ESTIMATES ON THE CONDITION NUMBER OF RANDOM, RANK-DEFICIENT MATRICES.

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Abstract. Let $r \leq m \leq n \in \mathbb{N}$ and let A be a rank r matrix of size $m \times n$, with entries in $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. The condition number of A, which measures the sensitivity of $Ker(A)$ to small perturbations of A, is defined as $\kappa(A) = \|A\|\|A^\dagger\|$, where † denotes Moore-Penrose pseudoinversion. In this paper, we prove sharp lower and upper bounds on the probability distribution of this condition number, when the set of rank r, $m \times n$ matrices is endowed with the natural probability measure coming from the Gaussian measure in $\mathbb{K}^{m\times n}$. We also prove an upper bound estimate for the expected value of $\log \kappa$ in this setting.

1. Introduction

Let $m, n \in \mathbb{N}$ be two positive integer numbers, and let $\mathcal{M}_{m,n}(\mathbb{K})$ be the set of $m \times n$ matrices with coefficients in $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given a rank r matrix $A \in \mathcal{M}_{m,n}(\mathbb{K})$, we denote by $\sigma_1(A), \ldots, \sigma_r(A)$ its non-zero singular values in decreasing order. The condition number of a (possibly rank-deficient) matrix is $\kappa(A) = \sigma_1(A)/\sigma_r(A)$, or equivalently

$$\kappa(A) = \|A\|_2\|A^\dagger\|_2,$$

where $\|\cdot\|_2$ is the operator norm. The condition number for $m = n = r$ was introduced in Turing's seminal paper [Tur48], as a measure of the sensitivity of the solution of a linear system $Ax = b$ to small perturbations of A. For rank-deficient matrices, the condition number $\kappa(A)$ as defined above measures the sensitivity of $Ker(A)$ and $A^\dagger$ to small perturbations of A. See, for example, [Kah00], [SS90, p. 145] and [BP07, Prop. 34]. Note that κ is invariant under transposing, and $\kappa = 1$ if $n = 1$ or $m = 1$. Thus, it makes sense to consider just the case $2 \leq m \leq n$.

A series of results begun in the 80's studied the probability distribution of κ for full-rank matrices with Gaussian coefficients, or equivalently, random matrices in the unit sphere $\mathbb{S}(\mathcal{M}_{m\times n}(\mathbb{K}))$ or the projective space $\mathbb{P}(\mathcal{M}_{m\times n}(\mathbb{K}))$. The first estimations are due to Smale [Sma85], Renegar [Ren87] and Demmel [Dem88]. Edelman [Ede88, Ede89, Ede92] showed the exact distribution of a scaled variant of κ in the case that $m = n = r$ and $\mathbb{K} = \mathbb{C}$, as well as limiting distributions and expected logarithms of κ in the case that $m = r \leq n$. Very sharp estimates on the probability distribution of κ in the case that $m = r \leq n$ have been obtained by Chen and Dongarra [CD05]. Using the notations in the following table,

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	β	c	C	u
$\mathbb{K}=\mathbb{R}$	1	0.245	6.414	2.258
$\mathbb{K}=\mathbb{C}$	2	0.319	6.298	2.24

the main result in [CD05] reads:

Proposition 1. *[Chen & Dongarra, 2005] Let $t \geq n-m+1$. Then,*

$$\frac{1}{(2\pi)^{\beta/2}}\left(\frac{c}{t}\right)^{\beta(n-m+1)} \leq \mathrm{P}\left[\frac{\kappa(A)}{n/(n-m+1)} > t : A \in \mathbb{GL}_{m\times n}\right] \leq \frac{1}{(2\pi)^{\beta/2}}\left(\frac{C}{t}\right)^{\beta(n-m+1)}.$$

Moreover,

$$\mathrm{E}[\log(\kappa(A)) : A \in \mathbb{GL}_{m\times n}] \leq \log\frac{n}{n-m+1} + u$$

Edelman and Sutton [ES05] proved asymptotically tight estimates for this case. Other works use techniques similar to those of [Ren87, BP07] to estimate the so-called smooth condition number. See the paper by Bürgisser, Cucker and Lotz [BCL06] and references therein.

Much less is known in the case of rank-deficient matrices. For fixed $2 \leq r \leq m \leq n$, let

$$\Sigma = \Sigma^r_{m\times n} = \{A \in \mathbb{S}(\mathcal{M}_{m\times n}(\mathbb{K})) : \mathrm{rank}(A) = r\}.$$

That is, Σ is the set of rank r matrices of size $m \times n$, lying in the unit sphere of $\mathcal{M}_{m\times n}(\mathbb{K})$ (for the Frobenius norm $\|A\|_F = tr(A^*A)^{1/2}$). For example, if $m = n$ and $r = n-1$, then Σ is, up to a lower-dimensional set, the set of singular $n \times n$ matrices with unit Frobenius norm.

The set Σ is a submanifold of $\mathcal{M}_{m\times n}(\mathbb{K})$, and $\mathrm{codim}_{\mathbb{K}}(\Sigma) = (n-r)(m-r)$ (see for example [AVGZ86]). Hence, it inherits from $\mathcal{M}_{m\times n}(\mathbb{K})$ a structure of Riemannian manifold with total finite volume (see Theorem 2 below). After normalization by the total volume, this yields an associated probability measure on Σ.

A natural object of study is the quantity

$$\mathrm{P}[A \in \Sigma : \kappa(A) > t], \quad t \geq 1. \tag{1.1}$$

Luis M. Pardo and I obtained upper bounds for this quantity in the complex case. See [BP05, BP07], where the following inequality was proved:

$$\mathrm{P}[A \in \Sigma : \kappa_D(A) > t] \leq C(n,m,r)t^{-2(n+m-2r+1)} \quad \mathbb{K} = \mathbb{C}.$$

Here $C(n,m,r)$ is a constant independent of t. The number κ_D is a scaled version of $\kappa(A)$. It satisfies $\kappa \leq \kappa_D \leq \sqrt{m}\kappa$, and has a well-known geometric interpretation as the inverse of the distance to the set where A loses rank. In particular, for the case that $r = n-1$ and $m = n$, this yields

$$\mathrm{P}[A \in \Sigma^{n-1}_{n\times n} : \kappa_D(A) > t] \leq \left(\frac{n^{10/3}}{t}\right)^6 \quad \mathbb{K} = \mathbb{C}. \tag{1.2}$$

However, we were not able to extend the methods of [BP07] to the case that $\mathbb{K} = \mathbb{R}$, and this remained as an open problem. Since [BP05, BP07], an increasing interest on these type of estimates has been expressed by several of our colleagues, but no new results have appeared yet. During Peter Bürgisser's plenary talk at FoCM'08 conference, Felipe Cucker described the real case study of the quantity (1.1) as, "a wide open question."

In this paper, we give a precise answer to this question for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, by reducing the problem to the full-rank case. As a result, we can improve our previous bounds of [BP07] for $\mathbb{K}=\mathbb{C}$, as well as introduce lower bounds. For example, our upper bound for equation (1.2) (for the non-scaled condition number κ) is

$$\mathrm{P}[A \in \Sigma_{n\times n}^{n-1} : \kappa(A) > t] \le \left(\frac{2.91(n+1)}{t}\right)^6, \quad \mathbb{K}=\mathbb{C}.$$

A similar bound for the real case also holds.

Our main result follows.

Theorem 1. *The probability distribution of $\kappa(A)$ for $A \in \Sigma$ equals the probability distribution of $\kappa(A)$ for $A \in \mathbb{GL}_{r,n+m-r}$. Namely, for every $t \in \mathbb{R}$,*

$$\mathrm{P}[\kappa(A) > t : A \in \Sigma] = \mathrm{P}[\kappa(A) > t : A \in \mathbb{GL}_{r\times(n+m-r)}].$$

Thanks to Theorem 1, we can translate every known result about the behavior of the condition number of random, full-rank matrices into the case of rank-deficient matrices. For example, Proposition 1 and Theorem 1 immediately imply the following result.

Corollary 1. *Let $t \ge n+m-2r+1$. Then,*

$$\frac{1}{(2\pi)^{\beta/2}}\left(\frac{c}{t}\right)^{\beta(n+m-2r+1)} \le \mathrm{P}\left[\frac{\kappa(A)}{\frac{n+m-r}{n+m-2r+1}} > t : A \in \Sigma\right] \le \frac{1}{(2\pi)^{\beta/2}}\left(\frac{C}{t}\right)^{\beta(n+m-2r+1)}.$$

Moreover,

$$\mathrm{E}[\log(\kappa(A)) : A \in \Sigma] \le \log\frac{n+m-r}{n+m-2r+1} + u.$$

As a by-product of our method of proof, we will be able to relate the total volume of Σ to the total volume of $\mathbb{GL}_{r,n+m-r}$. This will yield our second theorem.

Theorem 2. *The volume of Σ satisfies:*

$$Vol[\Sigma] = \frac{Vol[\mathcal{U}_n]Vol[\mathcal{U}_m]Vol[\mathcal{U}_{n+m-2r}]}{Vol[\mathcal{U}_{n-r}]Vol[\mathcal{U}_{m-r}]Vol[\mathcal{U}_r]Vol[\mathcal{U}_{n+m-r}]} Vol[\mathbb{S}(\mathbb{K}^{r(n+m-r)})],$$

where for $k \ge 1$, $\mathbb{S}(\mathbb{K}^k)$ is the unit sphere in $\mathbb{K}^k$ and $Vol[\mathcal{U}_k]$ is the volume of the unitary group (if $\mathbb{K}=\mathbb{C}$) or that of the orthogonal group (if $\mathbb{K}=\mathbb{R}$). Here, we use the convention $Vol[\mathcal{U}_0]=1$.

Formulae for $Vol[\mathcal{U}_k]$, $k \ge 1$ are known; see for example [Hua63].

$$\begin{array}{lll} Vol[\mathcal{U}_k] = \frac{(2\pi)^{\frac{k(k+1)}{2}}}{1!\cdot 2!\cdot 3!\cdots(k-1)!} & if & \mathbb{K}=\mathbb{C}, \\ Vol[\mathcal{U}_k] = 2\cdot(8\pi)^{\frac{k(k-1)}{4}}\prod_{j=1}^{k-1}\frac{\Gamma(j/2)}{\Gamma(j)} & if & \mathbb{K}=\mathbb{R}. \end{array}$$

For example, taking $m=n\ge 2$ and $r=n-1$, Theorem 2 yields the volume of the set of singular matrices,

$$Vol[A \in \mathbb{S}(\mathbb{K}^{n^2}) : \det(A)=0] = \begin{cases} 2n\frac{\pi^{n^2-1}}{\Gamma(n^2-2)} & \mathbb{K}=\mathbb{C}, \\ 2\frac{\pi^{\frac{n^2}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n^2-1}{2}\right)} & \mathbb{K}=\mathbb{R}. \end{cases}$$

In the case that $\mathbb{K}=\mathbb{C}$, the quantity $Vol[\Sigma]$ studied in Theorem 2 was already known, since Σ, or more precisely, its complex projective version $\mathbb{P}(\Sigma)$, is up to a lower-dimensional set a complex algebraic subvariety of the projective space of matrices. Hence its (projective) volume equals its algebraic degree times the volume of the complex projective space of dimension $\dim_{\mathbb{C}}\mathbb{P}(\Sigma)$. (This is a classical fact;

the reader may find a modern proof in [BP07, Corollary 14].) The algebraic degree and dimension of the set of rank r matrices of size $m\times n$ are known; see for example [BV88] and [Ful84, p. 261]. This way, the volume of Σ can be computed in the complex case, and the reader may check that the same number is obtained as in Theorem 2. I am not aware of any previous reference where the volume of Theorem 2 is computed for the real case. However, being such an elementary question, I would not be surprised if this result was already known.

Finally, exact formulas for the expected value of powers of the product of the non-zero singular values of $A\in\Sigma$ are given in Corollary 2.

2. An integral formula

Federer's coarea formula is an integral formula which generalizes the change of variables formula and Fubini's Theorem. The most general version we know may be found in [Fed69], but for our purposes a smooth version as used in [BCSS98] or [How93] suffices.

Definition 1. *Let X and Y be Riemannian manifolds, and let $F: X\longrightarrow Y$ be a C^1 surjective map. Let $k=\dim(Y)$ be the real dimension of Y. For every point $x\in X$ such that the differential $dF(x)$ is surjective, let $v_1^x,\ldots,v_k^x$ be an orthogonal basis of $Ker(dF(x))^\perp$. Then, we define the Normal Jacobian of F at x, $\mathrm{NJ}F(x)$, as the volume in the tangent space $T_{F(x)}Y$ of the parallelepiped spanned by $dF(x)(v_1^x),\ldots,dF(x)(v_k^x)$. In the case that $dF(x)$ is not surjective, we define $\mathrm{NJ}F(x)=0$.*

Theorem 3 (Federer's coarea formula)**.** *Let X,Y be two Riemannian manifolds of respective dimensions $k_1\geq k_2$. Let $F: X\longrightarrow Y$ be a C^1 surjective map, such that the differential mapping $dF(x)$ is surjective for almost all $x\in X$. Let $\psi: X\longrightarrow\mathbb{R}$ be an integrable mapping. Then, the following equality holds:*

$$\int_{x\in X}\psi(x)\mathrm{NJ}F(x)\,dX=\int_{y\in Y}\int_{x\in F^{-1}(y)}\psi(x)\;d(F^{-1}(y))\;dY. \tag{2.1}$$

The integral on the right-hand side of equation (2.1) must be read as follows: From Sard's Theorem, every $y\in Y$ out of a zero measure set is a regular value of F. Then, $F^{-1}(y)$ is a differentiable manifold of dimension k_1-k_2, and it inherits from X a structure of Riemannian manifold. Thus, it makes sense to integrate functions on $F^{-1}(y)$.

3. Proofs of the main theorems.

For a positive integer $r\in\mathbb{N}$, let $\mathbb{S}_r^+=\{Diag(\sigma_1>\ldots>\sigma_r):\sigma_j>0,\sigma_1^2+\cdots+\sigma_r^2=1\}$ be the unit sphere in the set of diagonal matrices with positive, real entries in decreasing order. Let

$$\begin{array}{rccc}\varphi: & \mathcal{U}_m\times\mathbb{S}_r^+\times\mathcal{U}_n & \longrightarrow & \Sigma\\ & (U,D,V) & \mapsto & U\begin{pmatrix}D&0\\0&0\end{pmatrix}V\end{array} \tag{3.1}$$

where $\mathcal{U}_k$ is the set of unitary matrices (if $\mathbb{K}=\mathbb{C}$) or the set of orthogonal matrices (if $\mathbb{K}=\mathbb{R}$). We want to use the coarea formula (2.1) for φ, so we must compute its normal jacobian and level sets. We do this task in propositions 2 and 3 below. Proposition 2 has at least a precedent in [Hua63, Chapter III], where similar quantities are computed for square, non-singular matrices.

Proposition 2. *Let* $\det(D) = \sigma_1 \cdots \sigma_r$ *and* $\Delta(D) = \prod_{k<j}(\sigma_k^2 - \sigma_j^2)$. *Then,*

$$\mathrm{NJ}\varphi(U,D,V) = \left(\frac{\det(D)^{n+m-2r+1-1/\beta}\Delta(D)}{\sqrt{2}^{r(n+m-r-1/2-\beta/2)}} \right)^{\beta}.$$

Proof. We write the proof for the complex case, as the real case is almost identical. By unitary invariance, the normal jacobian depends only on the diagonal matrix D. Hence, it suffices to consider the point (I,D,I). Now, observe that

$$d\varphi(I,D,I)(A,R,B) = A\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} B.$$

We compute the value of $d\varphi(I,D,I)$ in a orthogonal basis of $T_{(I,D,I)}\mathcal{U}_m \times \mathbb{S}_r \times \mathcal{U}_n$:

(A,R,B)	$d\varphi(I,D,I)(A,R,B)$	Cases
$(\delta_{jl}-\delta_{lj},0,0)$	$\delta_{jl}\sigma_l - \delta_{lj}\sigma_j$	$j<l\le r$
$(\delta_{jl}-\delta_{lj},0,0)$	$-\delta_{lj}\sigma_j$	$j\le r<l$
$(\delta_{jl}-\delta_{lj},0,0)$	0	$r<j<l$
$(0,0,\delta_{jl}-\delta_{lj})$	$\sigma_j\delta_{jl}-\sigma_l\delta_{lj}$	$j<l\le r$
$(0,0,\delta_{jl}-\delta_{lj})$	$\sigma_j\delta_{jl}$	$j\le r<l$
$(0,0,\delta_{jl}-\delta_{lj})$	0	$r<j<l$
$(i\delta_{jl}+i\delta_{lj},0,0)$	$i\delta_{jl}\sigma_l + i\delta_{lj}\sigma_j$	$j<l\le r$
$(i\delta_{jl}+i\delta_{lj},0,0)$	$i\delta_{lj}\sigma_j$	$j\le r<l$
$(i\delta_{jl}+i\delta_{lj},0,0)$	0	$r<j<l$
$(0,0,i\delta_{jl}+i\delta_{lj})$	$i\sigma_j\delta_{jl}+i\sigma_l\delta_{lj}$	$j<l\le r$
$(0,0,i\delta_{jl}+i\delta_{lj})$	$i\sigma_j\delta_{jl}$	$j\le r<j$
$(0,0,i\delta_{jl}+i\delta_{lj})$	0	$r<j<l$
$(i\delta_{jj},0,0)$	$i\sigma_j\delta_{jj}$	$j\le r$
$(i\delta_{jj},0,0)$	0	$r<j$
$(0,0,i\delta_{jj})$	$i\sigma_j\delta_{jj}$	$j\le r$
$(0,0,i\delta_{jj})$	0	$r<j$
$(0,R,0)$	$\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$	For every $R\in T_D\mathbb{S}_r^+$

We conclude that $d\varphi(I,D,I)$ preserves the orthogonality of the following orthonormal frame of $Ker(d\varphi(I,D,I))^{\perp}$:

$\frac{1}{2}(\delta_{jl}-\delta_{lj},0,\delta_{jl}-\delta_{lj})$, $j<l\le r$
$\frac{1}{2}(\delta_{jl}-\delta_{lj},0,-\delta_{jl}+\delta_{lj})$, $j<l\le r$
$\frac{1}{2}(i\delta_{jl}+i\delta_{lj},0,i\delta_{jl}+i\delta_{lj})$, $j<l\le r$
$\frac{1}{2}(i\delta_{jl}+i\delta_{lj},0,-i\delta_{jl}-i\delta_{lj})$, $j<l\le r$
$\frac{1}{\sqrt{2}}(\delta_{jl}-\delta_{lj},0,0)$, $j\le r<l\le m$
$\frac{1}{\sqrt{2}}(0,0,\delta_{jl}-\delta_{lj})$, $j\le r<l\le n$
$\frac{1}{\sqrt{2}}(i\delta_{jl}+i\delta_{lj},0,0)$, $j\le r<l\le m$
$\frac{1}{\sqrt{2}}(0,0,i\delta_{jl}+i\delta_{lj})$, $j\le r<l\le n$
$\frac{1}{\sqrt{2}}(i\delta_{jj},0,i\delta_{jj})$, $1\le j\le r$
$(0,R_j,0)$, where $\{R_j\}_{j=1\ldots r-1}$, is an orthonormal frame of $T_D\mathbb{S}_r^+$.

Hence, we can compute the normal jacobian of φ as the product of the norms of the images under $d\varphi(I,D,I)$ of these vectors:

$$\mathrm{NJ}\varphi(I,D,I) = \prod_{j<l\le r}\frac{(\sigma_j^2-\sigma_l^2)^2}{4} \prod_{j\le r<l\le n}\frac{\sigma_j^2}{2} \prod_{j\le r<l\le m}\frac{\sigma_j^2}{2} \prod_{1\le j\le r}\sqrt{2}\sigma_j =$$

$$\frac{1}{2^{r(n+m-r-3/2)}} \prod_{j<l\leq r} (\sigma_j^2 - \sigma_l^2)^2 \prod_{1\leq j\leq r} \sigma_j^{2n+2m-4r+1},$$

and the proposition follows. □

Proposition 3. *The quantity $Vol[\varphi^{-1}(A)]$ is constant a.e. Indeed, let $A \in \Sigma$ be such that its r non-zero singular values are different. Then,*

$$Vol[\varphi^{-1}(A)] = \left(2^{\frac{\beta+1}{2}}\pi^{\beta-1}\right)^r Vol[\mathcal{U}_{n-r}]Vol[\mathcal{U}_{m-r}].$$

Here, the convention $Vol[\mathcal{U}_0] = 1$ is used.

Proof. We prove the proposition for $\mathbb{K} = \mathbb{C}$, being very similar for $\mathbb{K} = \mathbb{R}$. By unitary invariance it suffices to prove this formula in the case that $A = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, with $D \in \mathbb{S}_r^+$ such that all its entries are different. Now, note that

$$\varphi^{-1}(A) = \left\{(U, \Lambda, V) \in \mathcal{U}_m \times \mathbb{S}_r^+ \times \mathcal{U}_n : \Lambda = D, UAV = A\right\}.$$

Let $(U, D, V) \in \varphi^{-1}(A)$. Write $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ where the blocks are compatible with those of A. Then, the equality $UA^{2k}U^* = A^{2k}$, which holds for every positive integer k, reads

$$\begin{pmatrix} U_1D^{2k}U_1^* & U_1D^2kU_3^* \\ U_3D^{2k}U_1^* & U_3D^{2k}U_3^* \end{pmatrix} = \begin{pmatrix} D^{2k} & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, $U_1D^{2k}U_1^* = D^{2k}$ for $k \geq 1$, so that U_1 is invertible and

$$(U_1D^2U_1^*)(U_1D^2U_1^*) = D^4 = U_1D^4U_1^*.$$

After left and right multiplication by U_1^{-1} and $(U_1^*)^{-1}$, we get $D^2U_1^*U_1D^2 = D^4$, namely $U_1^*U_1 = I_r$ and U_1 is unitary. As U is itself unitary, this implies $U_2 = U_3 = 0$. A similar argument for V shows that

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_4 \end{pmatrix}, \quad V = \begin{pmatrix} V_1 & 0 \\ 0 & V_4 \end{pmatrix},$$

where $U_1, V_1 \in \mathcal{U}_r$, $U_4 \in \mathcal{U}_{n-r}$ and $V_4 \in \mathcal{U}_{m-r}$. Hence,

$$Vol[\varphi^{-1}(A)] = Vol\left[\begin{pmatrix} U_1 & 0 \\ 0 & U_4 \end{pmatrix}, \begin{pmatrix} V_1 & 0 \\ 0 & V_4 \end{pmatrix} : U_1DV_1 = D\right] =$$

Finally, note that $U_1DV_1 = D$ is a singular value decomposition of D. Since all the entries of D are different, the singular value decomposition is unique up to simultaneous multiplication of U_1 and V_1 by a scalar of modulus 1 and its complex conjugate, respectively. Hence, $U_1DV_1 = D$ implies $U_1 = Diag(e^{i\theta_1}, \ldots, e^{i\theta_r})$ and $V_1 = Diag(e^{-i\theta_1}, \ldots, e^{-i\theta_r})$. Thus,

$$Vol[\varphi^{-1}(A)] = Vol\left[\begin{pmatrix} Diag(e^{i\theta_j}) & 0 \\ 0 & U_4 \end{pmatrix}, \begin{pmatrix} Diag(e^{-i\theta_j}) & 0 \\ 0 & V_4 \end{pmatrix} : \theta_j \in [0, 2\pi)\right] =$$

$$\sqrt{2}^r(2\pi)^r Vol[\mathcal{U}_{n-r}]Vol[\mathcal{U}_{m-r}].$$

This finishes the proof. □

Theorem 4. *For a matrix $A \in \Sigma$, let $\sigma(A) = (\sigma_1(A), \dots, \sigma_r(A)) \in \mathbb{R}^k$ be set of (ordered) singular values of A. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a measurable mapping. Then,*

$$\int_{A\in\Sigma} \phi(\sigma(A))\, d\Sigma = H \int_{D\in\mathbb{S}_r^+} \phi(D) \left(\det(D)^{n+m-2r+1-1/\beta} \Delta(D)\right)^\beta \, d\mathbb{S}_r^+,$$

where

$$H = \frac{Vol[\mathcal{U}_n] Vol[\mathcal{U}_m]}{Vol[\mathcal{U}_{n-r}] Vol[\mathcal{U}_{m-r}] \sqrt{2}^{\beta r(n+m-r+1/2-\beta/2)+r} \pi^{r(\beta-1)}}.$$

Proof. From (2.1),

$$\int_{(U,D,V)\in\mathcal{U}_n\times\mathbb{S}_r^+\times\mathcal{U}_m} \phi(D) \mathrm{NJ}\varphi(U,D,V)\, d(\mathcal{U}_m \times \mathbb{S}_r^+ \times \mathcal{U}_n) =$$

$$\int_{A\in\Sigma} Vol[\varphi^{-1}(A)] \phi(\sigma(A))\, d\Sigma.$$

Now, $\mathrm{NJ}\varphi(U,D,V)$ is known from Proposition 2 and $Vol[\varphi^{-1}(A)]$ is known from Proposition 3. Apply Fubini's theorem to the left hand side to get

$$Vol[\mathcal{U}_n] Vol[\mathcal{U}_m] \int_{D\in\mathbb{S}_r^+} \phi(D) \left(\frac{\det(D)^{n+m-2r+1-1/\beta}\Delta(D)}{\sqrt{2}^{r(n+m-r-1/2-\beta/2)}}\right)^\beta d\mathbb{S}_r^+ =$$

$$\left(2^{\frac{\beta+1}{2}} \pi^{\beta-1}\right)^r Vol[\mathcal{U}_{n-r}] Vol[\mathcal{U}_{m-r}] \int_{A\in\Sigma} \phi(\sigma(A))\, d\Sigma.$$

That is,

$$\int_{A\in\Sigma} \phi(\sigma(A))\, d\Sigma = H \int_{D\in\mathbb{S}_r^+} \phi(D) \left(\det(D)^{n+m-2r+1-1/\beta} \Delta(D)\right)^\beta \, d\mathbb{S}_r^+.$$

□

3.1. **Proof of Theorem 1.** Let

$$\phi_1(x_1 \dots, x_r) = 1, \quad \phi_2(x_1 \dots, x_r) = \begin{cases} 1 & \frac{x_1}{x_r} > t \\ 0 & \textit{otherwise} \end{cases}.$$

From Theorem 4 applied to ϕ_1,

$$Vol[\Sigma] = H \int_{D\in\mathbb{S}_r^+} (\det(D)^{n+m-2r+1-1/\beta} \Delta(D))^\beta \, d\mathbb{S}_r^+.$$

From Theorem 4 applied to ϕ_2,

$$Vol[A \in \Sigma : \kappa(A) > t] = H \int_{D\in\mathbb{S}_r^+, \kappa(D)>t} (\det(D)^{n+m-2r+1-1/\beta} \Delta(D))^\beta \, d\mathbb{S}_r^+.$$

Thus,

$$\mathrm{P}[A \in \Sigma : \kappa(A) > t] = \frac{\int_{D\in\mathbb{S}_r^+, \kappa(A)>t} (\det(D)^{n+m-2r+1-1/\beta} \Delta(D))^\beta \, d\mathbb{S}_r^+}{\int_{D\in\mathbb{S}_r^+} (\det(D)^{n+m-2r+1-1/\beta} \Delta(D))^\beta \, d\mathbb{S}_r^+}.$$

Now, apply again Theorem 4 to the set of full-rank matrices of size $r \times (n+m-r)$, and the same formula is obtained. Hence, both quantities are equal, and Theorem 1 follows.

3.2. **Proof of Theorem 2.** From Theorem 4,

$$Vol[\Sigma] = H_1 \int_{D\in\mathbb{S}_r^+} (\det(D)^{n+m-2r+1-1/\beta}\Delta(D))^\beta \, d\mathbb{S}_r^+,$$

where

$$H_1 = \frac{Vol[\mathcal{U}_n]Vol[\mathcal{U}_m]}{Vol[\mathcal{U}_{n-r}]Vol[\mathcal{U}_{m-r}]\sqrt{2}^{\beta r(n+m-r+1/2-\beta/2)+r}\pi^{r(\beta-1)}}.$$

Also from Theorem 4,

$$Vol[\mathbb{S}(\mathbb{GL}_{r,n+m-r})] = H_2 \int_{D\in\mathbb{S}_r^+} (\det(D)^{n+m-2r+1-1/\beta}\Delta(D))^\beta \, d\mathbb{S}_r^+,$$

where

$$H_2 = \frac{Vol[\mathcal{U}_r]Vol[\mathcal{U}_{n+m-r}]}{Vol[\mathcal{U}_{n+m-2r}]\sqrt{2}^{\beta r(n+m-r+1/2-\beta/2)+r}\pi^{r(\beta-1)}}.$$

We conclude that

$$Vol[\Sigma] = \frac{H_1}{H_2}Vol[\mathbb{S}(\mathbb{GL}_{r,n+m-r})] = \frac{H_1}{H_2}Vol[\mathbb{S}(\mathbb{K}^{r(n+m-r)})].$$

Finally,

$$\frac{H_1}{H_2} = \frac{Vol[\mathcal{U}_n]Vol[\mathcal{U}_m]Vol[\mathcal{U}_{n+m-2r}]}{Vol[\mathcal{U}_{n-r}]Vol[\mathcal{U}_{m-r}]Vol[\mathcal{U}_r]Vol[\mathcal{U}_{n+m-r}]}.$$

3.3. **Expected value of the product of non-zero singular values.**

Corollary 2. *Let $k\in\mathbb{N}$ and let $DET_k : \Sigma \longrightarrow \mathbb{R}$, $DET_k(A) = (\sigma_1(A)\cdots\sigma_r(A))^{\beta k}$. Then,*

$$\mathrm{E}[DET_k(A) : A\in\Sigma] = \frac{Vol[\mathcal{U}_{n+m-r}]Vol[\mathcal{U}_{n+m+k-2r}]Vol[\mathbb{S}(\mathbb{K}^{r(n+m+k-r)})]}{Vol[\mathcal{U}_{n+m-2r}]Vol[\mathcal{U}_{n+m+k-r}]Vol[\mathbb{S}(\mathbb{K}^{r(n+m-r)})]}\sqrt{2}^{\beta rk}.$$

Proof. Let

$$\phi(x_1\ldots,x_r) = (x_1\cdots x_r)^{\beta k}.$$

From Theorem 4,

$$\int_\Sigma DET_k \, d\Sigma = H_1 \int_{D\in\mathbb{S}_r^+} (\det(D)^{n+m+k-2r+1-1/\beta}\Delta(D))^\beta \, d\mathbb{S}_r^+,$$

where

$$H_1 = \frac{Vol[\mathcal{U}_n]Vol[\mathcal{U}_m]}{Vol[\mathcal{U}_{n-r}]Vol[\mathcal{U}_{m-r}]\sqrt{2}^{\beta r(n+m-r+1/2-\beta/2)+r}\pi^{r(\beta-1)}}.$$

Also from Theorem 4,

$$Vol[\mathbb{S}(\mathbb{GL}_{r,n+m+k-r})] = H_2 \int_{D\in\mathbb{S}_r^+} (\det(D)^{n+m+k-2r+1-1/\beta}\Delta(D))^\beta \, d\mathbb{S}_r^+,$$

where

$$H_2 = \frac{Vol[\mathcal{U}_r]Vol[\mathcal{U}_{n+m+k-r}]}{Vol[\mathcal{U}_{n+m+k-2r}]\sqrt{2}^{\beta r(n+m+k-r+1/2-\beta/2)+r}\pi^{r(\beta-1)}}.$$

Thus,

$$\int_\Sigma DET_k \, d\Sigma = \frac{H_1}{H_2}Vol[\mathbb{S}(\mathbb{GL}_{r,n+m+k-r})] = \frac{H_1}{H_2}Vol[\mathbb{S}(\mathbb{K}^{r(n+m+k-r)})] =$$

$$\frac{Vol[\mathcal{U}_n]Vol[\mathcal{U}_m]Vol[\mathcal{U}_{n+m+k-2r}]\sqrt{2}^{\beta rk}}{Vol[\mathcal{U}_{n-r}]Vol[\mathcal{U}_{m-r}]Vol[\mathcal{U}_r]Vol[\mathcal{U}_{n+m+k-r}]}Vol[\mathbb{S}(\mathbb{K}^{r(n+m+k-r)})].$$

Finally, divide by $Vol[\Sigma]$ (see Theorem 2) to get the result. □

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