A CONTINUATION METHOD TO SOLVE POLYNOMIAL SYSTEMS, AND ITS COMPLEXITY.

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Abstract. In a recent work [Shu09], Mike Shub obtained a new upper bound for the number of steps needed to continue a known zero η_0 of a system f_0, to a zero η_T of an input system f_T, following the path of pairs (f_t, η_t), where $f_t, t \in [0, T]$ is a polynomial system and $f_t(\eta_t) = 0$. He proved that, if one can choose the step–size in an optimal way, then the number of steps is essentially bounded by the length of the path of (f_t, η_t) in the so–called condition metric. However, the proof of that result in [Shu09] is not constructive. We give an explicit description of an algorithm which attains that complexity bound, including the choice of step–size.

1. Introduction

Continuation methods (also called homotopy or path–following methods) for solving systems of polynomial equations try to approximate solutions of a target system f by continuing one or more known solutions of a "simple" system g. They have been studied and used for years. A very brief list of references describing many practical and theoretical aspects of these methods is [GZ79, Ren87, Li93, Mor87, MS87, LVZ08, SW05, LT09]. In these works a path $f_t, t \in [0, T]$ is defined with extremes g and f (for example, $f_t = (1-t)g + tf$, $t \in [0, 1]$), so $f_0 = g$ and $f_T = f$. The space of polynomial systems with degrees bounded by some quantity has a natural structure of finite–dimensional complex vector space, and f_t is just a curve in that vector space. Under some widely satisfied regularity hypotheses, a known zero $\zeta_0 = \eta_0$ of f_0 can be continued to a zero η_t of f_t. Namely, we have $f_t(\eta_t) = 0$ which readily implies

$$\dot{f}_t(\eta_t) + Df_t(\eta_t)\dot{\eta}_t = 0, \text{ or equivalently } \dot{\eta}_t = Df_t(\eta_t)^{-1}\dot{f}_t.$$

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Continuation methods attempt to lift the solution path η_t to produce an approximation to η_T, a zero of $f_T = f$. In practice, a first "homotopy step" t_0 is chosen and a predictor method (for example Euler's approximation applied to the differential equality above) is used to produce an approximation z_{t_0} to $\zeta_1 = \eta_{t_0}$, and then a corrector method (like Newton's method) is used to get a better approximation of ζ_1. If no convergence of the corrector method is achieved, then t_0 is changed by a smaller step. This idea is repeated, generating $t_1, t_2, \ldots$ until we reach f_T.

In some of the papers cited above there are very impressive experimental results showing that methods based on this idea can produce approximations of one or more zeros of the target system very quickly. However, theoretical guarantees for the performance of these methods are still not known. We consider the two following questions:

- Q1 Can we describe analytically a choice of the homotopy steps $t_0, t_1, \ldots$ above? Can we guarantee that z_{t_i} is approximating the continued solution η_{t_i} of f_{t_i}, and not some other solution of f_{t_i}?
- Q2 Given a path f_t, can we control the total number of homotopy steps? Namely, what is the complexity of homotopy methods, in terms of some geometric or algebraic invariant of f_t?

In the list of references above, there is no general theory that can give satisfactory answers to these two questions, save for [Ren87] where they are addressed with probabilistic arguments.

During the nineties, a series of papers by Shub and Smale [SS93a, SS93b, SS93c, SS96, SS94] established the basis for the study of the complexity of homotopy methods. They started by considering the homogeneous (projective) version of the problem: fix $n \geq 1$ and for $r \in \mathbb{N}$, let $\mathcal{H}_r$ be the vector space of homogeneous polynomials of degree r with complex coefficients and unknowns $X_0, \ldots, X_n$. Then, let $(d) = (d_1, \ldots, d_n)$ be a list of positive degrees and let

$$\mathcal{H}_{(d)} = \mathcal{H}_{d_1} \times \cdots \times \mathcal{H}_{d_n},$$

namely $\mathcal{H}_{(d)}$ is the vector space of systems of n homogeneous polynomial equations of respective degrees $d_1, \ldots, d_n$, and elements in $\mathcal{H}_{(d)}$ are n–tuples $f = (f_1, \ldots, f_n)$. From now on, we let $d = \max\{d_1, \ldots, d_n\}$. Note that if an element in $\mathbb{C}^{n+1}$ is a zero of $f \in \mathcal{H}_{(d)}$, then every complex multiple of that element is also a zero of f. Thus, we consider zeros of f as points in the complex projective space $\mathbb{P}(\mathbb{C}^{n+1})$. It is useful to endow $\mathcal{H}_{(d)}$ with a Hermitian product and its associated norm. As in [SS93a], we choose Bombieri-Weyl inner product (sometimes called

Kostlan inner product): let $f = (f_1, \ldots, f_n), g = (g_1, \ldots, g_n) \in \mathcal{H}_{(d)}$. Consider f as a high–dimensional vector, containing the list of the coefficients the monomials of $f_1, \ldots, f_n$, and similarly for g. Then, the Bombieri-Weyl inner product $\langle f, g \rangle$ is the weighted Hermitian product of these two vectors, where the weight

$$\left(\frac{d_i!}{\alpha_0! \cdots \alpha_n!} \right)^{-1},$$

is associated with each monomial $X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ of f_i, g_i. A more detailed description of this Hermitian product including some interesting properties can be found in [BCSS98, Chapter 12.1]. Let

$$\mathbb{S} = \{ f \in \mathcal{H}_{(d)} : \|f\| = 1 \}$$

be the unit sphere in $\mathcal{H}_{(d)}$. Note that the (projective) zeros of a system $f \in \mathcal{H}_{(d)}$ are also those of $f/\|f\|$, and hence we may assume that our input systems f are always in $\mathbb{S}$.

Key to the works of Shub and Smale cited above is the so–called normalized condition number (sometimes denoted μ_{norm}, μ_{proj} or simply μ), defined as follows: given $f \in \mathcal{H}_{(d)}$ and $x \in \mathbb{C}^{n+1}$, let

$$\mu(f, x) = \|f\| \|(Df(x)|_{x^\perp})^{-1} Diag(\|x\|^{d_i - 1} d_i^{1/2})\|. \tag{1.1}$$

Here, $Df(x)|_{x^\perp}$ is just the differential matrix of f at x, restricted to the orthogonal complement $x^\perp$ of x. The quantity $\mu(f, x)$ depends only on the projective class of f and x, and it satisfies $\mu(f, x) \geq 1$ whenever $f(x) = 0$.

Finding decimal approximations to algebraic points sufficiently precise to distinguish a putative solution from some other algebraic point is a subtle problem. An elegant approach is provided by the concept of *approximate zero* (cf. [SS93a] or [Sma81] for the affine version.) First, recall projective Newton's method from [Shu93]:

$$N_{\mathbb{P}}(f)(z) = z - \left(Df(z)\big|_{z^\perp} \right)^{-1} f(z), \quad f \in \mathcal{H}_{(d)}, z \in \mathbb{P}\left(\mathbb{C}^{n+1}\right).$$

A point $z \in \mathbb{P}(\mathbb{C}^{n+1})$ is a (projective) approximate zero of a system $f \in \mathbb{S}$ if there exists an exact nondegenerate zero ζ of f such that all the iterates of projective Newton's method $N_{\mathbb{P}}(f)^i(z) = N_{\mathbb{P}}(f) \circ \cdots \circ N_{\mathbb{P}}(f)(z)$ (i times) exist and they satisfy $d_R(N_{\mathbb{P}}(f)^i(z), \zeta) \leq d_R(z, \zeta)/2^{2^i - 1}$ where d_R is the Riemannian distance in $\mathbb{P}(\mathbb{C}^{n+1})$. Namely, an approximate zero guarantees fast and secure convergence of the sequence of projective Newton's method iterates to an actual, exact zero of f.

It can be proven that if z is close enough to ζ in terms of $\mu(f, \zeta)$, then z is an approximate zero of f with associated zero ζ (in the spirit of Smale's γ–theory, with μ instead of γ.) See Lemma 6 below.

Now we describe with some more detail the general structure of the homotopy method proposed by Shub and Smale. Let $f \in \mathcal{H}_{(d)}$ be a target system to be solved, and let $g \in \mathcal{H}_{(d)}$ be another system that has a known approximate zero z_0 with associated exact zero some $\eta_0 \in \mathbb{P}(\mathbb{C}^{n+1})$. Consider some piecewise $\mathcal{C}^1$ curve $\{f_t : t \in [0,T]\}$ joining g and f, so that $f_0 = g$, $f_T = f$.

Under some widely satisfied regularity hypothesis (if no singular solution is found, or $\mathcal{C}_0(f_t,\eta_0) < \infty$ in the notation below), the curve f_t can be lifted to a piecewise C^1 curve of pairs

$$(f_t,\eta_t) \subseteq V = \{(f,\zeta) \in \mathbb{S} \times \mathbb{P}\left(\mathbb{C}^{n+1}\right) : \zeta \text{ is a zero of } f\}.$$

The set V is called the solution variety. This curve is completely determined from (f_t,η_0) so we denote it by $\Gamma(f_t,\eta_0)$.

Shub and Smale's homotopy method works as follows:

(1) Set $h_0 = g$, $z_0 = z$.

(2) Choose a small step $t_0 > 0$. Let $h_1 = f_{t_0} \in \mathbb{S}$. Let

$$z_1 = N_{\mathbb{P}}(h_1)(z_0).$$

The step t_0 must be small enough to guarantee that z_1 is an approximate zero of h_1, with associated zero $\zeta_1 = \eta_{t_0}$, the unique zero of h_1 which lies in the lifted path $\Gamma(f_t,\eta_0)$.

(3) For $i \geq 2$ define h_i, z_i, ζ_i inductively as follows. Choose a small step $t_{i-1} > 0$. Let $h_i = f_{t_0+\cdots+t_{i-1}} \in \mathbb{S}$. Let

$$z_i = N_{\mathbb{P}}(h_i)(z_{i-1}).$$

Again, the step t_{i-1} must be small enough to guarantee that z_i is an approximate zero of h_i, with associated zero $\zeta_i = \eta_{t_0+\cdots+t_{i-1}}$, the unique zero of h_i which lies in the lifted path $\Gamma(f_t,\eta_0)$.

Note that this description does not include the use of a predictor–corrector step, only projective Newton's method is used.

In [SS94, Theorem 6.1] and assuming that $f_t = (1-t)g + tf$ (i.e. linear homotopy), Shub and Smale proved that one may choose the steps t_i in such a way that the first question Q_1 above is answered in the affirmative, and that the total number of homotopy steps is at most

$$Cd^{3/2}\max\{\mu(f_t,\eta_t) : t \in [0,T]\}L, \tag{1.2}$$

where L is the length of the curve η_t, thus giving a satisfactory answer to question Q_2 above in the case of linear homotopy paths. In [BP08, BP09], this result is used to prove that randomized linear homotopy paths require a small (polynomial in the size of the input) number of homotopy steps to run, on the average. The algorithm in [BP09] works as follows: first, an initial pair (g,η_0) is chosen using a certain randomized procedure. Then, the path–following method of [SS94, Theorem 6.1] is used to approximate the solution path η_t associated to the linear homotopy $f_t = (1-t)g + tf$ and thus find an approximate zero of the input system f.

The average running time of the algorithm in [BP09] is already polynomial in the size of the input. However Mike Shub has pointed out in [Shu09] that the path–following method can be done much faster. The main outcome of [Shu09] is the following result, which generalizes and improves [SS94, Theorem 6.1]. Here and in the rest of the paper, $\lceil \lambda \rceil$ denotes the smallest integer greater than or equal to λ for $\lambda \in \mathbb{R}$.

Theorem 1 ([Shu09])**.** *Assume that $t \to f_t, t \in [0,T]$ is a piecewise C^1 curve. The number k of (projective) Newton's method steps necessary to guarantee that z_k is an approximate zero of f is bounded above by*

$$k \leq \lceil C d^{3/2} \mathcal{C}_0(f_t, \eta_0) \rceil, \tag{1.3}$$

where $C > 0$ is a universal constant and

$$\mathcal{C}_0(f_t, \eta_0) = \int_0^T \mu(f_t, \eta_t) \|(\dot{f}_t, \dot{\zeta}_t)\|_2 \, dt.$$

Theorem 1 says that, if the homotopy steps t_i are chosen properly, then $h_k = f$ for some k satisfying (1.3) and hence z_k is an approximate zero of $h_k = f$, the target system.

Remark 1. *We have defined the solution variety V as the set of pairs $(f, \zeta) \in \mathbb{S} \times \mathbb{P}(\mathbb{C}^{n+1})$ such that ζ is a zero of f. It turns out that V is a smooth submanifold of $\mathbb{S} \times \mathbb{P}(\mathbb{C}^{n+1})$, see* [BCSS98, p.193], *and hence it has a natural Riemannian structure (let us denote it by $\langle \cdot, \cdot \rangle_V$) inherited from the inner product Riemannian structure in $\mathbb{S} \times \mathbb{P}(\mathbb{C}^{n+1})$. Now, consider a new Riemannian structure in $W = V \setminus \{(f, \zeta) \in V : \mu(f, \zeta) = \infty\}$ given by*

$$\langle (\dot{f}, \dot{\zeta}), (\dot{g}, \dot{\eta}) \rangle_{\kappa, (f, \zeta)} = \mu(f, \zeta)^2 \langle (\dot{f}, \dot{\zeta}), (\dot{g}, \dot{\eta}) \rangle_V.$$

This new Riemannian structure defines a new metric in W, called the condition number metric, or condition metric for short, see [Shu09]. *The quantity $\mathcal{C}_0(f_t, \eta_0)$ is the length of the path (f_t, η_t) in the condition metric, which gives a nice geometrical interpretation of Theorem 1. In* [BS09, BP] *it is shown that the use of (1.3) instead of (1.2) yields a great improvement on the complexity of path–following methods, both for randomized algorithms and for theoretically optimal ones.*

However, the proof of Theorem 1 in [Shu09] is not constructive, and it does not provide an explicit, constructive description of the homotopy steps t_i. Thus, Q_1 above is not answered by Theorem 1 and an algorithm does not immediately follow from [Shu09]. The goal of this paper is to give a constructive version of Theorem 1, namely to describe analytically how to choose the t_i. As a drawback, we will need to ask our curves to be piecewise C^{1+Lip}, namely C^1 and with Lipschitz derivative, instead of only C^1 as in Theorem 1.

Remark 2. *Recall that a mapping $\beta : I \to \mathbb{R}^m$, $I = [a,b] \subseteq \mathbb{R}$, is Lipschitz if there exists a constant $K \geq 0$ such that $\|\beta(t) - \beta(t')\| \leq K|t - t'|$ for every $t, t' \in I$. The smallest of such K is called the Lipschitz constant of the map β. From Rademacher's Theorem (see for example* [EG92, p. 81]*), this implies that $\beta'(t)$ is defined a.e. in $[a,b]$. Moreover, clearly $\|\beta'(t)\| \leq K$ where defined. Any Lipschitz function $\beta : I \to \mathbb{R}$ where $U \subseteq \mathbb{R}$ is absolutely continuous and hence the following holds (see for example* [Rud87, Th. 7.18]*):*

$$\beta(t) = \beta(0) + \int_a^t \beta'(s)\, ds. \tag{1.4}$$

A function $\beta : I \to \mathbb{R}^m$, $I = [a,b) \subseteq \mathbb{R}$ is locally Lipschitz if it is Lipschitz in every compact subinterval $[a,b'] : a \leq b' < b$. Locally Lipschitz functions also satisfy (1.4) for $t < t_0$.

In [BL] we present an implementation (included in NAG4M2, the numerical algebraic geometry package of the computer algebra system Macaulay 2) of the algorithm described in this paper, and address other practical issues.

Remark 3. *Most commonly used homotopy paths f_t are certainly piecewise C^{1+Lip} (or even C^∞), so we believe that including this extra hypotheses is a minor drawback. Moreover, in view of Theorem 1, one should choose (if possible) paths f_t whose lifts (f_t, η_t) minimize $\mathcal{C}_0(f_t, \eta_t)$, namely length–minimizing geodesics with respect to the condition metric. These optimal paths, whose study has been started in* [Shu09, BS09, BD, BDMSa, BDMSb]*, are known from the arguments in* [BD, BDMSb] *to be of class C^{1+Lip}. Hence, they can be approximated using the algorithm described in this paper.*

I have (unsuccessfully) tried to produce an algorithm which only requires the curve to be of class C^1 and which uses no other extra hypotheses. This may be a difficult goal, for if only C^1 is assumed the integrand in the formula above might be an arbitrary continuous function, even a very pathological one. This question thus remains open.

The rest of this paper is organized as follows. In Section 2 we give the formal statement of our main results. Section 3 contains several technical lemmas used in our main proofs. Sections 4 and 5 contain the proofs of our two main theorems. A short Conclussions section is included at the end of the paper.

Notation 1. *The letters g, f, h, ζ, z are reserved for the meanings they have in the description above. We will also use the letters ℓ, v for polynomial systems; η for zeroes of systems; and x (resp. y) for projective (resp. affine) points.*

2. Main results

Now we give a practical version of the main theorem of [Shu09]. For $i \geq 0$ and $t \in [0, T]$ such that $h_i = f_t \in \mathbb{S}$, let $\dot{h}_i = \dot{f}_t = \frac{d}{dt} f_t \in T_{h_i}\mathbb{S}$ be the tangent vector to the curve $t \to f_t$ at h_i. Note that $\dot{f}_t$ (and thus $\dot{h}_i$) depends on the chosen parametrization of the path $t \to f_t$.

Recall that for any fixed $\ell \in \mathbb{S}$ and y in the unit sphere $\mathbb{S}(\mathbb{C}^{n+1})$ of $\mathbb{C}^{n+1}$, the differential matrix $Dl(y)$ is a $n \times (n+1)$ matrix with complex coefficients. Let $\mathcal{M}_{n+1}(\mathbb{C})$ be the set of $n+1$ square matrices with complex coefficients and define the diagonal matrix

$$\Lambda = Diag(d_1^{1/2}, \ldots, d_n^{1/2}, 1) \in \mathcal{M}_{n+1}(\mathbb{C}).$$

Then, consider the following mappings (where defined)

$$\begin{array}{rccl}
\phi: & \mathbb{S} \times \mathbb{S}(\mathbb{C}^{n+1}) & \to & \mathcal{M}_{n+1}(\mathbb{C}), \\
 & (\ell, y) & \mapsto & \Lambda^{-1}\binom{D\ell(y)}{y^*} \\
\chi_1: & \mathbb{S} \times \mathbb{S}(\mathbb{C}^{n+1}) & \to & \mathbb{R}, \\
 & (\ell, y) & \mapsto & \|\phi(\ell, y)^{-1}\| \\
\chi_2: & \mathbb{S} \times \mathcal{H}_{(d)} \times \mathbb{S}(\mathbb{C}^{n+1}) & \to & \mathbb{R}, \\
 & (\ell, v, y) & \mapsto & \left(\|v\|^2 + \left\| \binom{D\ell(y)}{y^*}^{-1} \binom{v(y)}{0} \right\|^2 \right)^{1/2} \\
\varphi: & \mathbb{S} \times \mathcal{H}_{(d)} \times \mathbb{S}(\mathbb{C}^{n+1}) & \to & \mathbb{R}, \\
 & (\ell, v, y) & \mapsto & \chi_1(\ell, y)\chi_2(\ell, v, y)
\end{array}$$

Here and throughout the paper, y^* is the conjugate transpose of y and $\|A\|$ denotes operator 2–norm of $A \in \mathcal{M}_{n+1}(\mathbb{C})$. Note that

$$\chi_1(\ell, y) \geq 1 \text{ for every } l, y. \tag{2.1}$$

Indeed, let $z \in \mathbb{C}^{n+1}$ be an element of the kernel of $Dl(y)$. Then,

$$\|\phi(\ell, y)z\| = \left\| \begin{pmatrix} 0 \\ \langle z, y \rangle \end{pmatrix} \right\| \leq \|z\|, \qquad \text{and hence } \chi_1(\ell, y) = \|\phi(\ell, y)^{-1}\| \geq 1.$$

The reader may check that χ_1, χ_2, φ only depend on the projective class of y in $\mathbb{S}(\mathbb{C}^{n+1})$. Thus, we will sometimes consider $\chi_1(\ell, x)$ with $x \in \mathbb{P}(\mathbb{C}^{n+1})$, meaning $\chi_1(\ell, y)$ for any representative y of x in $\mathbb{S}(\mathbb{C}^{n+1})$, and similarly for χ_2, φ.

If $t \to (\ell_t, \eta_t) \subseteq \mathbb{S} \times \mathbb{P}(\mathbb{C}^{n+1})$ is a C^1 curve such that η_t is a zero of ℓ_t, and if $t \to y_t$ is a horizontal lift[1] of $t \to \eta_t$ to the sphere $\mathbb{S}(\mathbb{C}^{n+1})$ then $\ell_t(y_t) = 0$ implies $\dot{\ell}_t(y_t) + D\ell_t(y_t)\dot{y}_t = 0$ and $\langle y_t, \dot{y}_t \rangle = 0$, that is $\dot{y}_t = \binom{D\ell_t(y_t)}{y_t^*}^{-1}\binom{-\dot{\ell}_t(y_t)}{0}$. Moreover, ℓ_t is a system of homogeneous polynomials and hence $D\ell_t(y_t)y_t = 0$ for every t. We hence conclude that

$$\chi_1(\ell_t, y_t) = \mu(\ell_t, y_t), \quad \varphi(\ell_t, \dot{\ell}_t, y_t) = \mu(\ell_t, y_t)\|(\dot{\ell}_t, \dot{y}_t)\|. \tag{2.2}$$

Note that the last formula is the length of the vector $(\dot{\ell}_t, \dot{y}_t)$ or equivalently $(\dot{\ell}_t, \dot{\eta}_t)$ in the condition metric. However, (2.2) does not hold in general (i.e. if η_t is not a zero of ℓ_t.)

2.1. **Explicit description of the algorithm.** We now describe the algorithm in its most general form. The particular case of linear homotopy paths will be addressed after the statements of the main theorems.

Assume that $t \to f_t \subseteq \mathbb{S}, 0 \leq t \leq T$ is a C^{1+Lip} curve (i.e. it is a C^1 curve and its derivative is Lipschitz.) Hence, $\dot{f}_t$ is Lipschitz and $\ddot{f}_t$ exists for almost every $t \in [0,T]$. Moreover, when defined, $\ddot{f}_t$ is bounded by the Lipschitz constant of $t \mapsto \dot{f}_t$. Assume moreover that

$$\|\ddot{f}_t\| \leq d^{3/2}H\|\dot{f}_t\|^2,$$

for almost every $t \in [0,T]$, where $H \geq 0$ is some constant. Note that if $\dot{f}_t \neq 0$ for $t \in [0,T]$ such a H exists. From now on, H (or an upper bound of H) is supposed to be known.

Let $P \geq 0$ be such that

$$P \geq \sqrt{2} + \sqrt{4 + 5H^2} \geq 2 + \sqrt{2}.$$

Let $c > 0$ be such that

$$c \leq \frac{(1 - \sqrt{2}u_0/2)^{\sqrt{2}}}{1 + \sqrt{2}u_0/2}\left(1 - \left(1 - \frac{u_0}{\sqrt{2} + 2u_0}\right)^{\frac{P}{\sqrt{2}}}\right), \tag{2.3}$$

where u_0 is as in Theorem 2 below. Set $h_0 = f_0, \dot{h}_0 = \dot{f}_0$ and let z_0 be an approximate zero of h_0 with associated exact zero η_0. As in the general scheme of homotopy methods described above, define h_i, z_i inductively as follows. Let

$$t_i \leq B_i = \frac{c}{Pd^{3/2}\varphi(h_i, \dot{h}_i, z_i)}, \quad i \geq 0. \tag{2.4}$$

[1] i.e. y_t is a representative of η_t and $\dot{y}_t$ is orthogonal to the complex line defined by y_t, for every t.

(If $t + B_i > T$, we just take $t_i = T - t$.) For the computation of $\varphi(h_i, \dot{h}_i, z_i)$ we choose any unit norm representative of z_i. Let $h_{i+1} = f_{t+t_i}$, $\dot{h}_{i+1} = \dot{f}_{t+t_i}$ and $z_{i+1} = N_{\mathbb{P}}(h_{i+1})(z_i)$.

Theorem 2. *With the notation and hypotheses above, assume that*

$$d_R(z_0, \eta_0) \leq \frac{u_0}{2d^{3/2}\mu(h_0, \eta_0)}, \quad u_0 = 0.17586...$$

where u_0 is the constant from Lemma 6 below. Then, for every $i \geq 0$, z_i is an approximate zero of h_i, with associated zero ζ_i, the unique zero of h_i that lies in the lifted path $\Gamma(f_t, \eta_0)$. Moreover,

$$d_R(z_i, \zeta_i) \leq \frac{u_0}{2d^{3/2}\mu(h_i, \zeta_i)}, \quad i \geq 1.$$

Theorem 3. *With the hypotheses of Theorem 2, assume moreover that*

$$\frac{c}{2Pd^{3/2}\varphi(h_i, \dot{h}_i, z_i)} \leq t_i \leq \frac{c}{Pd^{3/2}\varphi(h_i, \dot{h}_i, z_i)}, \quad i = 0, 1, 2...$$

namely t_i is within a factor of 2 of its upper bound (save possibly for the last step.) Then, if $\mathcal{C}_0(f_t, \eta_0) < \infty$, there exists $k \geq 0$ such that $f = h_{k+1}$. Namely the number of homotopy steps is at most k. Moreover,

$$k \leq \lceil C d^{3/2} \mathcal{C}_0(f_t, \eta_0) \rceil,$$

where

$$C = \frac{2P}{(1 - \sqrt{2}u_0/2)^{1+\sqrt{2}}} \left(\frac{1}{c} + \frac{1 + \sqrt{2}u_0/2}{\left(1 - \sqrt{2}u_0/2\right)^{\sqrt{2}}} \right).$$

In particular, if $\mathcal{C}_0(f_t, \eta_0) < \infty$ the algorithm finishes and outputs z_k, an approximate zero of $f = h_{k+1}$ with associated zero ζ_{k+1}, the unique zero of f that lies in the lifted path $\Gamma(f_t, \eta_0)$.

Remark 4. *Computing $\varphi(h_i, \dot{h}_i, z_i)$ involves computing the norm of a vector (for χ_2) and the norm of a matrix (for χ_1.) However, from Theorem 3 we only need to do this second and more difficult task approximately, for we just need to compute a quantity contained in the interval*

$$[\varphi(h_i, \dot{h}_i, z_i), 2\varphi(h_i, \dot{h}_i, z_i)].$$

Remark 5. *In particular, the number of steps is at most $1 + Cd^{3/2}\mathcal{C}_0(f_t, \eta_0)$. If the curve $t \to f_t$ is piecewise C^{1+Lip} we may divide the curve in L pieces, each of them of class C^{1+Lip} and satisfying a.e. $\|\ddot{\ell}_t\| \leq d^{3/2}H\|\dot{\ell}_t\|^2$ for a suitable $H \geq 0$. The theorem may then be applied to each of these pieces. The total number of steps is thus at most*

$$L + Cd^{3/2}\mathcal{C}_0(f_t, \eta_0),$$

by linearity of the integral.

Remark 6. *If more than one approximate zero of $g = f_0$ is known, the algorithm described above may be used to follow each of the homotopy paths starting at those zeros. By theorems 2 and 3, if the approximate zeros of g correspond to different exact zeros of g, and if $\mathcal{C}_0$ is finite for all the paths (i.e. if the algorithm finishes for every initial input), then the exact zeros associated with the output of the algorithm correspond to different exact zeros of $f = f_T$.*

2.2. **Example: linear homotopy.** We consider now the case of linear homotopy: Let $g, f \in \mathbb{S}$ be given, and let z_0 be an approximate zero of g with associated exact zero η_0. Consider the (short) arc of great circle joining g and f. That is, a portion of the unit circle (for the Bombieri–Weyl norm) in the real plane defined by the origin, g and f. We may parametrize this arc by arc–length as follows,

$$t \to f_t = g\cos(t) + \frac{f - Re(\langle f, g\rangle)g}{\sqrt{1 - Re(\langle f, g\rangle)^2}}\sin(t), \quad t \in \left[0, \arcsin\sqrt{1 - Re\langle f, g\rangle^2}\right],$$

where $Re(\cdot)$ stands for real part. Note that f_t is the projection on $\mathbb{S}$ of a segment, thus the name "linear homotopy". As f_t is arc–length parametrized, we have $\|\dot{f}_t\| \equiv 1$. Moreover, f_t is regular enough to be C^{1+Lip} and $\ddot{f}_t = -f_t$ yields

$$\|\ddot{f}\| = \|f_t\| = 1.$$

Hence, we may choose

$$H = \frac{1}{d^{3/2}}, \text{ and thus we need a } P \text{ such that } P \geq \sqrt{2} + \sqrt{4 + \frac{5}{d^3}}.$$

In particular, as $d \geq 2$ it suffices to take

$$P = \sqrt{2} + \sqrt{4 + \frac{5}{2^3}} = 3.56479487...$$

Moreover, from inequality (2.3) we just need

$$c \leq \frac{(1 - \sqrt{2}u_0/2)^{\sqrt{2}}}{1 + \sqrt{2}u_0/2}\left(1 - \left(1 - \frac{u_0}{\sqrt{2} + 2u_0}\right)^{\frac{3.56479487...}{\sqrt{2}}}\right) = 0.17126872...$$

so in the case of linear homotopy we may take $c = 0.17126872$ and we must thus choose the homotopy step in such a way that

$$\frac{0.04804448}{2d^{3/2}\varphi(h_i, \dot{h}_i, z_i)} \leq t_i \leq \frac{0.04804448}{d^{3/2}\varphi(h_i, \dot{h}_i, z_i)}.$$

The estimate of the number of steps given by Theorem 3 is then

$$\lceil 70.68842056 d^{3/2} \mathcal{C}_0(f_t, \eta_0) \rceil.$$

3. Technical lemmas

The proofs of theorems 2 and 3 will follow from the long and subtle computation of the rates of change of the functions χ_1, χ_2, φ studied in this section.

Recall first the higher derivative estimate from [SS93a] (see also [BCSS98, Prop. 1, p. 267]): let ℓ be a homogeneous polynomial of degree p. Let $0 \le k \le p$ and let $y, w_1, \dots, w_k \in \mathbb{C}^{n+1}$. Then,

(3.1)

$$\left| D^k l(y)(w_1, \dots, w_k) \right| \le p(p-1)\cdots(p-k+1)\|l\|\|y\|^{p-k}\|w_1\|\cdots\|w_k\|.$$

By abuse of notation, given a curve $t \to (\ell_t, v_t, y_t) \subseteq \mathbb{S} \times \mathcal{H}_{(d)} \times \mathbb{S}(\mathbb{C}^{n+1})$ we will denote

$$\phi(t) = \phi(\ell_t, y_t), \quad \chi_1(t) = \chi_1(\ell_t, y_t),$$

$$\chi_2(t) = \chi_2(\ell_t, v_t, y_t), \quad \varphi(t) = \varphi(\ell_t, v_t, y_t).$$

Lemma 1. *Let $\mathbb{S}(\mathbb{C}^{n+1})$ be the unit sphere in $\mathbb{C}^{n+1}$. Let $t \to (\ell_t, y_t) \in \mathbb{S} \times \mathbb{S}(\mathbb{C}^{n+1})$ be a C^1 curve, $0 \le t \le T$. Then,*

$$\left\| \frac{d}{dt}\phi(t) \right\| \le \sqrt{2d\|\dot{\ell}_t\|^2 + Q\|\dot{y}_t\|^2}, \quad 0 < t < T,$$

where $Q = 1 + 2d(d-1)^2$.

Proof. Recall that that $\phi(t) = \Lambda^{-1}\binom{D\ell_t(y_t)}{y_t^*}$. Hence,

$$\frac{d}{dt}\phi(t) = \Lambda^{-1}\begin{pmatrix} D\dot{\ell}_t(y_t) + D^2\ell_t(y_t)(\dot{y}_t) \\ \dot{y}_t^* \end{pmatrix},$$

where $D^2\ell_t(y_t)(\dot{y}_t)$ is the matrix satisfying $D^2\ell_t(y_t)(\dot{y}_t)y = D^2\ell_t(y_t)(\dot{y}_t, y)$, $y \in \mathbb{C}^{n+1}$. We consider the i–th row of that matrix, $1 \le i \le n$,

$$\left(\frac{d}{dt}\phi(t)\right)_i = d_i^{-1/2}(D(\dot{\ell}_t)_i(y_t) + D^2(\ell_t)_i(y_t)(\dot{y}_t)),$$

where $(\ell_t)_i$ is the i–th polynomial of the system ℓ_t, $1 \le i \le n$. From inequality (3.1) we conclude,

$$\left\| \left(\frac{d}{dt}\phi(\ell_t, y_t)\right)_i \right\|^2 \le \left(d_i^{1/2}\|(\dot{\ell}_t)_i\| + d_i^{1/2}(d_i - 1)\|(\ell_t)_i\|\|\dot{y}_t\| \right)^2 \le$$

$$\le 2d\|(\dot{\ell}_t)_i\|^2 + 2d(d-1)^2\|(\ell_t)_i\|^2\|\dot{y}_t\|^2.$$

Hence,

$$\left\|\frac{d}{dt}\phi(t)\right\|^2 \leq \sum_{i=1}^{n+1}\left\|\left(\frac{d}{dt}\phi(\ell_t,y_t)\right)_i\right\|^2 \leq$$

$$\|\dot{y}_t^*\|^2+\sum_{i=1}^{n}\left(2d\|(\dot{\ell}_t)_i\|^2+2d(d-1)^2\|(\ell_t)_i\|^2\|\dot{y}_t\|^2\right)=$$

$$\|\dot{y}_t\|^2+2d\|\dot{\ell}_t\|^2+2d(d-1)^2\|\ell_t\|^2\|\dot{y}_t\|^2=\|\dot{y}_t\|^2+2d\|\dot{\ell}_t\|^2+2d(d-1)^2\|\dot{y}_t\|^2,$$

and the lemma follows. □

Lemma 2. *Let $t\to(\ell_t,y_t)\in\mathbb{S}\times\mathbb{S}(\mathbb{C}^{n+1})$ be a C^1 curve, $0\leq t\leq T$. Let $t\to v_t\in\mathcal{H}_{(d)}$ be Lipschitz and let $K_t=\|\dot{v}_t\|$ where $\dot{v}_t$ is defined. $t\in[0,T]$. Assume that $\chi_1(0)<+\infty$. Then, $\chi_1(t)$ is a locally Lipschitz function in $[0,t_0)$ where $t_0=\sup\{t\in[0,T]:\chi_1(s)<+\infty\,\forall\, s\in[0,t]\}$ and*

$$|\chi_1'(t)|\leq\chi_1(t)^2\sqrt{2d\|\dot{\ell}_t\|^2+Q\|\dot{y}_t\|^2},\quad \textit{a.e. in } [0,t_0) \tag{3.2}$$

where $Q=1+2d(d-1)^2$. Moreover, $\chi_2(t)$ is a locally Lipschitz function in $[0,t_0)$ and

$$|\chi_2'(t)|\leq\sqrt{2\chi_1(t)^2\chi_2(t)^2(2d\|\dot{\ell}_t\|^2+Q\|\dot{y}_t\|^2)+5\chi_1(t)^2K_t^2}, \tag{3.3}$$

a.e. in $[0,t_0)$.

Proof. Note that χ_1 is a locally Lipschitz function for it is the composition of locally Lipschitz functions. Let $s,t\in[0,T]$. Then,

$$|\chi_1(t)-\chi_1(s)|=\left|\|\phi(t)^{-1}\|-\|\phi(s)^{-1}\|\right|\leq\|\phi(t)^{-1}-\phi(s)^{-1}\|.$$

On the other hand, $t\to\phi(t)$ is a C^1 map and hence,

$$\begin{array}{rcl}
\frac{\|\phi(t)^{-1}-\phi(s)^{-1}\|}{|t-s|} & \leq & \frac{1}{|t-s|}\int_s^t\left\|\frac{d}{du}\phi(u)^{-1}\right\|\,du\\
& = & \frac{1}{|t-s|}\int_s^t\left\|\phi(u)^{-1}\left(\frac{d}{du}\phi(u)\right)\phi(u)^{-1}\right\|\,du\\
& \leq & \frac{1}{|t-s|}\int_s^t\|\phi(u)^{-1}\|^2\left\|\frac{d}{du}\phi(u)\right\|\,du\\
& \underset{\textit{Lemma 1}}{\leq} & \frac{1}{|t-s|}\int_s^t\chi_1(u)^2\sqrt{2d\|\dot{\ell}_u\|^2+Q\|\dot{y}_u\|^2}\,du\\
& \leq & \max_{u\in[s,t]}\left(\chi_1(u)^2\sqrt{2d\|\dot{\ell}_u\|^2+Q\|\dot{y}_u\|^2}\right).
\end{array}$$

As $\chi_1(t)$ is locally Lipschitz, by Rademacher's Theorem it is differentiable a.e. in $[0,t_0)$ and satisfies

$$|\chi_1'(t)|=\lim_{s\to t}\frac{|\chi_1(t)-\chi_1(s)|}{|t-s|}\leq\chi_1(t)^2\sqrt{2d\|\dot{\ell}_t\|^2+Q\|\dot{y}_t\|^2},\quad \textit{a.e.}$$

Equation (3.2) follows.

On the other hand, $\chi_1 < \infty$ implies $\chi_2 < \infty$ is well defined in $[0, t_0)$. As v_t is Lipschitz, it is differentiable a.e. in $[0, t_0)$. Hence, $\chi_2(t)$ is also differentiable a.e. and

$$\chi_2'(t) = \frac{\mathbb{R}e\langle v_t, \dot{v}_t\rangle + \mathbb{R}e\langle \left(\begin{smallmatrix} D\ell_t(y_t) \\ y_t^* \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} v_t(y_t) \\ 0 \end{smallmatrix}\right), \frac{d}{dt}\left(\left(\begin{smallmatrix} D\ell_t(y_t) \\ y_t^* \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} v_t(y_t) \\ 0 \end{smallmatrix}\right)\right)\rangle}{\chi_2(t)}, \quad a.e.$$

Hence, the following holds a.e.

$$|\chi_2'(t)| \leq \left(\|\dot{v}_t\|^2 + \left\| \frac{d}{dt}\left(\begin{pmatrix} D\ell_t(y_t) \\ y_t^* \end{pmatrix}^{-1} \begin{pmatrix} v_t(y_t) \\ 0 \end{pmatrix}\right) \right\|^2 \right)^{1/2}.$$

Note that

$$\left\| \frac{d}{dt}\left(\begin{pmatrix} D\ell_t(y_t) \\ y_t^* \end{pmatrix}^{-1} \begin{pmatrix} v_t(y_t) \\ 0 \end{pmatrix}\right) \right\| =$$

$$\| - \left(\begin{smallmatrix} D\ell_t(y_t) \\ y_t^* \end{smallmatrix}\right)^{-1} \Lambda \frac{d}{dt}\left(\Lambda^{-1}\left(\begin{smallmatrix} D\ell_t(y_t) \\ y_t^* \end{smallmatrix}\right)\right) \left(\begin{smallmatrix} D\ell_t(y_t) \\ y_t^* \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} v_t(y_t) \\ 0 \end{smallmatrix}\right) +$$

$$\left(\begin{smallmatrix} D\ell_t(y_t) \\ y_t^* \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} \dot{v}_t(y_t) + Dv_t(y_t)\dot{y}_t \\ 0 \end{smallmatrix}\right) \|.$$

We find a bound for each of these summands. For the first one,

$$\left\| \begin{pmatrix} D\ell_t(y_t) \\ y_t^* \end{pmatrix}^{-1} \Lambda \frac{d}{dt}\left(\Lambda^{-1}\begin{pmatrix} D\ell_t(y_t) \\ y_t^* \end{pmatrix}\right) \begin{pmatrix} D\ell_t(y_t) \\ y_t^* \end{pmatrix}^{-1} \begin{pmatrix} v_t(y_t) \\ 0 \end{pmatrix} \right\| =$$

$$\left\| \phi(t)^{-1} \frac{d}{dt}(\phi(t)) \begin{pmatrix} D\ell_t(y_t) \\ y_t^* \end{pmatrix}^{-1} \begin{pmatrix} v_t(y_t) \\ 0 \end{pmatrix} \right\| \leq$$

$$\chi_1(t) \left\| \frac{d}{dt}(\phi(t)) \right\| \sqrt{\chi_2(t)^2 - \|v_t\|^2} \underset{Lemma\ 1}{\leq}$$

$$\chi_1(t)\sqrt{2d\|\dot{\ell}_t\|^2 + Q\|\dot{y}_t\|^2}\sqrt{\chi_2(t)^2 - \|v_t\|^2}.$$

For the second one,

$$\left\| \begin{pmatrix} D\ell_t(y_t) \\ y_t^* \end{pmatrix}^{-1} \begin{pmatrix} \dot{v}_t(y_t) + Dv_t(y_t)\dot{y}_t \\ 0 \end{pmatrix} \right\| \leq \chi_1(t) \left\| \Lambda^{-1} \begin{pmatrix} \dot{v}_t(y_t) + Dv_t(y_t)\dot{y}_t \\ 0 \end{pmatrix} \right\|.$$

We can bound this last expression using Inequality (3.1) as in the proof of Lemma 1, to get

$$\left\| \Lambda^{-1} \begin{pmatrix} \dot{v}_t(y_t) + Dv_t(y_t)\dot{y}_t \\ 0 \end{pmatrix} \right\| \leq \sqrt{2\|\dot{v}_t\|^2 + 2d\|v_t\|^2\|\dot{y}_t\|^2}.$$

We have thus proved:

$$\left\| \frac{d}{dt}\left(\begin{pmatrix} D\ell_t(y_t) \\ y_t^* \end{pmatrix}^{-1} \begin{pmatrix} v_t(y_t) \\ 0 \end{pmatrix} \right) \right\|^2 \leq$$

$$\left(\chi_1(t)\sqrt{\chi_2(t)^2 - \|v_t\|^2}\sqrt{2d\|\dot{\ell}_t\|^2 + Q\|\dot{y}_t\|^2} + \chi_1(t)\sqrt{2\|\dot{v}_t\|^2 + 2d\|v_t\|^2\|\dot{y}_t\|^2} \right)^2 \leq$$

$$2\chi_1(t)^2(\chi_2(t)^2 - \|v_t\|^2)(2d\|\dot{\ell}_t\|^2 + Q\|\dot{y}_t\|^2) + 2\chi_1(t)^2(2\|\dot{v}_t\|^2 + 2d\|v_t\|^2\|\dot{y}_t\|^2) \leq$$
$$2\chi_1(t)^2\chi_2(t)^2(2d\|\dot{\ell}_t\|^2 + Q\|\dot{y}_t\|^2) + 4\chi_1(t)^2\|\dot{v}_t\|^2$$

Hence,

$$|\chi_2'(t)|^2 \leq 2\chi_1(t)^2\chi_2(t)^2(2d\|\dot{\ell}_t\|^2 + Q\|\dot{y}_t\|^2) + (4\chi_1(t)^2 + 1)\|\dot{v}_t\|^2 \underset{(2.1)}{\leq}$$

$$2\chi_1(t)^2\chi_2(t)^2(2d\|\dot{\ell}_t\|^2 + Q\|\dot{y}_t\|^2) + 5\chi_1(t)^2\|\dot{v}_t\|^2,$$

and the second inequality of the lemma follows. □

Lemma 3. *Let $t \to (\ell_t, y_t) \in \mathbb{S}\times\mathbb{S}(\mathbb{C}^{n+1})$ be a C^{1+Lip} curve, $0 \leq t \leq T$. Assume that $t \to \dot{\ell}_t \in \mathcal{H}_{(d)}$ is Lipschitz and let $K_t = \|\ddot{\ell}_t\|$ where $\ddot{\ell}_t$ is defined. Assume that*

$$K_t \leq d^{3/2}H\|\dot{\ell}_t\|^2 \quad \textit{a.e. }, \qquad \textit{where } H \geq 0 \textit{ is some constant.}$$

Consider the curve $t \to (\ell_t, \dot{\ell}_t, y_t) \in \mathbb{S} \times \mathcal{H}_{(d)} \times \mathbb{S}(\mathbb{C}^{n+1})$. Assume that $\chi_1(0) < +\infty$. Assume moreover that

$$\|\dot{\ell}_t\|^2 + \|\dot{y}_t\|^2 \leq \chi_2(t)^2, \quad \forall\, t \in [0, T], \tag{3.4}$$

and let

$$P = \sqrt{2} + \sqrt{4 + 5H^2}.$$

Then, for $t < (P\, d^{3/2}\varphi(0))^{-1}$, we have

$$\frac{\varphi(0)}{1 + P\, d^{3/2}\varphi(0)t} \leq \varphi(t) \leq \frac{\varphi(0)}{1 - P\, d^{3/2}\varphi(0)t} < \infty.$$

Moreover,

$$d_R(y_t, y_0) \leq \frac{1}{\sqrt{2}d^{3/2}\chi_1(0)}\left(1 - \left(1 - Pd^{3/2}\varphi(0)t\right)^{\sqrt{2}/P}\right). \tag{3.5}$$

Proof. First, note that (3.4) and $Q = 1 + 2d(d-1)^2 \geq 2d$ implies

$$2d\|\dot{\ell}_t\|^2 + Q\|\dot{y}_t\|^2 \leq Q\chi_2(t)^2 \leq 2d^3\chi_2(t)^2. \tag{3.6}$$

Hence, if $t_0 = \sup\{t \in [0, T] : \chi_1(s) < +\infty, \forall s \in [0, t]\}$, (3.2) and (3.3) imply:

$$|\chi_1'(t)| \leq \sqrt{2}\, d^{3/2}\chi_1(t)^2\chi_2(t), \quad \text{a.e. in } [0, t_0),$$
$$|\chi_2'(t)| \leq \sqrt{4d^3\chi_1(t)^2\chi_2(t)^4 + 5\chi_1(t)^2K_t^2} \leq$$

$$\sqrt{4d^3\chi_1(t)^2\chi_2(t)^4+5\chi_1(t)^2d^3H^2\|\dot{\ell}_t\|^4}\leq$$
$$d^{3/2}\chi_1(t)\chi_2(t)^2\sqrt{4+5H^2},\quad \text{a.e. in } [0,t_0).$$

Now, φ is locally Lipschitz thus a.e. differentiable in $[0,t_0)$ and

$$|\varphi'(t)|\leq|\chi_1'(t)|\chi_2(t)+\chi_1(t)|\chi_2'(t)|\leq P\,d^{3/2}\varphi(t)^2,\quad \text{a.e. in } [0,t_0).$$

As $\varphi(t)>0$ for all $t\in[0,t_0)$, we conclude that $\varphi(t)^{-1}$ is also a locally Lipschitz function in $[0,,t_0)$ and

$$\left|\left(\frac{1}{\varphi(t)}\right)'\right|=\left|\frac{\varphi'(t)}{\varphi(t)^2}\right|\leq P\,d^{3/2},\quad \text{a.e. in } [0,t_0). \tag{3.7}$$

In particular, from (1.4) we conclude that

$$\left|\frac{1}{\varphi(t)}-\frac{1}{\varphi(0)}\right|\leq Pd^{3/2}t,\quad 0\leq t\leq t_0,$$

which yields the first claim of the lemma. For the second one, note that

$$\chi_2'(t)\leq\sqrt{4+5H^2}d^{3/2}\varphi(t)\chi_2(t)\leq\frac{(P-\sqrt{2})d^{3/2}\varphi(0)}{1-Pd^{3/2}t\varphi(0)}\chi_2(t),\quad \text{a.e. in } [0,t_0),$$

which from (1.4) implies

$$\chi_2(t)\leq\chi_2(0)+\int_0^t\frac{(P-\sqrt{2})d^{3/2}\varphi(0)}{1-Pd^{3/2}s\varphi(0)}\chi_2(s)\,ds.$$

Gronwall's Inequality (see for example [Fle80, Page 95]) then implies

$$\chi_2(t)\leq\frac{\chi_2(0)}{(1-Pd^{3/2}\varphi(0)t)^{\frac{P-\sqrt{2}}{P}}}.$$

Hence,

$$d_R(y_t,y_0)\leq\int_0^t\|\dot{y}_s\|\,ds\leq\int_0^t\chi_2(s)\,ds\leq\int_0^t\frac{\chi_2(0)}{(1-Pd^{3/2}\varphi(0)s)^{\frac{P-\sqrt{2}}{P}}}\,ds=$$
$$\frac{\chi_2(0)}{\sqrt{2}d^{3/2}\varphi(0)}\left(1-\left(1-Pd^{3/2}\varphi(0)t\right)^{\sqrt{2}/P}\right),$$

which proves the last assertion of the lemma. □

Lemma 4. *Let* $\ell_0,\ell\in\mathbb{S}$*,* $v\in\mathcal{H}_{(d)}$*,* $x_0,x\in\mathbb{P}(\mathbb{C}^{n+1})$*. Assume that* $\chi_1(\ell_0,x_0)<+\infty$*. Assume moreover that*

$$d_R(x_0,x)\leq\frac{a}{d^{3/2}\chi_1(\ell_0,x_0)},\ \text{some } a<1/\sqrt{2} \text{ and}$$

$$d_{\mathbb{S}}(\ell_0,\ell)\leq\frac{3a}{2d^{3/2}\chi_1(\ell_0,x_0)},$$

where $d_{\mathbb{S}}$ is the Riemannian distance in the sphere $\mathbb{S}$. Then,

$$\frac{\chi_1(\ell_0,x_0)}{1+\sqrt{2}a}\le\chi_1(\ell,x)\le\frac{\chi_1(\ell_0,x_0)}{1-\sqrt{2}a}\ \textit{and}$$

$$\varphi(\ell_0,v,x_0)\frac{(1-\sqrt{2}\,a)^{\sqrt{2}}}{1+\sqrt{2}\,a}\le\varphi(\ell,v,x)\le\frac{\varphi(\ell_0,v,x_0)}{(1-\sqrt{2}\,a)^{1+\sqrt{2}}},$$

for every $v\in\mathcal{H}_{(d)}$.

Proof. If $v=0$ the last assertion is trivial. We may thus consider that $v\neq 0$. Let $t\to(\ell_t,v,x_t)$, $0\le t\le T$ be C^1 curve with extremes (ℓ_0,v,x_0) and (ℓ,v,x) where $(\ell_T,v,x_T)=(\ell,v,x)$. From the assumptions on $d_{\mathbb{S}}(\ell_0,l)$ and $d_R(x_0,x)$ we can assume that the curve is parametrized such a way that

$$\|\dot\ell_t\|\le\frac{3}{2},\quad\|\dot x_t\|\le 1,\quad T\le\frac{a}{d^{3/2}\chi_1(0)}.$$

Let $t\to y_t$ be a horizontal lift of the curve $t\to x_t$ to the unit sphere $\mathbb{S}(\mathbb{C}^{n+1})$. Hence, $\|\dot y_t\|=\|\dot x_t\|\le 1$, and $\langle y_t,\dot y_t\rangle\equiv 0$. Note that we are under the hypotheses of Lemma 2 with $K_t\equiv 0$. Note that

$$2d\|\dot\ell_t\|^2+Q\|\dot y_t\|^2\le\frac{9d}{2}+Q=\frac{9d}{2}+1+2d(d-1)^2\underset{(d\ge2)}{\le}2d^3.$$

Let $t_0=\sup\{t\in[0,T]:\chi_1(s)<+\infty\,\forall s\in[0,t]\}$. Equations (3.2) and (3.3) then imply

$$|\chi_1'(t)|\le\sqrt{2}\,d^{3/2}\chi_1(t)^2,\quad\text{a.e. in }[0,t_0),$$
$$|\chi_2'(t)|\le 2\,d^{3/2}\chi_1(t)\chi_2(t),\quad\text{a.e. in }[0,t_0).$$

As in the proof of Lemma 3, the first inequality implies

$$\frac{\chi_1(0)}{1+\sqrt{2}\,d^{3/2}\chi_1(0)t}\le\chi_1(t)\le\frac{\chi_1(0)}{1-\sqrt{2}\,d^{3/2}\chi_1(0)t}. \tag{3.8}$$

Moreover,

$$|\chi_2'(t)|\le\frac{2\,d^{3/2}\chi_1(0)}{1-\sqrt{2}\,d^{3/2}\chi_1(0)t}\chi_2(t)\quad\text{a.e. in }[0,t_0).$$

As $\chi_2(t)$ is locally Lipschitz in $[0,t_0)$ and $\chi_2(t)\ge\|v\|>0$ is bounded away from 0, we have that $\log(\chi_2(t))$ is again locally Lipschitz and

$$\left|\frac{d\log(\chi_2(t))}{dt}\right|=\left|\frac{\chi_2'(t)}{\chi_2(t)}\right|\le\frac{2\,d^{3/2}\chi_1(0)}{1-\sqrt{2}\,d^{3/2}\chi_1(0)t}\quad\text{a.e. in }[0,t_0).$$

From (1.4), this implies that for $0\le t\le t_0$,

$$|\log(\chi_2(t))-\log(\chi_2(0))|\le\int_0^t\frac{2\,d^{3/2}\chi_1(0)}{1-\sqrt{2}\,d^{3/2}\chi_1(0)s}\,ds=$$

$$-\sqrt{2}\log(1-\sqrt{2}\,d^{3/2}\chi_1(0)t),$$

that is

$$\chi_2(0)(1-\sqrt{2}d^{3/2}\chi_1(0)t)^{\sqrt{2}} \le \chi_2(t) \le \frac{\chi_2(0)}{(1-\sqrt{2}d^{3/2}\chi_1(0)t)^{\sqrt{2}}}.$$

We conclude that $T = t_0$ (i.e. $\chi_1(t) < \infty$ for $t \in [0, T]$) and

$$\varphi(\ell, v, x) = \varphi(T) = \chi_1(T)\chi_2(T) \le \frac{\varphi(0)}{(1-\sqrt{2}d^{3/2}\chi_1(0)T)^{1+\sqrt{2}}}.$$

Finally,

$$\varphi(\ell, v, x) = \chi_1(T)\chi_2(T) \ge \varphi(0)\frac{(1-\sqrt{2}d^{3/2}\chi_1(0)T)^{\sqrt{2}}}{1+\sqrt{2}d^{3/2}\chi_1(0)T}.$$

The claims of the lemma follow from these last inequalities, (3.8) and the upper bound on T. □

Lemma 5. *Let $t \to \ell_t \in \mathbb{S}$, $0 \le t \le T$ be a C^{1+Lip} curve. Let $K_t = \|\ddot{\ell}_t\|$ where $\ddot{\ell}_t$ is defined, and assume that*

$$K_t \le d^{3/2}H\|\dot{\ell}_t\|^2 \quad a.e.\ , \qquad \textit{where } H \ge 0 \textit{ is some constant.}$$

Let $P = \sqrt{2} + \sqrt{4+5H^2}$. Let $\eta_0 \in \mathbb{P}(\mathbb{C}^{n+1})$ be a projective zero of ℓ_0 such that $\mu(\ell_0, \eta_0) < +\infty$. Let

$$t_0 = \frac{1}{Pd^{3/2}\varphi(\ell_0, \dot{\ell}_0, \eta_0)}.$$

Then, for $0 \le t < t_0$, η_0 can be continued to a zero $\eta_t \in \mathbb{P}(\mathbb{C}^{n+1})$ of ℓ_t in such a way that $t \to \eta_t$ is a C^{1+Lip} curve. Moreover, consider the curve $t \to (\ell_t, \dot{\ell}_t, \eta_t)$, $0 \le t < t_0$. Then, the following inequalities hold:

$$\frac{\varphi(0)}{1+P\,d^{3/2}\varphi(0)t} \le \varphi(t) \le \frac{\varphi(0)}{1-P\,d^{3/2}\varphi(0)t},$$

$$d_R(\eta_0, \eta_t) \le \frac{1}{\sqrt{2}d^{3/2}\chi_1(0)}\left(1-\left(1-Pd^{3/2}\varphi(0)t\right)^{\sqrt{2}/P}\right),$$

$$d_{\mathbb{S}}(\ell_0, \ell_t) \le \frac{1}{d^{3/2}H}\log\frac{1}{1-d^{3/2}H\chi_2(0)t}$$

Proof. Let $\pi : V \to \mathbb{S}$ be the projection on the first coordinate, defined from the solution variety V to the sphere of systems $\mathbb{S}$. It is known (see for example [BCSS98, Sections 12.3, 12.4]) that π admits a local inverse near $\pi(h, \eta)$ if and only if $\mu(h, \eta) = \chi_1(h, \eta) < +\infty$. We thus have that η_0 can be continued for $0 \le t < \varepsilon$, for some $0 < \varepsilon < t_0$. Now, consider the horizontally lifted path $y_t \in \mathbb{S}(\mathbb{C}^{n+1})$ where y_0 is some unit norm affine representative of η_0. Hence, the Hermitian product $\langle y_t, \dot y_t\rangle$ is equal to 0. Moreover, the equations $\ell_t(y_t) \equiv 0$ and $\langle y_t, \dot y_t\rangle \equiv 0$ imply:

$$\dot y_t = \begin{pmatrix} D\ell_t(y_t) \\ y_t^* \end{pmatrix}^{-1} \begin{pmatrix} -\dot\ell_t(y_t) \\ 0 \end{pmatrix},$$

which implies (3.4). Thus, all the conditions of Lemma 3 are satisfied for the curve $t \to (\ell_t, \dot\ell_t, y_t)$, $0 \le t < \varepsilon$. In particular, using Inequality (3.5) it is easy to see that for any sequence $t_i \overset{i\to\infty}{\to} \varepsilon$, the sequence y_{t_i} is a Cauchy sequence, which implies that the curve y_t converges to some $y_\varepsilon \in \mathbb{S}(\mathbb{C}^{n+1})$ as $t \to \varepsilon$. Moreover, y_ε is an (affine) zero of ℓ_ε and

$$\frac{\varphi(0)}{1 + P\, d^{3/2}\varphi(0)\varepsilon} \le \varphi(\ell_\varepsilon, y_\varepsilon) \le \frac{\varphi(0)}{1 - P\, d^{3/2}\varphi(0)\varepsilon} < +\infty$$

In particular, π is again locally invertible at $(\ell_\varepsilon, \eta_\varepsilon)$ where $\eta_\varepsilon \in \mathbb{P}(\mathbb{C}^{n+1})$ is the projective class of y_ε. Thus η_t can be continued for $0 < t < \varepsilon + \varepsilon'$. We conclude thus that η_t can be continued while $t < t_0$, and from Lemma 3, for $0 \le t < t_0$ we have

$$\frac{\varphi(0)}{1 + P\, d^{3/2}\varphi(0)t} \le \varphi(t) \le \frac{\varphi(0)}{1 - P\, d^{3/2}\varphi(0)t},$$

as wanted. Inequality (3.5) of Lemma 3 yields the bound for $d_R(\eta_0, \eta_t) = d_R(y_0, y_t)$. As for the last assertion of the lemma, note that $\|\dot\ell_t\|$ is locally Lipschitz, thus differentiable a.e. and

$$\frac{d}{dt}\|\dot\ell_t\| \le \|\ddot\ell_t\| = K_t \le d^{3/2} H \|\dot\ell_t\|^2, \quad \text{a.e. in } [0, t_0),$$

which as in the proof of Lemma 3 implies

$$\|\dot\ell_t\| \le \frac{\|\dot\ell_0\|}{1 - d^{3/2}H\|\dot\ell_0\|t}.$$

Finally,

$$d_{\mathbb{S}}(\ell_0, \ell_t) \le \int_0^t \|\dot\ell_s\|\, ds \le \int_0^t \frac{\|\dot\ell_0\|}{1 - d^{3/2}H\|\dot\ell_0\|s}\, ds \le$$

$$\frac{1}{d^{3/2}H} \log \frac{1}{1 - d^{3/2}H\|\dot\ell_0\|t} \le \frac{1}{d^{3/2}H} \log \frac{1}{1 - d^{3/2}H\chi_2(0)t},$$

as wanted. □

We will use the following result which is essentially included in [Shu09]. Recall that for $x, \eta \in \mathbb{P}(\mathbb{C}^{n+1})$, $d_R(x, \eta)$ is the Riemannian distance between these two points, namely the length of the shortest path joining x and η.

Lemma 6. *Let $\ell \in \mathbb{S}$ have a zero $\eta \in \mathbb{P}(\mathbb{C}^{n+1})$ and let $x \in \mathbb{P}(\mathbb{C}^{n+1})$ be such that $d_R(x,\eta) \leq u_0(d^{3/2}\mu(\ell,\eta))^{-1}$ where $u_0 = 0.17586...$ is a universal constant. Then x is an approximate zero of ℓ with associated zero η. That is, the sequence*

$$x_0 = x, \quad x_{i+1} = N_{\mathbb{P}}(\ell)(x_i), \quad i \geq 0,$$

is well–defined and it satisfies $d_R(x_i,\eta) \leq \frac{d_R(x,\eta)}{2^{2^i-1}}$. In particular, $d_R(x_1,\eta) \leq d_R(x,\eta)/2$.

[Shu09, Th. 2] is this same result but the constant in [Shu09, Th. 2] is $u_0 = 1 - \sqrt{7/8} \approx 0.06458$ instead of 0.17586 here.

Proof. The lemma is proved following the argument in [Shu09, Th. 2] and optimizing the constants there. We first prove that if $u \leq 2^{3/2}\arctan\left(\frac{3-\sqrt{7}}{2^{3/2}}\right)$ and $d_R(x,\eta) \leq u(d^{3/2}\mu(\ell,\eta))^{-1}$ then $d_R(N_{\mathbb{P}}(\ell)(x),\eta) \leq \frac{\lambda^2 u}{\psi(\lambda u)} d_R(x,\eta)$ where

$$\psi(r) = 1 - 4r + 2r^2, \quad \lambda = \frac{\frac{3-\sqrt{7}}{2^{3/2}}}{\arctan\left(\frac{3-\sqrt{7}}{2^{3/2}}\right)} = 1.00520714...$$

Indeed, note that $d \geq 2$ and $\mu \geq 1$ implies

$$d_R(x,\eta) \leq u(d^{3/2}\mu(\ell,\eta))^{-1} \leq u/2^{3/2} \leq \arctan\left(\frac{3-\sqrt{7}}{2^{3/2}}\right), \Longrightarrow$$

$$tan(d_R(x,\eta)) \leq \lambda d_R(x,\eta) \leq \frac{3-\sqrt{7}}{d^{3/2}\mu(\ell,\eta)}.$$

From [BCSS98, Lemma 1 and Remark 1, page 263] this implies that $N_{\mathbb{P}}(\ell)(x)$ is well–defined and

$$d_R(N_{\mathbb{P}}(\ell)(x),\eta) \leq \tan(d_R(N_{\mathbb{P}}(\ell)(x),\eta)) \leq$$

$$\frac{\lambda u}{\psi(\lambda u)} \tan(d_R(x,\eta)) \leq \frac{\lambda^2 u d_R(x,\eta)}{\psi(\lambda u)}.$$

We have thus proved a sharp version of [Shu09, Lemma 1] (where λ was chosen to be 2.) The rest of the proof of the lemma is an induction argument identical to the proof of [Shu09, Th. 2]. Our u_0 is the smallest positive number satisfying

$$\frac{\lambda^2 u_0}{\psi(\lambda u_0)} = \frac{1}{2}, \qquad \text{that is } u_0 \approx 0.17586...$$

Any lower bound of this number satisfies the claim of the lemma. □

4. Proof of Theorem 2

Recall that we have chosen

$$t_0 \leq \frac{c}{Pd^{3/2}\varphi(h_0, \dot{h}_0, z_0)}, \tag{4.1}$$

where c is a positive constant satisfying (2.3). The reader may check that (2.3) implies

$$c' = c\frac{1+\sqrt{2}u_0/2}{(1-\sqrt{2}u_0/2)^{\sqrt{2}}} < 1. \tag{4.2}$$

Moreover,

$$\frac{H\|\dot{h}_0\|}{P\varphi(h_0, \dot{h}_0, z_0)} \leq 1.$$

The proof is by induction on i. Thus, by our earlier hypotheses, the base case $i = 0$ of our induction follows. So,

$$d_R(z_0, \zeta_0) \leq \frac{u_0}{2d^{3/2}\mu(h_0, \zeta_0)} = \frac{u_0}{2d^{3/2}\chi_1(h_0, \zeta_0)} \tag{4.3}$$

From Lemma 4 we have,

$$\varphi(h_0, \dot{h}_0, \zeta_0)\frac{(1-\sqrt{2}u_0/2)^{\sqrt{2}}}{1+\sqrt{2}u_0/2} \leq \varphi(h_0, \dot{h}_0, z_0) \leq \frac{\varphi(h_0, \dot{h}_0, \zeta_0)}{(1-\sqrt{2}u_0/2)^{1+\sqrt{2}}}. \tag{4.4}$$

To simplify our notation, we just show how the first induction step goes. Note that

$$t_0 \leq \frac{c'}{Pd^{3/2}\varphi(h_0, \dot{h}_0, \zeta_0)} \underset{(4.2)}{<} \frac{1}{Pd^{3/2}\varphi(h_0, \dot{h}_0, \zeta_0)}. \tag{4.5}$$

From Lemma 5, for $0 \leq t \leq t_0$, ζ_0 can be continued to a unique zero $\eta_t \in \mathbb{P}(\mathbb{C}^{n+1})$ of f_t in such a way that $\zeta_0 = \eta_0$ and $t \to \eta_t$ is a C^{1+Lip} curve. Then, let ζ_1 from Theorem 2 be η_{t_0}.

The induction will be finished if we prove that

$$d_R(z_0, \zeta_1)\chi_1(h_1, \zeta_1) \leq \frac{u_0}{d^{3/2}}, \tag{4.6}$$

for in that case, from Lemma 6, z_0 is an approximate zero of h_1 with associated zero ζ_1, and so is $z_1 = N_{\mathbb{P}}(h_1)(z_0)$. Moreover,

$$d_R(z_1, \zeta_1) \leq \frac{d_R(z_0, \zeta_1)}{2} \leq \frac{u_0}{2d^{3/2}\mu(h_1, \zeta_1)}.$$

finishing the induction step and the proof of Theorem 2.

We thus have to prove (4.6). Let

$$\theta(t) = d_{\mathbb{S}(\mathbb{C}^{n+1})}(x_0, y_t)\chi_1(f_t, y_t), \quad 0 \leq t \leq t_0,$$

where x_0 is a unit norm representative of z_0, $d_{\mathbb{S}(\mathbb{C}^{n+1})}$ is the Riemannian distance in $\mathbb{S}(\mathbb{C}^{n+1})$ and y_t is a horizontal lift of η_t to $\mathbb{S}(\mathbb{C}^{n+1})$ such that

$$d_{\mathbb{S}(\mathbb{C}^{n+1})}(x_0, y_0) \le \frac{u_0}{2d^{3/2}\chi_1(h_0, \zeta_0)}, \tag{4.7}$$

whose existence is granted by (4.3). Then, $\theta(t)$ is a Lipschitz function of t and hence it is almost everywhere differentiable. Moreover, (4.3) and (4.7) imply that $\theta(0) \le u_0/(2d^{3/2})$. Finally, writing $\chi_1(t)$ (resp. $\varphi(t)$) for $\chi_1(f_t, y_t)$ (resp. $\varphi(f_t, \dot{f}_t, y_t)$),

$$\theta'(t) \le \|\dot{y}_t\|\chi_1(t) + d_{\mathbb{S}(\mathbb{C}^{n+1})}(x_0, y_t)\left|\frac{d}{dt}\chi_1(t)\right| \underset{(2.2),(3.2)}{\le}$$

$$\varphi(t) + \theta(t)\chi_1(t)\sqrt{2d\|\dot{\ell}_t\|^2 + Q\|\dot{y}_t\|^2} \underset{(3.6)}{\le} \varphi(t) + \theta(t)\varphi(t)\sqrt{2}d^{3/2}.$$

Thus, we get

$$\frac{\theta'(t)}{1 + \sqrt{2}d^{3/2}\theta(t)} \le \varphi(t), \quad \theta(0) \le u_0/(2d^{3/2}).$$

Gronwall's inequality applied to $\tilde{\theta}(t) = 1 + \sqrt{2}d^{3/2}\theta(t)$ then yields

$$\theta(t) \le \frac{1}{\sqrt{2}d^{3/2}}\left(\left(1 + \frac{u_0}{\sqrt{2}}\right) exp\left(\sqrt{2}d^{3/2}\int_0^t \varphi(s)\,ds\right) - 1\right).$$

From Lemma 5 we know that

$$\varphi(s) \le \frac{\varphi(0)}{1 - P\,d^{3/2}\varphi(0)s}, \quad 0 \le s \le t_0,$$

which yields

$$\theta(t) \le \frac{1}{\sqrt{2}d^{3/2}}\left(\left(1 + \frac{u_0}{\sqrt{2}}\right)\left(\frac{1}{1 - P\varphi(0)d^{3/2}t}\right)^{\sqrt{2}/P} - 1\right), \quad 0 \le t \le t_0.$$

In particular, from (4.5) we have,

$$\theta(t_0) \le \frac{1}{\sqrt{2}d^{3/2}}\left(\left(1 + \frac{u_0}{\sqrt{2}}\right)\left(\frac{1}{1 - c'}\right)^{\sqrt{2}/P} - 1\right).$$

Our choice of c is such that the right-hand term in this last equation is at most $u_0/d^{3/2}$. Thus, we get $\theta(t_0) \le u_0/d^{3/2}$, namely

$$d_{\mathbb{S}(\mathbb{C}^{n+1})}(x_0, y_{t_0})\chi_1(f_{t_0}, y_{t_0}) \le \frac{u_0}{d^{3/2}}.$$

The projective distance $d_R(z_0, \eta_{t_0})$ is the minimum of the distances between any unit norm affine representatives of z_0 and η_{t_0}. Thus, we conclude

$$d_R(z_0, \eta_{t_0})\chi_1(f_{t_0}, \eta_{t_0}) \le \frac{u_0}{d^{3/2}},$$

that is (4.6). The theorem is proved.

5. Proof of Theorem 3

The proof of Theorem 3 is similar to that of the main result of [Shu09]. We use the notation of Section 4.

From Lemma 5, (4.1) and (4.2),

$$\varphi(0) \leq \varphi(s)(1+c'), \quad 0 \leq s \leq t_0. \tag{5.1}$$

Then, if $h_1 \neq f$ (i.e. if the homotopy does not finish in one step),

$$t_0 \geq \frac{c}{2Pd^{3/2}\varphi(h_0,\dot{h}_0,z_0)} \underset{(4.4)}{\geq} \frac{c\,(1-\sqrt{2}u_0/2)^{1+\sqrt{2}}}{2Pd^{3/2}\varphi(h_0,\dot{h}_0,\zeta_0)} \underset{(5.1)}{\geq}$$

$$\frac{c\,(1-\sqrt{2}u_0/2)^{1+\sqrt{2}}}{2Pd^{3/2}\varphi(s)(1+c')}, \quad 0 \leq s \leq t_0.$$

This implies

$$\int_0^{t_0} \varphi(s)\,ds \geq \frac{c\,(1-\sqrt{2}u_0/2)^{1+\sqrt{2}}}{2Pd^{3/2}(1+c')}.$$

Similarly, as far as $h_{i+1} \neq f$ we have

$$\int_{t_0+\cdots+t_{i-1}}^{t_0+\cdots+t_i} \varphi(s)\,ds \geq \frac{c\,(1-\sqrt{2}u_0/2)^{1+\sqrt{2}}}{2Pd^{3/2}(1+c')}, \quad i \geq 1,$$

where η_s is the unique zero of f_s in $\Gamma(f_t,\zeta_0)$. We conclude that if $h_{i+1} \neq f$, necessarily

$$\int_0^{t_0+\cdots+t_i} \varphi(s)\,ds \geq \frac{ci\,(1-\sqrt{2}u_0/2)^{1+\sqrt{2}}}{2Pd^{3/2}(1+c')}.$$

As $t_0+\cdots+t_i < T$, we have that if $h_{i+1} \neq f$,

$$\frac{ci\,(1-\sqrt{2}u_0/2)^{1+\sqrt{2}}}{2Pd^{3/2}(1+c')} < \int_0^T \varphi(f_s,\dot{f}_s,\eta_s)\,ds \underset{(2.2)}{=} \mathcal{C}_0(f_t,\zeta_0),$$

namely,

$$i < Pd^{3/2}\mathcal{C}_0(f_t,\zeta_0)\frac{2(1+c')}{c\,(1-\sqrt{2}u_0/2)^{1+\sqrt{2}}}.$$

We conclude that for k greater than this quantity, necessarily $h_{k+1} = f$ and we are done.

6. Conclussions

We describe a new path–following algorithm which can be used to solve systems of homogeneous polynomial equations by continuation. Given a path $t \mapsto f_t$ where f_t is a polynomial system for $t \in [0,T]$ and given an approximate zero z_0 of the initial system f_0 with associated (exact) zero some η_0, we describe how to approximate the solution path η_t in such a way that η_t is a zero of f_t fot $t \in [0,T]$. The output of our algorithm is an approximate zero z of f_T with associated zero η_T. Two main features of our algorithm are certification of the output and analysis of the number of steps. Our algorithm is designed to lift paths $t \mapsto f_t$ which are of class C^{1+Lip}, that is C^1 with Lipschitz derivative. It attains the complexity bound of the main result in [Shu09], namely the number of homotopy steps is proportional to the length of the solution path in the condition metric. Our result opens the door to experimental research in complexity issues (to appear in [BL]) and justifies theoretical works on the complexity of Bezóut's Theorem as [BP]. The problem of designing this algorithm for general C^1 paths as stated in [Shu09] remains open.

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