{(k)}$.*

Proof. Let us consider the map

$$\begin{array}{rccc} \psi : & \mathcal{P}_{(k)} \times \mathbb{K}^{n+1} \setminus \{0\} & \to & \mathbb{K}^n \\ & (A, x) & \mapsto & Ax \end{array}$$

which is a smooth mapping between two smooth manifolds. Since 0 is a regular value of ψ, its preimage $\psi^{-1}(0)$ is a smooth submanifold of $\mathcal{P}_{(k)} \times \mathbb{K}^{n+1} \setminus \{0\}$. Moreover, σ_u is the composition of the projection onto the first coordinate $\mathcal{W}_{(k)} \to \mathcal{P}_{(k)}$ and the function σ_u which is smooth by Proposition

16. To check that σ_u is self–convex in $\mathcal{W}_{(k)}$ we use Corollary 15 and proceed as in the proof of Proposition 19. Let $G = \mathbb{U}_n \times \mathbb{U}_{n+1}$, and consider the action

$$\begin{array}{ccc} G \times \mathcal{W}_{(k)} & \to & \mathcal{W}_{(k)} \\ ((U,V),(A,x)) & \mapsto & (UAV^*, Vx) \end{array}$$

Let $p = (\Sigma, e_{n+1})$ where $e_{n+1}^T = (0,\ldots,0,1)$ and $\Sigma \in \mathcal{D}_{(k)}$ has ordered distinct singular values $\sigma_1 > \cdots > \sigma_u > 0$. Recall that $T_pG(p)$ is the tangent space in p of the orbit $G(p)$ of p by the Lie group G. As in Propositions 16 and 19, we have

$$T_pG(p) = \{(B_1\Sigma + \Sigma B_2^*, B_2 e_{n+1}) : (B_1, B_2) \in \mathcal{A}_n \times \mathcal{A}_{n+1}\},$$

$$T_pG(p)^\perp = \{(\dot\Sigma, 0) : \dot\Sigma \in \mathcal{P}_{(k)}, \dot\Sigma \text{ is diagonal}, \dot\Sigma e_{n+1} = 0\}.$$

Note that $T_pG(p)^\perp$ is isometric to the set of diagonal $n \times n$ matrices with eigenvalues $\sigma_1 > \ldots > \sigma_u > 0$ of respective multiplicities $k_1, \ldots, k_u$.

Let us check the conditions of Corollary 15. By unitary invariance, we can choose a pair $p = (\Sigma, e_{n+1})$ as above.

1. $D_\kappa^2 \log(\alpha)(p)$ is positive semi-definite in $(T_pG(p))^\perp$: let $(\dot\Sigma, 0) \in T_pG(p)^\perp$. Let γ be a condition geodesic in $T_pG(p)^\perp$ such that $\gamma(0) = (\Sigma, 0)$, $\dot\gamma(0) = (\dot\Sigma, 0)$. We have to check that

$$\frac{d^2}{dt^2} \log \alpha(\gamma(t)) \mid_{t=0} \geq 0.$$

This is true as α is log-convex in the set of diagonal $n \times n$ matrices with eigenvalues $\sigma_1 > \ldots > \sigma_u > 0$ from Proposition 16.

2. We have to check that for small enough t, and for

$$b = (\dot\Sigma, 0) \in T_pG(p)^\perp,$$

$D\phi_t(p)b$ is perpendicular to

$$T_{\phi_t(p)}G(\phi_t(p)),$$

where ϕ_t is the flow of $\mathrm{grad}_\kappa \alpha$ in $\mathcal{W}_{(k)}$. Now, as in the proof of Proposition 19, the operator grad preserves the diagonal form of (Σ, e_n) and hence $D\phi_t(p)b$ is of the form $(\Sigma', 0)$ where Σ' is diagonal with $\Sigma' e_n = 0$. In particular, it is orthogonal to $T_{\phi_t(p)}G(\phi_t(p))$. Thus, the second condition of Corollary 15 applies to our case.

3. For $(B_1, B_2) \in \mathcal{A}_n \times \mathcal{A}_m$, the vector field K on $\mathcal{W}_{(k)}$ generated by (B_1, B_2) is

$$K(A,x) = \frac{d}{dt}\left(e^{tB_1}Ae^{tB_2^*}, e^{tB_2}x\right)|_{t=0} = (B_1A + AB_2^*, B_2x).$$

Note that

$$\|K(A,x)\|^2 = \|B_1A + AB_2^*\|^2 + \|B_2x\|^2.$$

Thus,

$$D(\|K\|^2)(A,x)(C,v) = \frac{d}{dt}|_{t=0}\left(\|K(A+tC, x+tv)\|^2\right) =$$

$$2\mathrm{Re}\langle B_1B_1^*A + AB_2B_2^* + 2B_1^*AB_2^*, C\rangle + 2\mathrm{Re}\langle B_2^*B_2x, v\rangle.$$

Moreover,

$$\mathrm{grad}\ \alpha(\Sigma, e_{n+1}) = \left(-\frac{2}{k_u\sigma_u^2}E, 0\right) \text{ where } E = \mathrm{diag}\ (0,\ldots,0,\overbrace{1,\ldots,1}^{k_u}).$$

Thus,

$$\alpha(\Sigma, e_{n+1})D(\|K\|^2)(\Sigma, e_{n+1})(\mathrm{grad}\ \alpha(\Sigma, e_{n+1})) + \|K(\Sigma, e_{n+1})\|^2\|\mathrm{grad}\ \alpha(\Sigma, e_{n+1})\|^2 =$$

$$\frac{2}{k_u\sigma_u^6}\left(-\sigma_u\mathrm{Re}\langle B_1B_1^*\Sigma + \Sigma B_2B_2^* - 2B_1\Sigma B_2^*, E\rangle + \|B_1\Sigma - \Sigma B_2\|^2 + \|B_2x\|^2\right).$$

This is positive from the proof of Proposition 19.

Hence, all the conditions of Corollary 15 are fulfilled and the Proposition follows. □

7.2 Proof of Theorem 31

Now we can prove Theorem 31 using Theorem 29 and Proposition 32. Note that we have $\mathcal{W} = \cup_{(k)}\mathcal{W}_{(k)}$ and α is smooth and self–convex in each $\mathcal{W}_{(k)}$ by Proposition 32. From Theorem 29 we just need to check that α is regular in $\mathcal{W}$ and that $\overline{\mathcal{SD}^2}\alpha > -\infty$. Since

$$\alpha = \sigma_n^{-2} \circ \pi_1,$$

where π_1 is the projection on the first coordinate, α is a smooth function. Now, consider the chart locally given by π_1^{-1}, and note that $\alpha \circ \pi_1^{-1} = \sigma_n^{-2}$ is regular in $\mathbb{GL}_{n,m}$ from the proof of Theorem 1. By definition, this means that α is regular in $\mathcal{W}_{(k)}$. Using the same argument, $\overline{\mathcal{SD}^2}\sigma_n^{-2} > -\infty$ in $\mathbb{GL}_{n,m}$ also implies that $\overline{\mathcal{SD}^2}\alpha > -\infty$ in $\mathcal{W}$ and we are done. □

8 Appendix

In this appendix we prove the following which gives a sufficient condition for the image of a submanifold under a group action to be a submanifold.

Lemma 33. *Let G be a Lie group acting on a smooth manifold $\mathcal{M}$, and $\mathcal{D}$ a smooth submanifold in $\mathcal{M}$ Define on $G \times \mathcal{D}$ the equivalence relation $(g, d)\mathcal{R}(g', d')$ when $gd = g'd'$. Let us denote by*

$$\pi : G \times \mathcal{D} \to (G \times \mathcal{D})/\mathcal{R}$$

the canonical surjection onto the quotient space, by i the map

$$i : (G \times \mathcal{D})/\mathcal{R} \to \mathcal{M}, \quad i(\pi(g, d)) = gd,$$

and by $\mathcal{P} = i((G\times\mathcal{D})/\mathcal{R})$ the image of i. When the three following conditions are satisfied

1. *The graph of $\mathcal{R}$ is a closed embedded submanifold in $(G\times\mathcal{D})\times(G\times\mathcal{D})$,*
2. *i is an immersion,*
3. *For every sequence $(x_k) \in (G \times \mathcal{D})/\mathcal{R}$ such that $(i(x_k))$ converges to $y \in \mathcal{P}$ the sequence (x_k) converges,*

then, $\mathcal{P}$ is an embedded submanifold in $\mathcal{M}$.

Proof. Let $\mathcal{X}$ be a manifold and let $\mathcal{R}$ denote an equivalence relation defined on $\mathcal{X}$. A classical necessary and sufficient condition to define on the quotient space $\mathcal{X}/\mathcal{R}$ a unique quotient manifold structure making the canonical surjection $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{R}$ a submersion is the following: the graph $\mathcal{G}$ of the relation is an embedded submanifold in $\mathcal{X} \times \mathcal{X}$ and the first projection $\mathrm{pr}_1 : \mathcal{G} \to \mathcal{X}$ is a submersion (See [1, Th.3.5.25]).

In the context of our lemma this condition comes from the first hypothesis and from the definition of the equivalence relation via the group action: let $((g, d),(h, e)) \in \mathcal{G}$. Let $(\dot{g}, \dot{d}) \in T_{(g,d)}(\mathcal{G} \times \mathcal{D})$. Let $a(t)$ be a curve in G and $b(t)$ a curve in $\mathcal{D}_{(k)}$ such that:

$$a(0) = g, \quad \dot{a}(0) = \dot{g}, \quad b(0) = d, \quad \dot{b}(0) = \dot{d}.$$

Then, consider the following curve contained in $(\mathcal{G} \times \mathcal{D}) \times (\mathcal{G} \times \mathcal{D})$ defined by:

$$\theta(t) = ((a(t), b(t)),(a(t)g^{-1}h, h^{-1}gb(t))).$$

It is clear that $\theta(0) = ((g, d), (h, e))$ because

$$h^{-1}gb(0) = h^{-1}gd = e.$$

It is also clear that $D\mathrm{pr}_1(\theta(0))\ \theta'(0) = (\dot{g}, \dot{d})$. Moreover, it is immediate that $\theta(t)$ is contained in $\mathcal{G}$. Thus, pr_1 is a submersion.

Let $f : \mathcal{Y} \to \mathcal{Z}$ be a smooth map between two manifolds. Its image $f(\mathcal{Y})$ is a submanifold in $\mathcal{Z}$ when f is an immersion and a homeomorphism onto its image.

By construction, i is smooth. It is a homeomorphism by the third hypothesis and an immersion by the second one. To check that it is injective, we have to show that if $gd = g'd'$, then $(g, d)\mathcal{R}(g', d')$. This follows from the construction of the relation $\mathcal{R}$.

□

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